PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 3, March 1983

AN EXTRAPOLATION THEOREM IN THE THEORY OF A_p WEIGHTS

JOSE GARCIA-CUERVA

ABSTRACT. A new proof is given of the extrapolation theorem of J. L. Rubio de Francia [7]. Unlike the proof in [7], the proof presented here is independent of the theory of vector valued inequalities.

I am indebted to J. L. Rubio de Francia for calling my attention to the following:

THEOREM. *Let T be a sublinear operator defined on a class of measurable functions in $\mathbf{R}^n$. Suppose that, for some p_0 with $1 \leq p_0 < \infty$, and for every weight w in the class A_{p_0} of B. Muckenhoupt (see* [**1** *and* **5**]*), T satisfies an inequality*

$$\int_{\mathbf{R}^n} |Tf(x)|^{p_0} w(x)\,dx \leq C \int_{\mathbf{R}^n} |f(x)|^{p_0} w(x)\,dx \tag{1}$$

for every f, where $C = C_{p_0}(w)$ depends only on the A_{p_0} constant for w. Then, for every p with $1 < p < \infty$, and every weight w in the class A_p, T satisfies an inequality

$$\int_{\mathbf{R}^n} |Tf(x)|^{p} w(x)\,dx \leq C \int_{\mathbf{R}^n} |f(x)|^{p} w(x)\,dx \tag{2}$$

for every f, where $C = C_p(w)$ depends only on the A_p constant for w.

The original proof is given in [**7**]. It is based upon the equivalence between weighted inequalities and vector valued inequalities obtained in [**6**]. The purpose of the present note is to give a new proof of the theorem stated above. This new proof is independent of vector valued inequalities and lies completely within the realm of the theory of A_p weights.

The proof will follow after two lemmas. The letter C will be used to denote a constant, not necessarily the same at each occurrence, as has been done already in the statement of the theorem.

LEMMA 1. *Let $1 < p < \infty$, $w \in A_p$, $0 < t \leq 1$. For a function $g \geq 0$, define $G = \{M(g^{1/t}w)w^{-1}\}^t$, where M stands for the Hardy-Littlewood maximal operator. Then*

(i) *$g \mapsto G$ is an operator bounded in $L^{p'/t}(w)$, where $p' = p/(p-1)$, the exponent conjugate to p;*

(ii) *$(gw, Gw) \in A_{p-tp/p'}$, the class of pairs of weights appearing in* [**5**].

Besides, the operator norm in (i) *and the constant for the pair of weights in* (ii) *depend only on the A_p constant for w.*

Received by the editors March 5, 1982.
1980 *Mathematics Subject Classification*. Primary 42B25.
Key words and phrases. A_p weights.

PROOF. (i)

$$\begin{aligned}\|G\|_{p'/t,w}^{p'/t} &= \int_{\mathbf{R}^n} |G(x)|^{p'/t} w(x)\,dx = \int_{\mathbf{R}^n} |M(g^{1/t}w)|^{p'} w^{1-p'} \\ &\leq C \int_{\mathbf{R}^n} g^{p'/t} w^{p'} w^{1-p'} = C\|g\|_{p'/t,w}^{p'/t},\end{aligned}$$

since $w^{1-p'} = w^{-1/(p-1)} \in A_{p'}$.

(ii) Let $p_0 = p - tp/p'$. Clearly $p_0 \geq 1$, since $p_0 - 1 = (1-t)p/p' \geq 0$. If $t = 1$, then $p_0 = 1$ and it is immediate that $(gw, Gw) \in A_1$ with constant 1, because $M(gw) = Gw$. Let $0 < t < 1$. We have to prove that

$$\left(\frac{1}{|Q|}\int_Q gw\right)\left(\frac{1}{|Q|}\int_Q (Gw)^{-1/(p_0-1)}\right)^{p_0-1} \leq C$$

with a constant C independent of the cube Q, where $|Q|$ stands for the Lebesgue measure of Q. The left-hand side of this inequality is

$$\left(\frac{1}{|Q|}\int_Q gw\right)\left(\frac{1}{|Q|}\int_Q M(g^{1/t}w)^{-tp'/(p(1-t))} w^{-p'/p}\right)^{(1-t)p/p'}.$$

By using Hölder's inequality with exponents $1/t$ and its conjugate $1/(1-t)$, we see that the last line is bounded by

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q g^{1/t}w\right)^t\left(\frac{1}{|Q|}\int_Q w\right)^{1-t}\left(\frac{1}{|Q|}\int_Q M(g^{1/t}w)^{-tp'/(p(1-t))} w^{-1/(p-1)}\right)^{(1-t)p/p'} \\ &\quad\leq \left(\frac{1}{|Q|}\int_Q g^{1/t}w\right)^t\left(\frac{1}{|Q|}\int_Q w\right)^{1-t}\left(\frac{1}{|Q|}\int_Q g^{1/t}w\right)^{-t} \\ &\quad\quad\times\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{(p-1)(1-t)} \leq C\end{aligned}$$

where C is precisely the A_p constant for w, raised to the power $1-t$. This finishes the proof. □

OBSERVATION. If we set $p - tp/p' = p_0 < p$, we shall have $1 - t/p' = p_0/p$, or, in other words, $p'/t = (p/p_0)'$. With this change of notation, Lemma 1 can be restated as follows: *Let $1 \leq p_0 < p$, $w \in A_p$. Then, for every $g \geq 0$ belonging to $L^{(p/p_0)'}(w)$, there is a $G \geq g$ such that $\|G\|_{(p/p_0)',w} \leq C\|g\|_{(p/p_0)',w}$ and $(gw, Gw) \in A_{p_0}$, with both C and the A_{p_0} constant for the pair depending only on the A_p constant for w.*

Now Lemma 1 can be used to obtain a stronger result:

LEMMA 2. (a) *If $1 \leq p_0 < p$ and $w \in A_p$, then, for every $g \geq 0$ belonging to $L^{(p/p_0)'}(w)$, there is a $G \geq g$ such that $\|G\|_{(p/p_0)',w} \leq C\|g\|_{(p/p_0)',w}$ and $Gw \in A_{p_0}$, with both C and the A_{p_0} constant for Gw depending only on the A_p constant for w.*

(b) *If $1 < p < p_0$ and $w \in A_p$, then, for every $g \geq 0$ belonging to $L^{p/(p_0-p)}(w)$, there is a $G \geq g$ such that $\|G\|_{p/(p_0-p),w} \leq C\|g\|_{p/(p_0-p),w}$ and $G^{-1}w \in A_{p_0}$, with both C and the A_{p_0} constant for $G^{-1}w$ depending only on the A_p constant for w.*

PROOF. (a) Let $g_0 = g$. According to the observation made, there is $g_1 \geq g$ such that $\|g_1\|_{(p/p_0)',w} \leq C\|g\|_{(p/p_0)',w}$ and the inequality,

$$\int_{\{x\,:\,Mf(x)>s\}} g_0 w \leq Cs^{-p_0}\int_{\mathbf{R}^n} |f(x)|^{p_0} g_1 w,$$

holds for every function f and every $s > 0$, with C depending only on the A_p constant for w. Then we proceed by induction. We can use g_1 in place of g_0 and continue in this way. In general, given g_j, we obtain $g_{j+1} \geq g_j$ such that $||g_{j+1}||_{(p/p_0)',w} \leq C||g_j||_{(p/p_0)',w} \leq \cdots \leq C^{j+1}||g||_{(p/p_0)',w}$ and the inequality

$$\int_{\{x:\, Mf(x)>s\}} g_j w \leq C s^{-p_0} \int_{\mathbf{R}^n} |f(x)|^{p_0} g_{j+1} w, \tag{3}$$

holds for every function f and every $s > 0$, with C depending only on the A_p constant for w. Now consider $G = \sum_{j=0}^{\infty}(C+1)^{-j} g_j$ with C the same as above. Since $(C+1)^{-j}||g_j||_{(p/p_0)',w} \leq (C/(C+1))^j ||g||_{(p/p_0)',w}$, the series converges in $L^{(p/p_0)'}(w)$ and we get $G \geq g$ with $||G||_{(p/p_0)',w} \leq (C+1)||g||_{(p/p_0)',w}$ and also, adding in j the inequalities (3),

$$\int_{\{x:\, Mf(x)>s\}} Gw \leq C(C+1)s^{-p_0} \int_{\mathbf{R}^n} |f(x)|^{p_0} Gw,$$

for every $s > 0$; in other words: $Gw \in A_{p_0}$ and its A_{p_0} constant depends only on the A_p constant for w.

(b) Now $1 < p < p_0$ and $w \in A_p$. This is the same as saying that $1 < p_0' < p'$ and $u \equiv w^{-1/(p-1)} \in A_{p'}$. Therefore, we can apply part (a) to conclude that, for every $h \geq 0$ belonging to $L^{(p'/p_0')'}(u)$, there is an $H \geq h$ such that $||H||_{(p'/p_0')',u} \leq C||h||_{(p'/p_0')',u}$ and $Hu \in A_{p_0'}$, with C and the $A_{p_0'}$ constant for Hu depending only on the A_p constant for w. But $(p'/p_0')' = (p_0-1)p/(p_0-p)$, so that $h \in L^{(p'/p_0')'}(u)$ if and only if $h^{p_0-1}w^{-(p_0-p)/(p-1)} \in L^{p/(p_0-p)}(w)$. Also $Hu \in A_{p_0'}$ if and only if $(Hu)^{-1/(p_0'-1)} = [H^{p_0-1}w^{-(p_0-p)/(p-1)}]^{-1}w \in A_{p_0}$. Thus, if we are given $g \geq 0$ in $L^{p/(p_0-p)}(w)$, we just write $g = h^{p_0-1}w^{-(p_0-p)/(p-1)}$ with $h \in L^{(p'/p_0')'}(u)$, obtain the corresponding H and then define $G = H^{p_0-1}w^{-(p_0-p)/(p-1)}$. This proves part (b). □

We are ready to give the

PROOF OF THE THEOREM. (a) Let $1 \leq p_0 < p$, $w \in A_p$ and $f \in L^p(w)$,

$$||Tf||_{p,w}^{p_0} = |||Tf|^{p_0}||_{p/p_0,w} = \int_{\mathbf{R}^n} |Tf|^{p_0} gw$$

for some $g \geq 0$ with $||g||_{(p/p_0)',w} = 1$. Associate with g a function G as in Lemma 2. Then, the last integral is bounded by

$$\int_{\mathbf{R}^n} |Tf|^{p_0} Gw \leq C \int_{\mathbf{R}^n} |f|^p Gw \leq C|||f|^{p_0}||_{p/p_0,w}||G||_{(p/p_0)',w} \leq C||f||_{p,w}^{p_0}.$$

(b) Let $1 < p < p_0$, $w \in A_p$ and $f \in L^p(w)$,

$$||f||_{p,w}^{p_0} = |||f|^{p_0}||_{p/p_0,w} = \int_{\mathbf{R}^n} |f|^{p_0} g^{-1} w$$

for some $g \geq 0$ such that $||g||_{p/(p_0-p),w} = 1$. Associate with g a function G as in part (b) of Lemma 2. Then

$$\begin{aligned} ||Tf||_{p,w}^{p_0} &= |||Tf|^{p_0}||_{p/p_0,w} \leq \left(\int_{\mathbf{R}^n} |Tf|^{p_0} G^{-1} w\right) ||G||_{p/(p_0-p),w} \\ &\leq C \int_{\mathbf{R}^n} |Tf|^{p_0} G^{-1} w \leq C \int_{\mathbf{R}^n} |f|^{p_0} G^{-1} w \leq C \int_{\mathbf{R}^n} |f|^{p_0} g^{-1} w = C||f||_{p,w}^{p_0}. \end{aligned}$$

This ends the proof. □

OBSERVATION. There is a weak-type version of the theorem, in which one substitutes for (1) the inequality

$$(1') \qquad \int_{\{x:\ |Tf(x)|>s\}} w(x)\,dx \le Cs^{-p_0}\int_{\mathbf{R}^n} |f(x)|^{p_0} w(x)\,dx, \qquad s>0,$$

and also changes the inequality (2) in the conclusion to

$$(2') \qquad \int_{\{x:\ |Tf(x)|>s\}} w(x)\,dx \le Cs^{-p}\int_{\mathbf{R}^n} |f(x)|^{p} w(x)\,dx, \qquad s>0.$$

The proof of this weak-type version is as follows. For $s>0$, let us call $E_s = \{x: |Tf(x)| > s\}$. For a set E, $w(E)$ will stand for $\int_E w(x)\,dx$ and χ_E will denote the characteristic function of E.

(a) Let $1 \le p_0 < p$, $w \in A_p$ and $f \in L^p(w)$. Then, for $s>0$,

$$s^{p_0} w(E_s)^{p_0/p} = s^{p_0} \|\chi_{E_s}\|_{p/p_0,w} = s^{p_0}\int_{\mathbf{R}^n} \chi_{E_s} g w$$

for some $g \ge 0$ with $\|g\|_{(p/p_0)',w} = 1$. Associate with g a function G as in Lemma 2, part (a). Then

$$s^{p_0} w(E_s)^{p_0/p} \le s^{p_0}\int_{\mathbf{R}^n} \chi_{E_s} G w \le C\int_{\mathbf{R}^n} |f|^{p_0} G w \le C\|f\|_{p,w}^{p_0}.$$

Thus, we have proved that $w(E_s) \le Cs^{-p}\|f\|_{p,w}^p$, which is precisely (2′).

(b) Let $1 < p < p_0$, $w \in A_p$ and $f \in L^p(w)$. Choose g as in the proof of the corresponding part of the strong type theorem. Associate with g the function G as in Lemma 2 part (b). Then, for $s>0$,

$$\begin{aligned} s^{p_0} w(E_s)^{p_0/p} &= s^{p_0}\|\chi_{E_s}\|_{p/p_0,w} \le Cs^{p_0}\int_{\mathbf{R}^n} \chi_{E_s} G^{-1} w \\ &\le C\int_{\mathbf{R}^n} |f|^{p_0} G^{-1} w \le C\int_{\mathbf{R}^n} |f|^{p_0} g^{-1} w = C\|f\|_{p,w}^{p_0}, \end{aligned}$$

which gives (2′) as before.

The two theorems we have proved are also valid for the weights associated to other maximal operators. Some interesting examples are given in [**7**]. Our approach works for all of them because the description of the classes A_p is formally the same. These general weights do not necessarily satisfy a reverse Hölder's inequality. However, in the particular situation we are dealing with, the A_p weights do satisfy a reverse Hölder's inequality (see [**1** and **5**]). This fact allows us to strengthen the weak type theorem so that the strong type theorem becomes a corollary. Indeed, we can get the strong type inequalities (2) from the weak type inequalities (1′). We just need to realize that, because of the reverse Hölder's inequality, if $w \in A_p$ and $1 < p < \infty$, there is $\epsilon > 0$ such that $w \in A_{p-\epsilon}$ also. Then, Marcinkiewicz interpolation theorem applies, and we obtain the strong type estimate.

The case $p=1$ has been excluded. In fact, it has to be excluded. We can always extrapolate from $p_0 = 1$ to any p; but extrapolation down to $p=1$ is not legitimate. For example, the maximal partial sum operator of Fourier series is bounded in $L^p(w)$ whenever $1 < p < \infty$ and $w \in A_p$ (see [**3**]), but it fails to be of weak type 1, 1 with respect to Lebesgue measure.

We should mention that the case $t = 1$ of our Lemma 1 has been used by R. Coifman, P. Jones and J. L. Rubio de Francia [**2**] to give a simple constructive proof of the factorization theorem of P. Jones [4]. Their proof simplifies a previous (nonconstructive) proof by J. L. Rubio de Francia [7], which used vector valued inequalities.

REFERENCES

1. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
2. R. Coifman, P. Jones and J. L. Rubio de Francia, *On a constructive decomposition of B.M.O. functions and factorizations of A_p weights*, Proc. Amer. Math. Soc. (to appear).
3. R. Hunt and W. S. Young, *A weighted norm inequality for Fourier series*, Bull. Amer. Math. Soc. **80** (1974), 274–277.
4. P. Jones, *Factorization of A_p weights*, Ann. of Math. **111** (1980), 511–530.
5. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
6. J. L. Rubio de Francia, *Weighted norm inequalities and vector valued inequalities*, Lecture Notes in Math., vol. 908, Springer-Verlag, Berlin and New York, 1982, pp. 86–101.
7. ——, *Factorization theory and A_p weights*, Amer. J. Math. (to appear).

UNIVERSIDAD AUTÓNOMA, CANTOBLANCO, MADRID 34, SPAIN

Current address: Facultad de Ciencias (Matemáticas), Universidad de Salamanca, Salamanca, Spain