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MAXIMAL OPERATORS FOR THE HOLOMORPHIC ORNSTEIN–UHLENBECK SEMIGROUP

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ABSTRACT

For each p in $[1,\infty)$ let $\mathbf{E}_p$ denote the closure of the region of holomorphy of the Ornstein–Uhlenbeck semigroup $\{\mathscr{H}_t : t > 0\}$ on L^p with respect to the Gaussian measure. Sharp weak type and strong type estimates are proved for the maximal operator $f \mapsto \mathscr{H}_p^* f = \sup\{|\mathscr{H}_z f| : z \in \mathbf{E}_p\}$ and for a class of related operators. As a consequence, a new and simpler proof of the weak type 1 estimate is given for the maximal operator associated to the Mehler kernel.

1. *Introduction*

In this paper we investigate the boundedness on L^p spaces of a class of maximal operators associated to the holomorphic Ornstein–Uhlenbeck semigroup on finite-dimensional Euclidean space. We shall be working with the Gaussian measure γ on $\mathbb{R}^d$ whose density with respect to Lebesgue measure is $\gamma_0(x) = \pi^{-d/2} e^{-|x|^2}$. The Ornstein–Uhlenbeck semigroup is the symmetric diffusion semigroup $\{\mathscr{H}_t : t \geqslant 0\}$ on $(\mathbb{R}^d, \gamma)$ whose integral kernel $h_t(x,y)$ with respect to the Gaussian measure is the Mehler kernel; see (2.2). The function $t \mapsto h_t$ has analytic continuation to a distribution-valued function $z \mapsto h_z$, which is holomorphic in $\operatorname{Re} z > 0$ and continuous in $\operatorname{Re} z \geqslant 0$. Let $\{\mathscr{H}_z : \operatorname{Re} z \geqslant 0\}$ denote the corresponding family of continuous operators from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$.

J. P. Epperson proved in [**2**] that the operator $\mathscr{H}_z$ extends to a bounded operator on $L^p(\gamma)$ if and only if z is in the closed $i\pi$-periodic set $\mathbf{E}_p$ defined by

$$\{x + iy \in \mathbb{C} : |\sin y| \leqslant \tan \phi_p \ \sinh x\}, \quad \phi_p = \arccos|2/p - 1|. \tag{1.1}$$

The map $z \mapsto \mathscr{H}_z$ from $\mathbf{E}_p$ to the space of bounded operators on $L^p(\gamma)$ is continuous in the strong operator topology and its restriction to the interior of $\mathbf{E}_p$ is analytic.

The aim of this paper is to analyse the boundedness properties in $L^p(\gamma)$ of the maximal operator

$$\mathscr{H}_p^* f(x) = \sup_{z \in \mathbf{E}_p} |\mathscr{H}_z f(x)|.$$

By the Banach principle (see [**5**]), it is well known that this maximal operator is a key tool to investigate the almost everywhere convergence of $\mathscr{H}_z f$ to $\mathscr{H}_{z_0} f$ as z tends to z_0, for f in $L^p(\gamma)$. We remark that, in particular, $\mathscr{H}_1^*$ is the maximal operator for the Mehler kernel, which is known to be of weak type 1 and of strong type p for each $p > 1$. The strong type result for this case is a consequence of the

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Littlewood–Paley–Stein theory for diffusion semigroups [**9**]; the weak type estimate is due to B. Muckenhoupt [**7**] in the one-dimensional case and to P. Sjögren [**8**] in higher dimensions. A different proof of the higher-dimensional result has been given by T. Menárguez, S. Pérez and F. Soria [**6**]. As part of our more general results, we shall give another, simpler proof of this result (see Theorems 3.2 and 4.3 below).

For $1 < p \leqslant 2$ we shall prove that the operator $\mathscr{H}_p^*$ is of strong type q for each q in (p, p'), but not of weak type p.

By the periodicity properties of the semigroup $\mathscr{H}_z$, we may restrict the parameter z to the set $\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leqslant \operatorname{Im} z \leqslant \pi/2\}$. In our analysis we use a change of variables τ introduced in [**3**]; see (2.6). With this change of variables, the set $\mathbf{F}_p \cup \{\infty\}$ is the image of the sector $\mathbf{S}_{\phi_p} = \{\zeta \in \mathbb{C} : |\zeta| \leqslant 1, 0 \leqslant \arg \zeta \leqslant \phi_p\}$ and the kernel of the operator $\mathscr{H}_{\tau(\zeta)}$ is described by a simpler formula; see (2.7). Hence we are led to study the maximal operator

$$\sup_{\zeta \in \mathbf{S}_{\phi_p}} |\mathscr{H}_{\tau(\zeta)} f(x)|. \tag{1.2}$$

As in previous papers on the subject (see for instance [**3**, **4**, **6**, **8**]), the positive results are proved by decomposing the operator into a 'local' part, given by a kernel living close to the diagonal, and the remaining or 'global' part. The local part can be controlled by the maximal operator associated to the Euclidean heat semigroup. The global part is bounded by an integral operator with positive kernel.

The failure of the strong type estimate at the end point p, for $1 < p \leqslant 2$, is only due to the behaviour of the Mehler kernel $h_{\tau(\zeta)}$ when ζ approaches the point $e^{i\phi_p}$ on the boundary of $\mathbf{S}_{\phi_p}$. We remark that at this point the operator $\mathscr{H}_{\tau(\zeta)}$ is isometrically equivalent to the Fourier transform from $L^p(dx)$ to $L^{p'}(dx)$.

Therefore it is natural to investigate the smaller maximal operator defined by taking in (1.2) the supremum only over the set obtained by deleting from $\mathbf{S}_{\phi_p}$ a small neighbourhood of $e^{i\phi_p}$. We shall prove, again with the method described above, that for each p in $(1, 2)$ this smaller operator is of weak type p, but not of strong type p. For $p = 2$ we prove a comparison result with the Schrödinger maximal operator, which allows us to prove the failure of the weak type 2 estimate by using a result of Carleson [**1**]; see Theorem 5.3.

The paper is organized as follows. In Section 2 we recall some basic properties of the Ornstein–Uhlenbeck semigroup and we summarize our results. In Section 3 we estimate the local parts and in Section 4 the global parts of the maximal operators. Negative results will be proved in Section 5.

2. *Preliminaries and statement of results*

The Ornstein–Uhlenbeck semigroup on $\mathbb{R}^d$ is the family of operators $\{\mathscr{H}_t : t \geqslant 0\}$ defined on test functions by

$$\begin{aligned} \mathscr{H}_t f(x) &= \int h_t(x, y) f(y) d\gamma(y), \qquad \forall\ t > 0, \\ \mathscr{H}_0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

where

$$h_t(x, y) = (1 - e^{-2t})^{-d/2} \exp\left(\frac{1}{2}\frac{1}{e^t + 1}|x + y|^2 - \frac{1}{2}\frac{1}{e^t - 1}|x - y|^2\right) \tag{2.2}$$

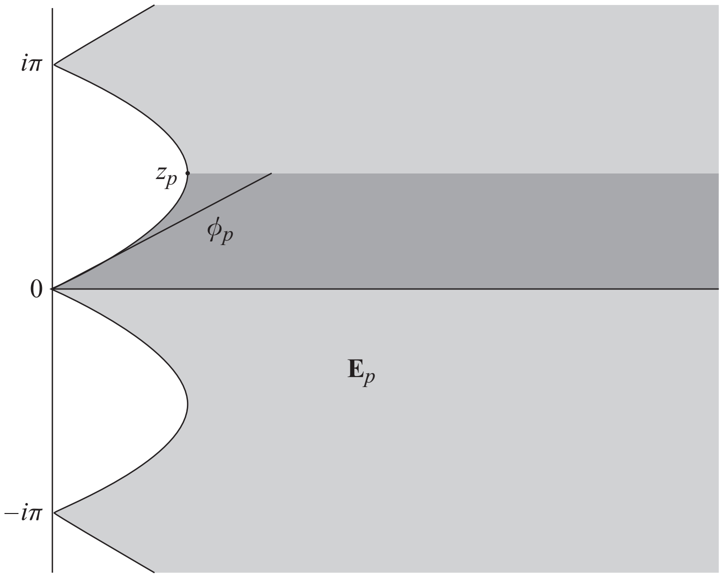

FIGURE 1. *The region* $\mathbf{E}_p$*; the darker area is* $\mathbf{F}_p$.

is the Mehler kernel with respect to the Gaussian measure. Using (2.1) and (2.2), it is easy to see that the Ornstein–Uhlenbeck semigroup has analytic continuation to a family $\{\mathscr{H}_z : \operatorname{Re} z \geqslant 0\}$ of continuous operators from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ such that

$$\mathscr{H}_{z+i\pi} f(x) = \mathscr{H}_z f(-x), \quad \mathscr{H}_{\bar{z}} f(x) = \overline{\mathscr{H}_z \overline{f(x)}}. \tag{2.3}$$

By [**2**] the operator $\mathscr{H}_z$ extends to a bounded operator on $L^p(\gamma)$, $1 \leqslant p \leqslant \infty$, if and only if z is in the set $\mathbf{E}_p$, defined in (1.1), in which case it is a contraction. The set $\mathbf{E}_p$ is a closed $i\pi$-periodic subset of the right half-plane. Note that if $1/p + 1/p' = 1$ then $\mathbf{E}_p = \mathbf{E}_{p'}$, and that $\mathbf{E}_p \subset \mathbf{E}_q$ if $1 < p < q < 2$. In particular $\mathbf{E}_1$ is the union of the rays $\{t + ik\pi : t \geqslant 0\}$, $k \in \mathbb{Z}$, and $\mathbf{E}_2 = \{z : \operatorname{Re} z \geqslant 0\}$.

To investigate the boundedness of the maximal operator $\mathscr{H}_p^*$ on $L^q(\gamma)$, $1 \leqslant q \leqslant \infty$, we may restrict the parameter z to the set $\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leqslant \operatorname{Im} z \leqslant \pi/2\}$. Indeed, define the maximal operator $\mathscr{M}_p$ by

$$\mathscr{M}_p f(x) = \sup_{z \in \mathbf{F}_p} |\mathscr{H}_z f(x)|.$$

By (2.3) the operators $\mathscr{H}_p^*$ and $\mathscr{M}_p$ on $L^q(\gamma)$ are simultaneously of weak or strong type. More generally, we shall consider the family of maximal operators $\mathscr{M}_{p,\sigma}$ defined as follows. Let z_p denote the point on the boundary of $\mathbf{F}_p$ whose imaginary part is $\pi/2$, that is, $z_p = 1/2\,\big(|\log(p-1)| + i\pi\big)$. For each σ, $0 \leqslant \sigma < |z_p|$, let $\mathbf{F}_{p,\sigma} = \{z \in \mathbf{F}_p : |z - z_p| \geqslant \sigma\}$. Define

$$\mathscr{M}_{p,\sigma} f(x) = \sup_{z \in \mathbf{F}_{p,\sigma}} |\mathscr{H}_z f(x)|.$$

Thus $\mathscr{M}_p = \mathscr{M}_{p,0}$. Since $\mathbf{F}_1 = \overline{\mathbb{R}^+}$, the operator $\mathscr{M}_1$ is the Mehler maximal operator. We now state our main results. Note that $\mathscr{M}_{p,\sigma} = \mathscr{M}_{p',\sigma}$ because $\mathbf{E}_p = \mathbf{E}_{p'}$. Thus we only need to study the boundedness of $\mathscr{M}_{p,\sigma}$ for $1 \leqslant p \leqslant 2$.

THEOREM 2.1. *The following hold*:

(1) *The operator* $\mathscr{M}_1$ *is of weak type* 1 *and of strong type* q *for every* q *in* $(1, \infty]$.

(2) *Let* $1 < p < 2$. *Then* $\mathscr{M}_p$ *is of strong type* q *whenever* $p < q < p'$, *but it is not of weak type* p. *For* $d \leqslant 2$ *the operator* $\mathscr{M}_p$ *is of weak type* p'.

(3) *If* $1 < p < 2$ *and* $0 < \sigma < |z_p|$, *the operator* $\mathscr{M}_{p,\sigma}$ *is of weak type* p *and* p', *but not of strong type* p.

(4) *The operators* $\mathscr{M}_2$ *and* $\mathscr{M}_{2,\sigma}$ *with* $0 < \sigma < \pi/2$ *are not of weak type* 2.

REMARK 2.2. We do not know if $\mathscr{M}_p$ is of weak type p' for $d \geqslant 3$. It is also an open question whether $\mathscr{M}_{p,\sigma}$ is of strong type p'. (These questions have recently been settled by P. Sjögren, who proved that for $1 < p < 2$ the operator $\mathscr{M}_{p,\sigma}$ is of weak type p', but not of strong type p'.)

Our method consists in decomposing the operator in a 'local' part, given by a kernel living close to the diagonal, and the remaining or 'global' part. To this end consider the set

$$L = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leqslant 1 \wedge \frac{1}{|x + y|} \right\},$$

and denote by G its complement. We shall call L and G the 'local' and the 'global' region, respectively. The local and the global parts of the operators $\mathscr{M}_{p,\sigma}$ are defined by

$$\mathscr{M}_{p,\sigma}^{\mathrm{loc}} f(x) = \sup_{z \in \mathbf{F}_{p,\sigma}} \left| \int h_z(x, y) \chi_L(x, y) f(y) d\gamma(y) \right| \tag{2.4}$$

$$\mathscr{M}_{p,\sigma}^{\mathrm{glob}} f(x) = \sup_{z \in \mathbf{F}_{p,\sigma}} \left| \int h_z(x, y) \chi_G(x, y) f(y) d\gamma(y) \right|, \tag{2.5}$$

where χ_L and χ_G are the characteristic functions of the sets L and G, respectively. Clearly

$$\mathscr{M}_{p,\sigma} f(x) \leqslant \mathscr{M}_{p,\sigma}^{\mathrm{loc}} f(x) + \mathscr{M}_{p,\sigma}^{\mathrm{glob}} f(x).$$

We shall prove separately the boundedness of $\mathscr{M}_{p,\sigma}^{\mathrm{loc}}$ and $\mathscr{M}_{p,\sigma}^{\mathrm{glob}}$. To estimate the Mehler kernel, both in the local and in the global region, it is convenient to simplify its expression by means of a change of variables in the complex parameter z. We denote by $\tau : \{\zeta \in \mathbb{C} : |\zeta| \leqslant 1, |\arg \zeta| \leqslant \pi/2\} \longrightarrow \mathbb{C} \cup \{\infty\}$ the function

$$\tau(\zeta) = \begin{cases} \log \dfrac{1 + \zeta}{1 - \zeta} & \text{if } \zeta \neq 1 \\ \infty & \text{if } \zeta = 1, \end{cases} \tag{2.6}$$

where the logarithm is real when its argument is positive. It is straightforward to check that τ is a homeomorphism of its domain onto the halfstrip $\{z \in \mathbb{C} : \mathrm{Re}\, z \geqslant 0,\ |\mathrm{Im}\, z| \leqslant \pi/2\} \cup \{\infty\}$, mapping the sector $\mathbf{S}_{\phi_p} = \{\zeta \in \mathbb{C} : |\zeta| \leqslant 1,\ 0 \leqslant \arg \zeta \leqslant \phi_p\}$ onto the set $\mathbf{F}_p \cup \{\infty\}$, the point $e^{i\phi_p}$ to the point z_p and the interval $[0, 1]$ to $[0, \infty]$. Moreover, if $\zeta \neq 1$,

$$h_{\tau(\zeta)}(x, y) = (4\zeta)^{-d/2}(1 + \zeta)^d \exp\left(\frac{|x|^2 + |y|^2}{2} - \frac{1}{4}\left(\zeta|x + y|^2 + \frac{1}{\zeta}|x - y|^2 \right) \right). \tag{2.7}$$

We define also $h_\infty(x, y) = 1$, for all x, y.

3. *Estimate of* $\mathscr{M}_{p,\sigma}^{\mathrm{loc}}$

In this section we shall prove that $\mathscr{M}_{p,\sigma}^{\mathrm{loc}}$ is of weak type 1 and of strong type q for each q in $(1, \infty]$. Since $\mathscr{M}_{p,\sigma}^{\mathrm{loc}} \leqslant \mathscr{M}_{p,0}^{\mathrm{loc}} = \mathscr{M}_{p}^{\mathrm{loc}}$, it is enough to consider the latter operator.

Lemma 3.1. *For every p in $[1, 2)$, there exists a constant C such that for all t in $(0, 1]$ and all (x, y) in the local region L*

$$\sup_{|\phi|\leqslant\phi_p} |h_{\tau(te^{i\phi})}(x, y)| \leqslant C\ t^{-d/2} e^{|y|^2} \exp\left(-\frac{\cos\phi_p}{4t}|x-y|^2\right).$$

Proof. By (2.7)

$$|h_{\tau(te^{i\phi})}(x, y)| \leqslant C\ t^{-d/2} \exp\left(\frac{|x|^2+|y|^2}{2} - \frac{\cos\phi}{4}\left(t|x+y|^2 + \frac{1}{t}|x-y|^2\right)\right). \tag{3.1}$$

Since $|x|^2 \leqslant 1 + |y|^2$ in the local region L, the desired estimate follows if we majorize $e^{(|x|^2+|y|^2)/2}$ by $C\, e^{|y|^2}$ and the exponential of $-(\cos\phi/4)t|x+y|^2$ by 1 in (3.1). □

Theorem 3.2. *For each p in $[1, 2)$ the operator $\mathscr{M}_p^{\mathrm{loc}}$ is of weak type 1 and strong type q whenever $1 < q \leqslant \infty$.*

Proof. By Lemma 3.1, one has for each $f \geqslant 0$

$$\begin{aligned}\mathscr{M}_p^{\mathrm{loc}} f(x) &\leqslant C \sup_{t>0} t^{-d/2} \int e^{-\cos\phi_p(|x-y|^2/4t)}\, \chi_L(x, y) f(y)dy \\ &= \mathscr{W} f(x),\end{aligned}$$

say. Thus $\mathscr{M}_p^{\mathrm{loc}}$ is bounded on $L^\infty(\gamma)$. It remains only to prove that it is also of weak type 1. Since the operator $\mathscr{W}$ is of weak type 1 with respect to Lebesgue measure and its kernel is supported in the local region L, the conclusion follows by well-known arguments (see for instance [**4**, Section 3]). □

4. *Estimate of* $\mathscr{M}_{p,\sigma}^{\mathrm{glob}}$

In this section we shall estimate the global maximal operators $\mathscr{M}_{p,\sigma}^{\mathrm{glob}}$, $1 \leqslant p < 2$, $\sigma \geqslant 0$. Our estimates will be based on the following inequality, which is a straightforward consequence of (2.7)

$$\sup_{|\phi|\leqslant\phi_p} |h_{\tau(te^{i\phi})}(x, y)| \leqslant C\ t^{-d/2}\ e^{(|x|^2+|y|^2)/2-(\cos\phi_p/4)(t|x+y|^2+(1/t)|x-y|^2)} \tag{4.1}$$

for all t in $(0, 1]$ and all (x, y) in $\mathbb{R}^d \times \mathbb{R}^d$. We give first two different expressions of the right-hand side of this inequality. Consider the quadratic form

$$Q_t(x, y) = |(1+t)x - (1-t)y|^2.$$

It is straightforward to check that

$$\begin{aligned}t|x+y|^2 + \frac{1}{t}|x-y|^2 &= \frac{1}{t}Q_t(x, y) - 2|x|^2 + 2|y|^2 \\ &= \frac{1}{t}Q_{-t}(x, y) - 2|y|^2 + 2|x|^2.\end{aligned}$$

Since $\cos\phi_p = 2/p - 1$, it follows that

$$\exp\left(\frac{|x|^2+|y|^2}{2} - \frac{\cos\phi_p}{4}\left(t|x+y|^2 + \frac{1}{t}|x-y|^2\right)\right)$$

$$= \exp\left(\frac{|x|^2}{p} + \frac{|y|^2}{p'} - \frac{\cos\phi_p}{4t} Q_t(x,y)\right) \tag{4.2}$$

$$= \exp\left(\frac{|x|^2}{p'} + \frac{|y|^2}{p} - \frac{\cos\phi_p}{4t} Q_{-t}(x,y)\right). \tag{4.3}$$

For every x in $\mathbb{R}^d$ let $G(x)$ denote the x-section of the global region G, that is, the set $\{y : (x,y) \in G\}$. With $\delta > 0$ we set

$$J_\pm(x,t) = \int_{G(x)} \exp\left(-\frac{\delta}{t} Q_{\pm t}(x,y)\right) dy. \tag{4.4}$$

LEMMA 4.1. *Let $\theta = \theta(x,y)$ denote the angle between the non-zero vectors x and y. Then for each $\delta > 0$, the following hold*:

(i) *There exists a constant C such that*

$$\sup_{0<t\leqslant 1} t^{-d/2} \exp\left(-\frac{\delta}{t} Q_t(x,y)\right) \leqslant C[(1+|x|)^d \wedge (|x|\sin\theta)^{-d}].$$

(ii) *For each p in $(1,\infty)$ and each η in $(0,1)$, there exists a constant C such that*

$$\sup_{0<t\leqslant 1-\eta} t^{-pd/2} \exp\left(-\frac{\delta}{t} Q_{\pm t}(x,y)\right) J_\pm^{p/p'}(x,t) \leqslant C[(1+|x|)^d \wedge (|x|\sin\theta)^{-d}].$$

REMARK 4.2. If $d = 1$ then $\sin\theta = 0$ and $(1+|x|)^d \wedge (|x|\sin\theta)^{-d}$ should be interpreted as $(1+|x|)^d$.

Proof of Lemma 4.1. We claim that for each η in $(0,1)$ there exists a positive constant c such that for all non-zero x, y and all $t \geqslant -1+\eta$

$$Q_t(x,y) \geqslant c|x|^2 \sin^2\theta \tag{4.5}$$

and for all (x,y) and t such that $(x,y) \in G$ and $|t| < (1+|x|)^{-2}/8$

$$Q_t(x,y) \geqslant c\frac{1}{(1+|x|)^2}. \tag{4.6}$$

Assuming this claim for the moment, we prove the lemma. To obtain (i), we observe that by (4.5)

$$t^{-d/2} \exp\left(-\frac{\delta}{t} Q_t(x,y)\right) \leqslant C(|x|\sin\theta)^{-d} \tag{4.7}$$

for all $t > 0$. It remains to estimate the left-hand side of (4.7) by $C(1+|x|)^d$. If $t > (1+|x|)^{-2}/8$, it is enough to majorize the exponential by 1. In the opposite case, we observe that by (4.6)

$$t^{-d/2} \exp\left(-\frac{\delta}{t} Q_t(x,y)\right) \leqslant C(1+|x|)^d$$

as desired. Next we prove (ii). Performing the change of variables $(1+t)x-(1-t)y = z$

in J_+ and $(1-t)x-(1+t)y=z$ in J_-, we see that

$$J_{\pm}^{p/p'}(x,t)\leqslant C\left(\frac{t^{d/2}}{(1\mp t)^d}\right)^{p/p'}\leqslant Ct^{pd/(2p')}, \tag{4.8}$$

since $1\mp t>\eta$. Thus by (4.5) and (4.8)

$$t^{-pd/2}\exp\left(-\frac{\delta}{t}Q_{\pm t}(x,y)\right)J_{\pm}^{p/p'}(x,t)\leqslant C(|x|\sin\theta)^{-d}. \tag{4.9}$$

It remains to estimate the left-hand side of (4.9) by $C(1+|x|)^d$. If $t>(1+|x|)^{-2}/8$ it is enough to majorize the exponential by 1 and to apply (4.8). In the opposite case, we observe that by (4.6)

$$J_{\pm}^{p/p'}(x,t)\leqslant C\left(\int_{|z|>c(1+|x|)^{-1}}e^{-c|z|^2/t}dz\right)^{p/p'}\leqslant C\,t^{pd/(2p')}\big(\sqrt{t}(1+|x|)\big)^{pm/p'}$$

for any $m>0$. We choose $m=dp'/p$ and get $t^{-pd/2}J_{\pm}^{p/p'}(x,t)\leqslant C(1+|x|)^d$, as desired.

We must finally prove the claim. To obtain (4.5), it is enough to minorize $|(1+t)x-(1-t)y|$ by the length of the projection of $(1+t)x$ on the hyperplane orthogonal to y. For (4.6), we first verify that for $x,y\in G$

$$|x-y|\geqslant\tfrac{1}{2}(1+|x|)^{-1}. \tag{4.10}$$

When $|y|\geqslant 1+|x|$, this follows from $|x-y|\geqslant 1$. If $|y|<1+|x|$, one has $|x+y|\leqslant 2(1+|x|)$, and so $\min(1,|x+y|^{-1})>\frac{1}{2}(1+|x|)^{-1}$. Since $(x,y)\in G$, this implies (4.10). To obtain (4.6), observe that $|t|\leqslant(1+|x|)^{-2}/8$ implies that $1-t\geqslant 7/8$ and so $|(1+t)x-(1-t)y|=|(1-t)(x-y)+2tx|\geqslant\frac{7}{16}(1+|x|)^{-1}-\frac{1}{4}(1+|x|)^{-1}\geqslant\frac{3}{16}(1+|x|)^{-1}$. This completes the proof of the claim and of the lemma. □

We prove first that $\mathscr{M}_1^{\text{glob}}=\mathscr{M}_{1,0}^{\text{glob}}$ is of weak type 1. The result is well known (see [**6**, **8**]), but the proof we give here is new and simpler.

Theorem 4.3. *The operator $\mathscr{M}_1^{\text{glob}}$ is of weak type 1 and of strong type q for every q in $(1,\infty]$.*

Proof. Since the operator $\mathscr{M}_1^{\text{glob}}$ is obviously bounded on $L^\infty(\gamma)$, we only need to prove that it is of weak type 1. Using (4.1), (4.2) and Lemma 4.1(i), we see that $\mathscr{M}_1^{\text{glob}}$ is controlled by the operator

$$\mathscr{T}f(x)=e^{|x|^2}\int(1+|x|)^d\wedge\big(|x|\sin\theta\big)^{-d}f(y)d\gamma(y),$$

where $\theta=\theta(x,y)$ is the angle between the vectors x and y. Therefore the theorem is an immediate consequence of the following lemma. □

Lemma 4.4. *Let $\theta=\theta(x,y)$ denote the angle between the vectors x and y. The operator*

$$\mathscr{T}f(x)=e^{|x|^2}\int(1+|x|)^d\wedge(|x|\sin\theta)^{-d}f(y)d\gamma(y)$$

is of weak type 1.

Proof. Assume that $\|f\|_1 = 1$. Choose C_0 so large that $\lambda > C_0$ implies that the positive root r_0 of the equation

$$e_0^{r^2}(1+r_0)^d = \lambda$$

is greater than 1. Fix $\lambda > 0$ and let E be the level set $\{x : \mathcal{T} f(x) \geqslant \lambda\}$. To prove that there exists a constant $C > 0$ such that $\gamma(E) \leqslant C\lambda^{-1}$, we can assume that $\lambda > C_0$, since otherwise the trivial estimate $\gamma(E) \leqslant \gamma(\mathbb{R}^d)$ will do. Since $\mathcal{T} f(x) \leqslant e^{|x|^2}(1+|x|)^d$, we see that E does not intersect the ball $\{|x| < r_0\}$. We need only consider the intersection of the set E with the ring $R = \{r_0 \leqslant |x| \leqslant 2r_0\}$. Indeed, the elementary fact that $\int_R^\infty e^{-\rho^2}\rho^{d-1}\,d\rho \sim e^{-R^2}R^{d-2}$ for $R > 1$ implies that

$$\gamma\{|x| > 2r_0\} \leqslant Ce^{-4r_0^2}r_0^{d-2} \leqslant Ce^{-r_0^2}(1+r_0)^{-d} = C\lambda^{-1}.$$

Write $x = \rho x'$ with $\rho > 0$ and $|x'| = 1$, and let dx' be the area measure on S^{d-1}. If $d = 1$ then $S^0 = \{-1, 1\}$ and dx' is the sum of unit point masses at -1 and 1. We let E' denote the set of $x' \in S^{d-1}$ for which there exists a $\rho \in [r_0, 2r_0]$ with $\rho x' \in E$. For $x' \in E'$ we let $r(x')$ be the smallest such ρ. Then $\mathcal{T} f(r(x')x') = \lambda$ by continuity, and this implies that

$$e^{r(x')^2}r_0^{-d}\int r_0^{2d} \wedge (\sin\theta)^{-d} f(y)d\gamma(y) \sim \lambda. \tag{4.11}$$

Clearly

$$\gamma(E\cap R) \leqslant \int_{E'} dx' \int_{r(x')}^{2r_0} e^{-\rho^2}\rho^{d-1}\,d\rho \leqslant C\int_{E'} e^{-r(x')^2}r_0^{d-2}\,dx'.$$

Combining this with (4.11), we get

$$\gamma(E\cap R) \leqslant C\lambda^{-1}r_0^{-2}\int_{E'} dx' \int r_0^{2d} \wedge (\sin\theta)^{-d} f(y)\,d\gamma(y).$$

It is now enough to change the order of integration and observe that

$$\int_{S^{d-1}} r_0^{2d} \wedge (\sin\theta)^{-d}\,dx' \leqslant Cr_0^2$$

to obtain the desired estimate of $\gamma(E)$. □

Now we turn to study the global parts of the operators $\mathcal{M}_p$ and $\mathcal{M}_{p,\sigma}$. First we need a lemma. For each η in $[0, 1)$, consider the maximal operator

$$\mathcal{A}_{p,\eta} f(x) = \sup_{0<t\leqslant 1-\eta} t^{-d/2}\int_{G(x)} e^{(|x|^2+|y|^2)/2}\, e^{-(\cos\phi_p/4)(t|x+y|^2+(1/t)|x-y|^2)} f(y)d\gamma(y). \tag{4.12}$$

LEMMA 4.5. *Suppose that $1 < p < 2$. Then the following hold*:
(i) *If $\eta > 0$ the operator $\mathcal{A}_{p,\eta}$ is of weak type p and p'.*
(ii) *The operator $\mathcal{A}_{p,0}$ is of strong type q whenever $p < q < p'$.*

Proof. We shall prove that there exists a constant C such that for each $f \geqslant 0$

$$\mathcal{A}_{p,\eta} f(x) \leqslant C\min(|\mathcal{T} f^p(x)|^{1/p}, |\mathcal{T} f^{p'}(x)|^{1/p'}),$$

whence (i) will follow, by Lemma 4.4. By (4.2)

$$\mathcal{A}_{p,\eta} f(x) = \sup_{0<t\leqslant 1-\eta} t^{-d/2}e^{|x|^2/p}\int_{G(x)} \exp\left(-\frac{\delta}{t}Q_t(x,y)\right) f(y)\,e^{-|y|^2/p}\,dy,$$

where δ is a positive constant which depends on p. Applying Hölder's inequality, we see that the right-hand side is bounded by

$$\sup_{0<t\leqslant 1-\eta} t^{-d/2} e^{|x|^2/p} \left(\int_{G(x)} \exp\left(-\frac{\delta}{t} Q_t(x,y)\right) f^p(y) d\gamma(y) \right)^{1/p} J_+^{1/p'}(x,t),$$

where $J_+(x,t)$ is defined in (4.4). Thus, by Lemma 4.1(ii),

$$\begin{aligned}\mathscr{A}_{p,\eta} f(x) &\leqslant C e^{|x|^2/p} \left(\int_{G(x)} (1+|x|)^d \wedge (|x| \sin\theta)^{-d} f^p(y) d\gamma(y) \right)^{1/p} \\ &= C \left(\mathscr{T} f^p(x)\right)^{1/p}.\end{aligned}$$

To prove the inequality $\mathscr{A}_{p,\eta} f(x) \leqslant C(\mathscr{T} f^{p'}(x))^{1/p'}$ we observe that, by (4.3),

$$\mathscr{A}_{p,\eta} f(x) = \sup_{0<t\leqslant 1-\eta} t^{-d/2} e^{|x|^2/p'} \int_{G(x)} \exp\left(-\frac{\delta}{t} Q_{-t}(x,y)\right) f(y)\, e^{-|y|^2/p'} dy.$$

The rest of the proof is similar.

Next we prove (ii). Fix r in (p,p'). Then $\lambda = \cos\phi_p / \cos\phi_r > 1$ and

$$\begin{aligned}\cos\phi_p \left(t|x+y|^2 + \frac{1}{t}|x-y|^2 \right) &= \lambda \cos\phi_r \left(t|x+y|^2 + \frac{1}{t}|x-y|^2 \right) \\ &= \cos\phi_r \left(\lambda^2 \frac{t}{\lambda} |x+y|^2 + \frac{\lambda}{t}|x-y|^2 \right) \\ &\geqslant \cos\phi_r \left(\frac{t}{\lambda}|x+y|^2 + \frac{\lambda}{t}|x-y|^2 \right).\end{aligned}$$

Thus

$$\begin{aligned}\mathscr{A}_{p,0} f(x) &\leqslant \sup_{0<t\leqslant 1} t^{-d/2} \int_{G(x)} e^{(|x|^2+|y|^2)/2} e^{-(\cos\phi_r/4)((t/\lambda)|x+y|^2+(\lambda/t)|x-y|^2)} f(y) d\gamma(y) \\ &= \lambda^{d/2} \mathscr{A}_{r,1-1/\lambda} f(x).\end{aligned}$$

Hence, by (i), the operator $\mathscr{A}_{p,0}$ is of weak type r and of weak type r' for each r in (p,p'). Therefore, by interpolation, it is of strong type q whenever $p<q<p'$. □

THEOREM 4.6. *Suppose that* $1<p<2$, $0<\sigma<|z_p|$. *Then the following hold*:
(i) $\mathscr{M}_p^{\mathrm{glob}}$ *is of strong type* q *whenever* $p<q<p'$.
(ii) $\mathscr{M}_{p,\sigma}^{\mathrm{glob}}$ *is of weak type* p *and* p'.

Proof. We prove first that $\mathscr{M}_p^{\mathrm{glob}}$ is of strong type q whenever $p<q<p'$. Since the transformation τ, defined in (2.6), maps the sector $\mathbf{S}_{\phi_p}$ onto the set $\mathbf{F}_p$,

$$\mathscr{M}_p^{\mathrm{glob}} f(x) \leqslant \sup_{0<t\leqslant 1} \int_{G(x)} \sup_{|\phi|\leqslant\phi_p} |h_{\tau(te^{i\phi})}(x,y)| f(y) d\gamma(y)$$

for each $f \geqslant 0$. Thus, by (4.1) and (4.12)

$$\mathscr{M}_p^{\mathrm{glob}} f(x) \leqslant C \mathscr{A}_{p,0} f(x).$$

The conclusion follows by Lemma 4.5(ii).

Next we prove that $\mathscr{M}_{p,\sigma}^{\mathrm{glob}}$ is of weak type p and p'. The function τ maps the point $e^{i\phi_p}$ to the point z_p. Since $\tau'(e^{i\phi_p}) \neq 0$, for each $\sigma>0$ there exists a small $\eta>0$ such that the set $\mathbf{F}_{p,\sigma}$ is contained in the τ-image of the union of the two sets $\mathbf{S}_{\phi_p-\eta}$ and $\mathbf{T}_{p,\eta} = \{\zeta \in \mathbf{S}_{\phi_p} : |\zeta| \leqslant 1-\eta\}$. Let $\mathscr{B}_{p,\eta}$ denote the operator defined by

$$\mathscr{B}_{p,\eta} f(x) = \sup_{0<t\leqslant 1} \int_{G(x)} \sup_{|\phi|\leqslant\phi_p-\eta} |h_{\tau(te^{i\phi})}(x,y)| f(y) d\gamma(y).$$

Hence, for each non-negative function f,

$$\begin{aligned}\mathscr{M}^{\text{glob}}_{p,\sigma} f(x) &\leqslant \sup_{\zeta\in\mathbf{T}_{p,\eta}} \left| \int_{G(x)} h_{\tau(\zeta)}(x,y) f(y) d\gamma(y) \right| \\ &\quad + \sup_{\zeta\in\mathbf{S}_{\phi_p-\eta}} \left| \int_{G(x)} h_{\tau(\zeta)}(x,y) f(y) d\gamma(y) \right| \\ &\leqslant C\mathscr{A}_{p,\eta} f(x) + \mathscr{B}_{p,\eta} f(x).\end{aligned}$$

By Lemma 4.5(i) the operator $\mathscr{A}_{p,\eta}$ is of weak type p and p'. We claim that the operator $\mathscr{B}_{p,\eta}$ is of strong type p and p'. Indeed let $r \in (1,p)$ be such that $\phi_r = \phi_p - \eta$. Then, by (4.1) and (4.12)

$$\mathscr{B}_{p,\eta} f(x) \leqslant C\mathscr{A}_{r,0} f(x).$$

By Lemma 4.5(ii) the latter operator is of strong type q whenever $r < q < r'$. This concludes the proof of the theorem. □

To complete the proof of the positive results of Theorem 2.1, it remains to prove the following theorem.

THEOREM 4.7. *If $d \leqslant 2$ and $1 < p < 2$, the operator $\mathscr{M}^{\text{glob}}_p$ is of weak type p'.*

Proof. By Theorem 4.6 we only need to prove that for any fixed σ, with $0 < \sigma < |z_p|$, the operator $\mathscr{N}_{p,\sigma} = \mathscr{M}^{\text{glob}}_p - \mathscr{M}^{\text{glob}}_{p,\sigma}$ is of weak type p'. Now

$$\mathscr{N}_{p,\sigma} f(x) \leqslant \sup_{1-\sigma\leqslant t\leqslant 1} \int_{G(x)} \sup_{|\phi|\leqslant\phi_p} |h_{\tau(te^{i\phi})}(x,y)|\, f(y) d\gamma(y).$$

Write $f(y) = g(y)\exp(|y|^2/p')$. Then $\|f\|_{L^{p'}(\gamma)} = \|g\|_{L^{p'}(dx)}$. By using (4.1), the identities $1/p - 1/2 = 1/2 - 1/p' = (\cos\phi_p)/2$ and a little algebra, we obtain

$$\mathscr{N}_{p,\sigma} f(x) \leqslant C(\sigma)\, e^{|x|^2/p'} \sup_{1-\sigma\leqslant t\leqslant 1} \int k_p(t,x,y)\, g(y) dy,$$

where

$$k_p(t,x,y) = \exp\left(-\frac{\cos\phi_p}{4} \left[\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) x + \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) y \right]^2 \right).$$

The function $y \mapsto k_p(t,x,y)$ is in $L^p(dy)$, uniformly with respect to t and x. Hence

$$\mathscr{N}_{p,\sigma} f(x) \leqslant C\, e^{|x|^2/p'} \|g\|_{L^{p'}(dy)} = C\, e^{|x|^2/p'} \|f\|_{L^{p'}(\gamma)}.$$

The conclusion follows since, for $d \leqslant 2$, the function $x \mapsto e^{|x|^2/p'}$ is in $L^{p',\infty}(\gamma)$. □

5. *Negative results*

In this section we shall prove that $\mathcal{M}_{p,\sigma}$, $0 < \sigma < |z_p|$, is unbounded on $L^p(\gamma)$, $1 \leqslant p < 2$, and that $\mathcal{M}_2$ is not of weak type $(2,2)$. We shall also prove that $\mathcal{M}_p$ is not of weak type p.

For each σ, $0 < \sigma < |z_p|$, the set $\mathbf{F}_{p,\sigma}$ contains an arc $\{\tau(te^{i\phi_p}) : \alpha \leqslant t \leqslant \beta\}$, with $0 < \alpha < \beta < 1$. Thus the maximal operator $\mathcal{M}_{p,\sigma}$ is bounded from below by the operator

$$f \mapsto \mathcal{M}_{p,\alpha,\beta} f = \sup_{\alpha \leqslant t \leqslant \beta} |\mathcal{H}_{\tau(te^{i\phi_p})} f|.$$

To prove that $\mathcal{M}_{p,\sigma}$ is unbounded on $L^p(\gamma)$, $1 \leqslant p < 2$, we shall prove the following sharper result.

THEOREM 5.1. *The operator $\mathcal{M}_{p,\alpha,\beta}$ is unbounded on $L^p(\gamma)$, $1 \leqslant p < 2$, for each $0 < \alpha < \beta < 1$.*

Proof. By restricting the operator to functions which depend only on the first variable in $\mathbb{R}^d$, one sees that it suffices to consider the one-dimensional case $d = 1$. Let $\mathcal{U}_p$ be the isometry $f \mapsto f\gamma^{1/p}$ of $L^p(\gamma)$ onto $L^p(dx)$. We shall prove that the operator $\widetilde{\mathcal{M}}_{p,\alpha,\beta} = \mathcal{U}_p \mathcal{M}_{p,\alpha,\beta} \mathcal{U}_p^{-1}$ is unbounded on $L^p(dx)$. Note that by (2.7)

$$\widetilde{\mathcal{M}}_{p,\alpha,\beta} g(x) \geqslant C \sup_{\alpha \leqslant t \leqslant \beta} \left| \int \exp(q_{te^{i\phi_p}}(x,y))\, g(y) dy \right|,$$

where

$$q_\zeta(x,y) = \frac{x^2+y^2}{2} - \tfrac{1}{4}(\zeta(x+y)^2 + \zeta^{-1}(x-y)^2) - x^2/p - y^2/p'$$
$$= \tfrac{1}{2}\cos\phi_p (y^2 - x^2) - \tfrac{1}{4}(\zeta(x+y)^2 + \zeta^{-1}(x-y)^2)$$

and C is a positive constant which depends on α and β. Fix a smooth function ϕ such that $\phi(0) = 1$ and the support of ϕ is contained in the interval $[-1,1]$. For $y_0 \geqslant 2$ and $1/p' < \delta < 1/p$ let $g(y) = |y - y_0|^{-\delta}\phi(y - y_0)$.

Consider first the case $1 < p < 2$, that is, $0 < \phi_p < \pi/2$. We shall prove that there exist positive constants $c, C_1 < C_2$ such that, if y_0 is large and $x \in [C_1 y_0, C_2 y_0]$, then

$$|\widetilde{\mathcal{M}}_{p,\alpha,\beta} g(x)| \geqslant c y_0^{\delta-1}. \tag{5.1}$$

Hence $\|\widetilde{\mathcal{M}}_{p,\alpha,\beta} g\|_{L^p(dx)} \geqslant C y_0^{\delta-1/p'}$. Since $\|g\|_{L^p(dx)} \leqslant C$, the unboundedness of $\widetilde{\mathcal{M}}_{p,\alpha,\beta}$ on $L^p(dx)$ will follow if we let y_0 tend to infinity.

To prove (5.1), define $t(x) = (y_0 - x)/(y_0 + x)$ and choose two constants C_1 and C_2 such that $(1-\beta)(1+\beta)^{-1} \leqslant C_1 < C_2 \leqslant (1-\alpha)(1+\alpha)^{-1}$. Then $\alpha \leqslant t(x) \leqslant \beta$ for all $x \in [C_1 y_0, C_2 y_0]$. Let $\mathcal{Q}(x,y) = q_{t(x)e^{i\phi_p}}(x,y)$. Then

$$|\widetilde{\mathcal{M}}_{p,\alpha,\beta} g(x)| \geqslant c \left| \int \exp \mathcal{Q}(x,y)\, g(y) dy \right|. \tag{5.2}$$

Write $\mathcal{Q}(x,y) = \mathcal{R}(x,y) + i\mathcal{I}(x,y)$, $\mathcal{R}$, $\mathcal{I}$ real. The functions $\mathcal{R}$ and $\mathcal{I}$ are quadratic polynomials in y. Let

$$\mathcal{R}(x,y) = a_0(x) + a_1(x)(y-y_0) + a_2(x)(y-y_0)^2, \tag{5.3}$$

$$\mathcal{I}(x,y) = b_0(x) + b_1(x)(y-y_0) + b_2(x)(y-y_0)^2 \tag{5.4}$$

be their expansions in powers of $y - y_0$. We claim that

$$a_0 = a_1 = b_0 = 0, \quad b_1(x) = x \sin \phi_p \tag{5.5}$$

and there exists a positive constant C such that

$$|a_2(x)| + |b_2(x)| \leqslant C, \tag{5.6}$$

for all x in $[C_1 y_0, C_2 y_0]$. The claim can be proved by using the fact that $a_k(x)$ and $b_k(x)$, $k = 0, 1, 2$, are the real and the imaginary parts, respectively, of

$$\frac{1}{k!} \partial_y^k \, q_\zeta(x, y)\big|_{\zeta = t(x) e^{i\phi_p}, y = y_0}.$$

Now we observe that, by (5.3) and (5.4),

$$\left| \int \exp \mathcal{Q}(x, y)\, g(y) dy \right| = \left| \int e^{ib_1(x)u} |u|^{-\delta} \Phi(x, u) du \right|, \tag{5.7}$$

where

$$\Phi(x, u) = \exp[(a_2(x) + ib_2(x))u^2]\, \phi(u).$$

Using (5.6) and the fact that ϕ is supported in $[-1, 1]$, it is easy to see that the function $u \mapsto \Phi(x, u)$ satisfies the estimates

$$\|\partial_u^2 \Phi(x, \cdot)\|_1 + \|\Phi(x, \cdot)\|_{H_2^s} \leqslant C, \tag{5.8}$$

uniformly for x in $[C_1 y_0, C_2 y_0]$. Here H_2^s denotes the Sobolev space of any order $s \geqslant 0$ and C may depend on s. To finish the proof of the theorem we need the following lemma.

LEMMA 5.2. *Suppose that* $0 < \delta < 1$. *Let* Φ *be a function in* $C_c^\infty(\mathbb{R})$. *Then for any* ξ *in* $\mathbb{R} \setminus \{0\}$

$$\int e^{i\xi u} |u|^{-\delta} \Phi(u) du = C_\delta \Phi(0)\, |\xi|^{\delta - 1} + E_\delta(\xi, \Phi),$$

where C_δ *is a positive constant and*

$$|E_\delta(\xi, \Phi)| \leqslant C(\delta, s)\, |\xi|^{\delta - 2} (\|\Phi''\|_1 + \|\Phi\|_{H_2^s})$$

for every $s > 3/2$.

Proof. The integral to be evaluated is $(2\pi)^{-1}$ times the value at $-\xi$ of the convolution of the Fourier transform of the function Φ and that of the distribution $|y|^{-\delta}$, which is $C|\xi|^{\delta - 1}$, for some positive constant $C = C(\delta)$. Thus

$$\begin{aligned} \int e^{i\xi u} |u|^{-\delta} \Phi(u) du &= C \int |\xi + \eta|^{\delta - 1} \hat{\Phi}(\eta) d\eta \\ &= C\Phi(0)\, |\xi|^{\delta - 1} + E_\delta(\xi, \Phi), \end{aligned}$$

where

$$E_\delta(\xi, \Phi) = C \int (|\xi + \eta|^{\delta - 1} - |\xi|^{\delta - 1}) \hat{\Phi}(\eta) d\eta.$$

To estimate $E_\delta(\xi, \Phi)$ we split the integral into the sum of an integral over the set $|\eta| \leqslant |\xi|/2$ and an integral over the set $|\eta| > |\xi|/2$. The first integral is bounded in absolute value by

$$C|\xi|^{\delta - 2} \int |\eta \hat{\Phi}(\eta)| d\eta \leqslant C(\delta, s)\, |\xi|^{\delta - 2}\, \|\Phi\|_{H_2^s}$$

for every $s > 3/2$. The integral over the set $|\eta| > |\xi|/2$ is bounded in absolute value by

$$C \int_{|\eta| \geqslant |\xi|/2} |\xi + \eta|^{\delta-1} |\hat{\Phi}(\eta)| d\eta + C_\delta\, |\xi|^{\delta-1} \int_{|\eta| \geqslant |\xi|/2} |\hat{\Phi}(\eta)| d\eta.$$

Fix γ with $\delta < \gamma < 2$. The first summand is majorized by

$$\begin{aligned} C(\delta,\gamma)\, |\xi|^{\gamma-2} & \int_{|\eta| \geqslant |\xi|/2} |\xi + \eta|^{\delta-1}\, |\eta|^{-\gamma}\, |\eta^2 \hat{\Phi}(\eta)| d\eta \\ & \leqslant C(\delta,\gamma)\, |\xi|^{\gamma-2} \int_{|\eta| \geqslant |\xi|/2} |\xi + \eta|^{\delta-1}\, |\eta|^{-\gamma} d\eta \;\; \max_\eta |\eta^2 \hat{\Phi}(\eta)| \\ & \leqslant C(\delta,\gamma)\, |\xi|^{\delta-2}\, \|\Phi''\|_1. \end{aligned}$$

The second summand is majorized by $2\,C\, |\xi|^{\delta-2}\, \|\eta \hat{\Phi}\|_1 \leqslant 2C(\delta,s)\, |\xi|^{\delta-2} \|\Phi\|_{H_2^s}$ for all $s > 3/2$. □

We can now finish the proof of the theorem. Because of (5.5), (5.7), (5.8) and Lemma 5.2, there exists a positive constant C such that

$$\left| \int \exp \mathcal{Q}(x,y)\, g(y) dy \right| = C|x|^{\delta-1} + O(|x|^{\delta-2}), \tag{5.9}$$

as $y_0 \to \infty$, uniformly for x in $[C_1 y_0, C_2 y_0]$. The desired estimate (5.1) follows from (5.2) and (5.9). Thus the proof of the theorem for $1 < p < 2$ is complete.

It remains to consider the case $p = 1$. We observe that in this case $\phi_p = 0$ and $\mathcal{Q}$ is real. The estimate (5.6) is still true, but now $b_1,\ b_2$ vanish identically. Hence (5.7) becomes

$$\left| \int \exp \mathcal{Q}(x,y)\, g(y) dy \right| = \int \exp(a_2(x) u^2)\; |u|^{-\delta}\; \phi(u) du.$$

Since the support of ϕ is contained in $[-1,1]$, by (5.2) and (5.6), there exists a positive constant C such that for y_0 large

$$|\widetilde{\mathcal{M}}_{1,\alpha,\beta} g(x)| \geqslant C$$

for all x in $[C_1 y_0, C_2 y_0]$. The unboundedness of $\widetilde{\mathcal{M}}_{1,\alpha,\beta}$ on $L^1(dx)$ follows by comparing the $L^1(dx)$ norms of $\widetilde{\mathcal{M}}_{1,\alpha,\beta} g$ and g when y_0 tends to infinity. This concludes the proof of the theorem. □

THEOREM 5.3. *The operator $\mathcal{M}_{2,\sigma}$ is not of weak type* $(2,2)$.

Proof. As in the proof of Theorem 5.1, we may reduce the problem to the one-dimensional case. The set $\mathbf{E}_2$ contains the imaginary axis. Thus it is enough to prove that there exists a continuous function f with compact support such that

$$\limsup_{t \to 0^+} \left| \int h_{it}(x,y)\, f(y) d\gamma(y) \right| = \infty$$

for all x in $\mathbb{R}$. Since $\tau^{-1}(it) = i \tan(t/2)$, by (2.7)

$$\begin{aligned} h_{it}(x,y) &= (4i \tan(t/2))^{-1/2} (1 + i \tan(t/2)) e^{(x^2+y^2)/2} \\ &\quad \times \exp\left(-\frac{i}{4} \left(\tan(t/2)(x+y)^2 - \frac{(x-y)^2}{\tan(t/2)} \right) \right). \end{aligned}$$

By a result of Carleson [**1**], there exists a continuous function with compact support g such that

$$\limsup_{t\to 0+} t^{-1/2}\left|\int e^{i(x-y)^2/t}\, g(y)dy\right| = \infty \tag{5.10}$$

for all x in $\mathbb{R}$. Set $f(y) = g(y)e^{y^2/2}$. Then

$$\int h_{it}(x,y)\, f(y)d\gamma(y) = e^{x^2/2}(4i\tan(t/2))^{-1/2}\int e^{i(x-y)^2/\tan(t/2)}\, g(y)dy + O(t^{1/2})$$

as $t \to 0^+$, for each x in $\mathbb{R}$. The conclusion follows from (5.10). □

Next we prove that $\mathscr{M}_p$ is not of weak type p for $1 < p \leqslant 2$. Since we already know that $\mathscr{M}_{2,\sigma}$ is not of weak type 2 and $\mathscr{M}_2 f \geqslant \mathscr{M}_{2,\sigma} f$ for each non-negative function f, we only need to consider the case $1 < p < 2$. The maximal operator $\mathscr{M}_p$ is bounded from below by the operator

$$f \mapsto \mathscr{N}_{p,\varepsilon} f = \sup_{1-\varepsilon\leqslant t<1} |\mathscr{H}_{\tau(te^{i\phi_p})} f|$$

for any $\varepsilon \in [0,1]$. Therefore it suffices to prove the following result.

THEOREM 5.4. *The operator $\mathscr{N}_{p,\varepsilon}$ is not of weak type p for any p in $(1,2]$ and ε in $(0,1)$.*

Proof. As in the proof of the previous theorem, it suffices to consider the one-dimensional case. Let $\mathscr{U}_p$ be as in the proof of Theorem 5.1. We shall prove that the operator $\widetilde{\mathscr{N}}_{p,\varepsilon} = \mathscr{N}_{p,\varepsilon}\mathscr{U}_p^{-1}$ is unbounded from $L^p(dx)$ to $L^{p,\infty}(\gamma)$. We claim that for each choice of x_0, y_0, with x_0 sufficiently large and $y_0 \geqslant x_0^\delta$, $\delta > 1$, there is a function g such that $\|g\|_{L^p(dx)} \leqslant C(y_0/x_0)^{1/2p}$ and

$$|\widetilde{\mathscr{N}}_{p,\varepsilon} g(x)| \geqslant Ce^{x_0^2/p}\left(\frac{y_0}{x_0}\right)^{1/2} \qquad \forall\ x \in \left[x_0, x_0 + \frac{1}{x_0}\right].$$

Assuming this claim for the moment, we complete the proof. Seeking a contradiction, we assume that the operator is of weak type p. Then

$$\begin{aligned}\frac{e^{-x_0^2}}{x_0} &\leqslant C\,\gamma\left(\left[x_0, x_0 + \frac{1}{x_0}\right]\right)\\ &\leqslant C\,\gamma\left\{x : |\widetilde{\mathscr{N}}_{p,\varepsilon} g(x)| \geqslant Ce^{x_0^2/p}\left(\frac{y_0}{x_0}\right)^{1/2}\right\}\\ &\leqslant C\, e^{-x_0^2}\left(\frac{y_0}{x_0}\right)^{-p/2}\|g\|_p^p\\ &\leqslant C\, e^{-x_0^2}\left(\frac{y_0}{x_0}\right)^{(1-p)/2}.\end{aligned}$$

Choosing $y_0 = x_0^\delta$, with $\delta > (p+1)/(p-1)$ and letting y_0 tend to infinity, we reach a contradiction.

It remains to prove the claim. Arguing as in the proof of Theorem 5.1, one sees that for any function g in $L^p(dx)$

$$|\widetilde{\mathcal{N}}_{p,\varepsilon} g(x)| \geqslant C\, e^{x^2/p} \sup_{1-\varepsilon \leqslant t \leqslant 1} \left| \int \exp(q_{te^{i\phi_p}}(x,y))\, g(y) dy \right|,$$

with q_ζ as in that proof and $C = C(\varepsilon) > 0$. Let $x \mapsto t(x)$ be a measurable function, to be chosen later, such that $1-\varepsilon \leqslant t(x) < 1$ for all x in $[x_0, x_0+1/x_0]$. Define $\mathcal{Q}(x,y) = q_{t(x)e^{i\phi_p}}(x,y)$. Then for all x in $[x_0, x_0+1/x_0]$

$$|\widetilde{\mathcal{N}}_{p,\varepsilon} g(x)| \geqslant C e^{x_0^2/p} \left| \int \exp \mathcal{Q}(x,y)\, g(y) dy \right|. \tag{5.11}$$

As in the proof of Theorem 5.1, we write $\mathcal{Q}(x,y) = \mathcal{R}(x,y) + i\mathcal{I}(x,y)$, $\mathcal{R}$, $\mathcal{I}$ real, and we consider the expansions

$$\mathcal{R}(x,y) = a_0(x) + a_1(x)(y-y_0) + a_2(x)(y-y_0)^2$$

$$\mathcal{I}(x,y) = b_0(x) + b_1(x)(y-y_0) + b_2(x)(y-y_0)^2$$

of $\mathcal{R}$, $\mathcal{I}$ in powers of $y-y_0$. Next we choose $t(x)$ so that $b_1(x) = x_0 \sin\phi_p$. Since

$$b_1(x) = -\frac{\sin\phi_p}{2}\left[\left(t(x) - \frac{1}{t(x)}\right) y_0 + \left(t(x) + \frac{1}{t(x)}\right) x\right],$$

this choice yields

$$t(x) = \frac{\sqrt{y_0^2 + x_0^2 - x^2} - x_0}{y_0 + x}.$$

A lengthy but straightforward computation shows that if $x \in [x_0, x_0+1/x_0]$, $y_0 \geqslant x_0^\delta$, $\delta > 1$, then

$$\begin{aligned} t(x) &= \frac{y_0 - x_0}{y_0 + x_0} + O(1/(x_0 y_0)) \\ a_0(x) &= O(1) \\ a_1(x) &= O(1/y_0) \\ a_2(x) &= O(x_0^2/y_0^2) \\ b_1(x) &= x_0 \sin\phi_p \\ b_2(x) &= \sin\phi_p \frac{x_0}{y_0} + O((x_0/y_0)^3) + O(1/(x_0 y_0)), \end{aligned} \tag{5.12}$$

as $x_0 \to \infty$. Thus $1-\varepsilon \leqslant t(x) < 1$ for x_0 sufficiently large. Let g be the function defined by

$$g(y_0+u) = \chi\left(\left(\frac{x_0}{y_0}\right)^{1/2} u\right) \exp(-iux_0 \sin\phi_p),$$

where χ is the characteristic function of the interval $[-1,1]$. Then $\|g\|_{L^p(dx)} = (y_0/x_0)^{1/2p}$ and

$$\left| \int \exp 2(x,y)\, g(y) dy \right| = \left(\frac{y_0}{x_0}\right)^{1/2} e^{a_0(x)} \left| \int_{-1}^{1} e^{(a_1(x)(y_0/x_0)^{1/2} v + (a_2(x) + ib_2(x))(y_0/x_0) v^2)} dv \right|.$$

By the asymptotic estimates (5.12), if x_0 is sufficiently large and $y_0 \geqslant x_0^\delta$, $\delta > 1$, there exists a positive constant c such that the right-hand side is bounded from below by $c(y_0/x_0)^{1/2}$ for all x in $x \in [x_0, x_0 + 1/x_0]$. By (5.11) this completes the proof of the claim and of the theorem. □

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