VECTOR-VALUED HAUSDORFF–YOUNG INEQUALITY ON COMPACT GROUPS

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Introduction

If $1 \leqslant p \leqslant 2$ and $p' = p/(p-1)$ denotes the conjugate exponent of p, the classical Hausdorff–Young inequality establishes the boundedness of the Fourier transform from $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$. Its proof is obtained by complex interpolation between the obvious case $p = 1$ and the case $p = 2$ given by Plancherel's theorem. In the same spirit, in 1958 Kunze applied new techniques of non-commutative integration introduced by Dixmier [6] and Segal [23, 24] to study this inequality on locally compact unimodular groups; see [15]. In particular, for a compact not-necessarily-abelian group G, he proved the boundedness of the Fourier transform from $L^p(G)$ to $\mathcal{L}^{p'}(\widehat{G})$. That is,

$$\left(\sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{S_{d_\pi}^{p'}}^{p'} \right)^{1/p'} \leqslant \left(\int_G |f(g)|^p \, d\mu(g) \right)^{1/p} \quad \text{for } 1 < p \leqslant 2,$$

with the obvious modifications for $p = 1$. Here $\pi \in \widehat{G}$ denotes an irreducible unitary representation of G, d_π is the degree of π, $S_n^{p'}$ stands for the $(n \times n)$-dimensional Schatten class of exponent p', and μ is the Haar measure of G normalized so that $\mu(G) = 1$. On the other hand, Peetre presented in 1969 the first work [19] analyzing the Hausdorff–Young inequality for Banach-valued functions $f : \mathbb{R} \to B$. In this case, the validity of the inequality for some fixed p depends on the Banach space B. This leads to the notion of Fourier type of a Banach space with respect to a locally compact abelian group, introduced by Milman in [17]. The theory of Fourier type with respect to locally compact abelian groups was further developed by several authors; see [1, 3, 4, 10, 11, 14].

However, as far as we know, in the non-commutative setting there is no analogous theory of Fourier type described in the literature. Our aim is to fill this gap. Namely, to analyze the validity of Kunze's results for vector-valued functions. In this work we investigate the validity of the Hausdorff–Young inequality for vector-valued functions defined on a compact group. As can be seen throughout the paper, compactness is an essential assumption in many of the results we present here. For a non-commutative compact group G, the vector-valued Fourier transform must be defined for irreducible unitary representations $\pi \in \widehat{G}$ and its values are vector-valued matrices. Therefore, just to start talking about the Hausdorff–Young inequality, one has to be able to define norms for vector-valued matrices. By

Ruan's theorem [**22**], this matricial structure leads us to consider an operator space structure on the vector space in which we are taking values. It appears clear that, in order to develop a theory of Fourier type in this context, we shall need to take values in operator spaces rather than Banach spaces. This crucial point is obviously at the root of the notion of Fourier type.

To conclude, we would like to point out that the theory initiated in this paper has been further developed in [**12**] and [**13**]. Roughly speaking, the paper [**12**] deals with the sharpness of Theorems 6.5 and 6.8 for compact semisimple Lie groups; see §6 for more on this topic. On the other hand, the notions of Fourier type and cotype of an operator space with respect to a compact group are extended in [**13**] to the more general setting of type and cotype with respect to a 'quantized orthonormal system'. This contains, for instance, the non-commutative versions of Rademacher or Gauss type and cotype. All this is used in [**13**] to obtain an operator space version of Kwapień's theorem [**16**] characterizing Hilbert spaces by means of vector-valued orthogonal series.

The organization of the paper is as follows. In §1 we recall the notions of operator space theory and vector-valued Schatten classes that will be used in the sequel. In §2 we define the Fourier transform on compact groups for vector-valued functions. We also study the spaces $\mathcal{L}_E^p(\widehat{G})$, where the Fourier transform takes values. Some results for which we have not found any reference have been proved for completeness. In particular, Proposition 2.4 and Corollary 2.5 are especially relevant since they show that the spaces $\mathcal{L}_E^\infty(\widehat{G})$ and $\mathcal{L}_E^1(\widehat{G})$ behave with respect to the minimal and projective tensor products as the classical L_E^∞ and L_E^1 do with respect to Grothendieck's tensor norms. Sections 3 and 4 are mainly devoted to showing that the notions of Fourier type and cotype are well defined, and also to proving some basic properties. Section 4 is especially far from the commutative theory since, as we show there, when dealing with abelian groups the notion of Fourier cotype reduces to the notion of Fourier type with respect to the dual group. In §5, given an operator space E, we investigate the Fourier type and cotype of some general operator spaces related to E such as subspaces, duals, interpolated spaces, etc. Finally, in §6, we investigate the main examples, that is, Lebesgue spaces and Schatten classes. In particular, for the vector-valued ones we prove some quantized Minkowski inequalities that we shall need.

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1. *Operator spaces and Schatten classes*

The basic theories behind this paper are the theory of operator spaces and the subsequent theory of vector-valued Schatten classes. The reader is referred to [**8**] and [**21**] for a basic background on these topics and their connection with the present work. We begin with a brief summary of the results of operator space theory that will be used in the sequel.

(a) *Definition of operator space.* We will denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on some Hilbert space $\mathcal{H}$. For our purposes an *operator space* E can be defined as a closed subspace of $\mathcal{B}(\mathcal{H})$. Given an operator space $E \subset \mathcal{B}(\mathcal{H})$ we write $M_n(E)$ for the space $M_n \otimes E$ of $n \times n$ matrices with entries in E and with the norm imposed by the natural embedding of $M_n \otimes E$ into $\mathcal{B}(l_\mathcal{H}^2(n))$. Here $l_\mathcal{H}^2(n)$

denotes the Hilbert space of all $\mathcal{H}$-valued n-tuples with its natural inner product. On the other hand, given a vector space E and a collection of norms $\| \cdot \|_n$ on the spaces $M_n \otimes E$, one can impose some extra conditions to obtain what is called an *operator space matrix norm* or *operator space structure* on E; see [**8**, Chapter 2]. One of the main results of the theory is the *abstract characterization of operator spaces* given by Ruan in [**22**]. Ruan's theorem can be rephrased by saying that for any operator space structure on a vector space E, there exists a unique Hilbert space $\mathcal{H}$ such that the norm from the operator space structure on $M_n \otimes E$ coincides with the norm induced by the space $\mathcal{B}(l_{\mathcal{H}}^2(n))$.

(b) *Complete boundedness.* A linear mapping between operator spaces $u : E_1 \to E_2$ is said to be *completely bounded* if the members of the family of maps $I_{M_n} \otimes u : M_n(E_1) \to M_n(E_2)$ satisfy

$$\|u\|_{\mathrm{cb}} = \sup_{n \geqslant 1} \|I_{M_n} \otimes u\|_{\mathcal{B}(M_n(E_1), M_n(E_2))} < \infty.$$

We write $\mathrm{cb}(E_1, E_2)$ for the Banach space of completely bounded maps from E_1 to E_2 with the cb-norm. Let $u \in \mathrm{cb}(E_1, E_2)$. We say that u is a *complete isometry* if the mappings $I_{M_n} \otimes u$ are isometries for all $n \geqslant 1$. Similarly u is called *completely contractive* if $\|u\|_{\mathrm{cb}} \leqslant 1$. We also say that u is a *complete isomorphism* if it is a completely bounded linear isomorphism whose inverse is also completely bounded. Finally u is a *completely isometric isomorphism* if it is also a complete isometry.

(c) *Duality.* Ruan's theorem was used by Blecher and Paulsen in [**2**] and by Effros and Ruan in [**7**] to get a *duality theory* in the category of operator spaces. It was shown that, by imposing on $M_n \otimes \mathrm{cb}(E_1, E_2)$ the norm induced by $\mathrm{cb}(E_1, M_n(E_2))$, we obtain an operator space structure on $\mathrm{cb}(E_1, E_2)$. In particular, we have an operator space structure on the dual space $E^* = \mathrm{cb}(E, \mathbb{C})$. This notion of duality behaves as Banach space duality in many senses. For instance, it can be proved that the natural isometric inclusion $E \subset E^{**}$ is a complete isometry.

(d) *Tensor products.* We are interested in two tensor norms that will be used repeatedly in this paper. Given two operator spaces $E_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $E_2 \subset \mathcal{B}(\mathcal{H}_2)$ we define their *minimal tensor product* $E_1 \otimes_{\min} E_2$ by the natural embedding of $E_1 \otimes E_2$ into $\mathcal{B}(\mathcal{H}_1 \otimes_2 \mathcal{H}_2)$, where $\otimes_2$ stands for the Hilbertian tensor product. The minimal tensor product plays the role of the *injective tensor product* of Banach spaces in the category of operator spaces. Similarly, there exists an analog for operator spaces of the *projective tensor product*. It is denoted by $E_1 \otimes^{\wedge} E_2$ and it was introduced in [**2**] and [**7**] independently. The tensor products $\otimes_{\min}$ and $\otimes^{\wedge}$ are associative and commutative. We now list some other properties which we will use in the sequel with no further reference:

$$
\begin{aligned}
E_1 \otimes^{\wedge} E_2 &\longrightarrow E_1 \otimes_{\min} E_2 & \text{is a complete contraction;} \\
E_1^* \otimes_{\min} E_2 &\longrightarrow \mathrm{cb}(E_1, E_2) & \text{is a complete isometry;} \\
E_1 \otimes_{\min} E_2 &\longrightarrow \mathrm{cb}(E_1^*, E_2) & \text{is a complete isometry;} \\
(E_1 \otimes^{\wedge} E_2)^* &\longrightarrow \mathrm{cb}(E_1, E_2^*) & \text{is a completely isometric isomorphism.}
\end{aligned}
$$

(e) *Complex interpolation.* Let $\{E_0, E_1\}$ be a compatible couple of Banach spaces in the sense of complex interpolation. Let us suppose that E_0 and E_1

have an operator space structure. In [**20**] Pisier showed that, if E_θ denotes the interpolation space $[E_0, E_1]_\theta$, one can define an operator space structure on E_θ by imposing on $M_n \otimes E_\theta$ the norm of the Banach space $[M_n(E_0), M_n(E_1)]_\theta$. He also proved the analog for operator spaces of the classical interpolation result for Banach spaces. Namely, if we assume that $u : E_0 + E_1 \to F_0 + F_1$ satisfies the inequalities $\|u\|_{\mathrm{cb}(E_0, F_0)} \leqslant C_0$ and $\|u\|_{\mathrm{cb}(E_1, F_1)} \leqslant C_1$, then for $0 < \theta < 1$ we have the estimate

$$\|u\|_{\mathrm{cb}(E_\theta, F_\theta)} \leqslant C_0^{1-\theta} C_1^\theta.$$

We now recall the definition and the main properties of the Schatten classes. The non-commutative analog of the n-dimensional Lebesgue space $l^p(n)$ is the Schatten class S_n^p which is defined as the space M_n of $n \times n$ complex matrices with the norm given by

(a) $\|A\|_{S_n^p} = (\mathrm{tr}\, |A|^p)^{1/p}$, if $1 \leqslant p < \infty$,

(b) $\|A\|_{S_n^\infty} = \sup\{\|Ax\|_{l^2(n)} : \|x\|_{\ell^2(n)} \leqslant 1\}$, if $p = \infty$.

Now we present the *vector-valued Schatten classes* $S_n^p(E)$, introduced by Pisier in [**21**]. The point here is that the space E where we take values has to be an operator space. If $p = \infty$, we have, by definition, $S_n^\infty = \mathcal{B}(l^2(n))$ and so we obtain a natural operator space structure for S_n^∞. We define $S_n^\infty(E)$ as the operator space $S_n^\infty \otimes_{\min} E$. It is obvious that $M_n(E)$ and $S_n^\infty(E)$ coincide. In what follows we shall write $S_n^\infty(E)$ for $M_n(E)$. If $p = 1$, the duality $S_n^1 = (S_n^\infty)^*$ gives a natural operator space structure on S_n^1. We set $S_n^1(E) = S_n^1 \otimes^\wedge E$. Finally, since the identity mapping $S_n^1(E) \to S_n^\infty(E)$ is contractive, we define the classes $S_n^p(E)$ by means of complex interpolation. Namely, $S_n^p(E) = [S_n^\infty(E), S_n^1(E)]_{1/p}$. The next theorem summarizes some properties of the vector-valued Schatten classes that will be used repeatedly throughout the paper; see [**21**, Chapter 1].

REMARK 1.1.   In the same fashion, it is possible to define a natural operator space structure on the Bochner–Lebesgue spaces; see [**21**, Chapter 2].

THEOREM 1.2 (Pisier).   *The vector-valued Schatten classes satisfy the following properties.*

1. *The cb-norm. Let E_1 and E_2 be operator spaces and let $1 \leqslant p \leqslant \infty$. Then the cb-norm of any linear mapping $u : E_1 \to E_2$ is given by*

$$\|u\|_{\mathrm{cb}} = \sup_{n \geqslant 1} \|I_{M_n} \otimes u\|_{\mathcal{B}(S_n^p(E_1), S_n^p(E_2))}.$$

2. *Duality. Let $1 \leqslant p \leqslant \infty$ and let p' denote the conjugate exponent of p. The map $A \in S_n^{p'}(E^*) \mapsto \mathrm{tr}(A \cdot) \in S_n^p(E)^*$ is completely isometric.*

3. *Complex interpolation. Let $1 \leqslant p_0$, $p_1 \leqslant \infty$ and $0 < \theta < 1$ and assume that $\{E_0, E_1\}$ is a compatible couple of operator spaces. Then, letting $p_\theta^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$, we have*

$$[S_n^{p_0}(E_0), S_n^{p_1}(E_1)]_\theta = S_n^{p_\theta}(E_\theta).$$

4. *Ordered norms. Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$. Then the identity map*

$$S_n^{p_1}(E) \longrightarrow S_n^{p_2}(E)$$

is a contraction.

5. Fubini type theorems. *Let* $1 \leqslant p \leqslant \infty$ *and let* $n_1, n_2, n \geqslant 1$. *Then, we have, completely isometrically,*

$$S_{n_1}^p(S_{n_2}^p(E)) \simeq S_{n_2}^p(S_{n_1}^p(E)) \quad \text{and} \quad S_n^p(L_E^p(\Omega)) \simeq L_{S_n^p(E)}^p(\Omega).$$

2. Vector-valued Fourier transform

We assume the reader is familiar with the language of non-commutative abstract harmonic analysis on compact groups. In any case all the results we use here can be found in [**9**]. In what follows we shall assume that G is a compact Hausdorff topological group endowed with its Haar measure μ normalized so that $\mu(G) = 1$. The mapping $\pi : G \to U(\mathbb{C}^{d_\pi})$ will denote an irreducible unitary representation of G of degree d_π. That is, $\pi \in \widehat{G}$ where the symbol $\widehat{G}$ stands for the dual object of G.

DEFINITION 2.1.   Given an operator space E, $f \in L_E^1(G)$ and $\pi \in \widehat{G}$, the vector-valued *Fourier coefficient* of f at π is defined as the operator

$$\widehat{f}(\pi) = \int_G f(g)\pi(g)^* \, d\mu(g) \in \mathcal{B}(\mathbb{C}^{d_\pi}, E^{d_\pi}).$$

We interpret this operator-valued integral in the weak sense. That is, given an orthonormal basis $\{v_1, v_2, \dots, v_{d_\pi}\}$ of $\mathbb{C}^{d_\pi}$ and $u \in \mathbb{C}^{d_\pi}$, we define the jth component of $\widehat{f}(\pi)u$, with respect to that basis, by the following element of E:

$$\int_G f(g)\langle \pi(g)^* u, v_j \rangle \, d\mu(g).$$

Since $\pi(g)$ is unitary, it follows that the vector-valued Fourier coefficients are well defined for all f in $L_E^1(G)$. Once we have fixed the basis of $\mathbb{C}^{d_\pi}$, we can identify $\mathcal{B}(\mathbb{C}^{d_\pi}, E^{d_\pi})$ with the space $M_{d_\pi} \otimes E$. This leads us to write the *Fourier transform* operator $\mathcal{F}_{G,E}$, in the form

$$\mathcal{F}_{G,E} : L_E^1(G) \longrightarrow \prod_{\pi \in \widehat{G}} M_{d_\pi} \otimes E.$$

The first step in studying the Hausdorff–Young inequality is to find a *natural* L^p-norm for this Cartesian product, which we denote by $\mathcal{M}_E(\widehat{G})$.

DEFINITION 2.2.   Let E be an operator space and $1 \leqslant p < \infty$. The *spaces* $\mathcal{L}_E^p(\widehat{G})$ are defined as follows:

$$\mathcal{L}_E^p(\widehat{G}) = \left\{ A \in \mathcal{M}_E(\widehat{G}) : \|A\|_{\mathcal{L}_E^p(\widehat{G})} = \left(\sum_{\pi \in \widehat{G}} d_\pi \|A^\pi\|_{S_{d_\pi}^p(E)}^p \right)^{1/p} < \infty \right\},$$

$$\mathcal{L}_E^\infty(\widehat{G}) = \left\{ A \in \mathcal{M}_E(\widehat{G}) : \|A\|_{\mathcal{L}_E^\infty(\widehat{G})} = \sup_{\pi \in \widehat{G}} \|A^\pi\|_{S_{d_\pi}^\infty(E)} < \infty \right\}.$$

We write $\mathcal{L}^p(\widehat{G})$ for the case $E = \mathbb{C}$. Note that we require the vector space E to be an operator space. This condition is necessary since we are making use of the spaces $S_n^p(E)$; see Pisier's monograph [**21**] for more on this topic. The family of spaces $\mathcal{L}_E^p(\widehat{G})$ is a particular case of a bigger family of spaces studied in [**21**, Chapter 2], the spaces $\ell_p(\mu, \{E_i\})$. This remark allows us to provide the spaces

$\mathcal{L}_E^p(\widehat{G})$ with the natural operator space structure induced by $\ell_p(\mu, \{E_i\})$. We now summarize the main properties of these spaces.

(a) *Duality.* Let $1 \leqslant p < \infty$ and let p' be the conjugate exponent of p. Then the following map is a completely isometric isomorphism:

$$A \in \mathcal{L}_{E^*}^{p'}(\widehat{G}) \longmapsto \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \cdot) \in \mathcal{L}_E^p(\widehat{G})^*.$$

(b) *Complex interpolation.* Let $1 \leqslant p_0, p_1 \leqslant \infty$. Assume that $\{E_0, E_1\}$ is a compatible couple of operator spaces. Then $\{\mathcal{L}_{E_0}^{p_0}(\widehat{G}), \mathcal{L}_{E_1}^{p_1}(\widehat{G})\}$ is also a compatible couple and, for $0 < \theta < 1$ and $p_\theta^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$, we have

$$[\mathcal{L}_{E_0}^{p_0}(\widehat{G}), \mathcal{L}_{E_1}^{p_1}(\widehat{G})]_\theta = \mathcal{L}_{E_\theta}^{p_\theta}(\widehat{G}).$$

(c) *Ordered norms.* The embedding $\mathcal{L}_E^{p_1}(\widehat{G}) \to \mathcal{L}_E^{p_2}(\widehat{G})$ is contractive whenever $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$.

(d) *Fubini type theorems.* Let $1 \leqslant p \leqslant \infty$ and let $n \geqslant 1$. Then the following are completely isometric isomorphisms:

$$S_n^p(\mathcal{L}_E^p(\widehat{G})) \simeq \mathcal{L}_{S_n^p(E)}^p(\widehat{G}) \quad \text{and} \quad \mathcal{L}_{L_E^p(\Omega)}^p(\widehat{G}) \simeq L_{\mathcal{L}_E^p(\widehat{G})}^p(\Omega).$$

We now present a couple of results concerning these spaces for which we have not found any reference. These will be especially useful in the study of the Fourier cotype; see for instance the proof of Proposition 4.3. We first need a technical result, which is an inequality of Hölder type.

LEMMA 2.3. *Let E be an operator space, $n_1, n_2 \geqslant 1$ and $1 \leqslant p \leqslant \infty$. Let us consider $A \in M_{n_1} \otimes E$ and $B_{ij} \in M_{n_1}$ for $1 \leqslant i, j \leqslant n_2$. Then*

$$\|(\operatorname{tr}(AB_{ij}))\|_{S_{n_2}^1(E)} \leqslant \|A\|_{S_{n_1}^{p'}(E)} \|(B_{ij})\|_{S_{n_1}^p(S_{n_2}^1)}.$$

Proof. If a_{kl} and b_{kl}^{ij} denote the entries of A and B_{ij} respectively, then we can write

$$(\operatorname{tr}(AB_{ij})) = \sum_{k,l=1}^{n_1} (b_{lk}^{ij}) \otimes a_{kl} \in M_{n_2} \otimes E.$$

Hence, recalling the completely isometric isomorphism from $\operatorname{cb}(S_n^1, E^*)$ onto $(S_n^1 \otimes^\wedge E)^*$ given by $\Psi(A \otimes e) = \Phi(A)(e)$, we obtain

$$\|(\operatorname{tr}(AB_{ij}))\|_{S_{n_2}^1(E)} = \sup_{\|\Phi\|_{\operatorname{cb}(S_{n_2}^1, E^*)} \leqslant 1} \left| \sum_{k,l=1}^{n_1} [\Phi(b_{lk}^{ij})](a_{kl}) \right|$$

$$= \sup_{\|\Phi\|_{\operatorname{cb}(S_{n_2}^1, E^*)} \leqslant 1} |\operatorname{tr}(CA)|$$

$$\leqslant \|A\|_{S_{n_1}^{p'}(E)} \sup_{\|\Phi\|_{\operatorname{cb}(S_{n_2}^1, E^*)} \leqslant 1} \|C\|_{S_{n_1}^p(E^*)}$$

$$\leqslant \|A\|_{S_{n_1}^{p'}(E)} \|(B_{ij})\|_{S_{n_1}^p(S_{n_2}^1)},$$

where $C = [I_{M_{n_1}} \otimes \Phi](B_{ij}) \in M_{n_1} \otimes E^*$. This completes the proof. $\qquad \square$

For fixed $\pi_0 \in \widehat{G}$ and $1 \leqslant i_0, j_0 \leqslant d_{\pi_0}$, we define $M(\pi_0, i_0, j_0) \in \mathcal{M}_{\mathbb{C}}(\widehat{G})$ by the relations $M(\pi_0, i_0, j_0)_{ij}^{\pi} = \delta_{\pi\pi_0} \delta_{ii_0} \delta_{jj_0}$.

PROPOSITION 2.4.   *The following is a completely isometric isomorphism:*

$$\Lambda : \mathcal{L}_E^{\infty}(\widehat{G}) \longrightarrow \mathrm{cb}(\mathcal{L}^1(\widehat{G}), E),$$
$$A \longmapsto \sum_{\pi \in \widehat{G}} d_{\pi} \operatorname{tr}(A^{\pi} \cdot).$$

Proof. We just need to show that Λ is an isometric isomorphism, since we have the natural isometric isomorphisms

$$S_n^{\infty}(\mathcal{L}_E^{\infty}(\widehat{G})) \sim \mathcal{L}_{S_n^{\infty}(E)}^{\infty}(\widehat{G}) \quad \text{and} \quad \mathrm{cb}(\mathcal{L}^1(\widehat{G}), S_n^{\infty}(E)) \sim S_n^{\infty}(\mathrm{cb}(\mathcal{L}^1(\widehat{G}), E)).$$

(1) Λ *is a contraction.* By expressing the cb-norm in terms of the Schatten class S^1, we have

$$\|\Lambda(A)\|_{\mathrm{cb}(\mathcal{L}^1(\widehat{G}),E)} \leqslant \sup_{n \geqslant 1} \left\{ \sum_{\pi \in \widehat{G}} d_{\pi} \|(\operatorname{tr}(A^{\pi} B_{ij}^{\pi}))\|_{S_n^1(E)} : \|(B_{ij})\|_{S_n^1(\mathcal{L}^1(\widehat{G}))} \leqslant 1 \right\}.$$

But Lemma 2.3 with $p = 1$ gives

$$\sum_{\pi \in \widehat{G}} d_{\pi} \|(\operatorname{tr}(A^{\pi} B_{ij}^{\pi}))\|_{S_n^1(E)} \leqslant \sum_{\pi \in \widehat{G}} d_{\pi} \|A^{\pi}\|_{S_{d_{\pi}}^{\infty}(E)} \|(B_{ij}^{\pi})\|_{S_{d_{\pi}n}^1}$$
$$\leqslant \|A\|_{\mathcal{L}_E^{\infty}(\widehat{G})} \|(B_{ij})\|_{S_n^1(\mathcal{L}^1(\widehat{G}))}.$$

(2) Λ *is an isometry.* For fixed $\pi \in \widehat{G}$, we define $B(\pi, i, j)$ to be the element of $\mathcal{L}^1(\widehat{G})$ given by $d_{\pi}^{-1} M(\pi, j, i)$ and we denote by $\mathbf{B}(\pi)$ the matrix with entries $B(\pi, i, j)$ where $1 \leqslant i, j \leqslant d_{\pi}$. Due to the natural complete isometry $\mathcal{L}^1(\widehat{G}) \to \mathrm{cb}(\mathcal{L}^{\infty}(\widehat{G}), \mathbb{C})$, it is not difficult to check that $\|\mathbf{B}(\pi)\|_{S_{d_{\pi}}^{\infty}(\mathcal{L}^1(\widehat{G}))} = 1$. Since this works for any $\pi \in \widehat{G}$, we get

$$\|\Lambda(A)\|_{\mathrm{cb}(\mathcal{L}^1(\widehat{G}),E)} \geqslant \sup_{\pi \in \widehat{G}} \|[I_{M_{d_{\pi}}} \otimes \Lambda(A)](\mathbf{B}(\pi))\|_{S_{d_{\pi}}^{\infty}(E)} = \|A\|_{\mathcal{L}_E^{\infty}(\widehat{G})}.$$

(3) Λ *is surjective.* Let $\Phi \in \mathrm{cb}(\mathcal{L}^1(\widehat{G}), E)$. Then we define $A \in \mathcal{M}_E(\widehat{G})$ by the relation

$$A^{\pi} = d_{\pi}^{-1}(\Phi(M(\pi, j, i))), \quad \text{where } \pi \in \widehat{G}.$$

The definition of A gives rise to the following expression:

$$\Phi(B) = \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_{\pi}} b_{ij}^{\pi} \Phi(M(\pi, i, j)) = \sum_{\pi \in \widehat{G}} d_{\pi} \operatorname{tr}(A^{\pi} B^{\pi})$$

where b_{ij}^{π} are the entries of B^{π}. Therefore it suffices to check that $A \in \mathcal{L}_E^{\infty}(\widehat{G})$. But following the notation of Step (2), we obtain

$$\|A^{\pi}\|_{S_{d_{\pi}}^{\infty}(E)} = d_{\pi}^{-1} \|(\Phi(M(\pi, j, i)))\|_{S_{d_{\pi}}^{\infty}(E)}$$
$$\leqslant \|\Phi\|_{\mathrm{cb}(\mathcal{L}^1(\widehat{G}),E)} \|\mathbf{B}(\pi)\|_{S_{d_{\pi}}^{\infty}(\mathcal{L}^1(\widehat{G}))} \leqslant \|\Phi\|_{\mathrm{cb}(\mathcal{L}^1(\widehat{G}),E)}.$$

Since $\Phi \in \mathrm{cb}(\mathcal{L}^1(\widehat{G}), E)$, we have a uniform upper bound. □

The space $\mathcal{L}_E^{\infty}(\widehat{G})$ behaves with respect to the minimal tensor product as $L_E^{\infty}(\Omega)$ does with respect to the injective tensor product in the category of Banach spaces. Namely, as a consequence of Proposition 2.4, we see that $\mathcal{L}^{\infty}(\widehat{G}) \otimes_{\min} E \hookrightarrow \mathcal{L}_E^{\infty}(\widehat{G})$

is a complete isometry. The space $\mathcal{L}_E^1(\widehat{G})$ behaves in the same fashion with respect to the projective tensor product.

COROLLARY 2.5. *The identity* $\mathcal{L}^1(\widehat{G}) \otimes^\wedge E \to \mathcal{L}_E^1(\widehat{G})$ *is completely isometric.*

Proof. By [**8**, Proposition 3.2.2] it suffices to check that the adjoint mapping is a complete isometric isomorphism. But Proposition 2.4 gives the chain

$$\mathcal{L}_E^1(\widehat{G})^* \simeq \mathcal{L}_{E^*}^\infty(\widehat{G}) \simeq \mathrm{cb}(\mathcal{L}^1(\widehat{G}), E^*) \simeq (\mathcal{L}^1(\widehat{G}) \otimes^\wedge E)^*$$

of completely isometric isomorphisms. This completes the proof. □

For the sake of completeness we introduce the *space* $C_0(\widehat{G}, E)$. It is defined as the collection of those $A \in \mathcal{L}_E^\infty(\widehat{G})$ satisfying the following condition:

for all $\varepsilon > 0$ we have $\|A^\pi\|_{S_{d_\pi}^\infty(E)} < \varepsilon$ except for finitely many $\pi \in \widehat{G}$.

As a subspace of $\mathcal{L}_E^\infty(\widehat{G})$ this space inherits a natural operator space structure. The only two results about the spaces $\mathcal{L}_E^p(\widehat{G})$ that fail at $p = \infty$ are the density of $\mathcal{L}^p(\widehat{G}) \otimes E$ in $\mathcal{L}_E^p(\widehat{G})$ and duality: the predual of $\mathcal{L}_{E^*}^\infty(\widehat{G})$ is not $\mathcal{L}_E^\infty(\widehat{G})$. However, it is easy to see the density of $C_0(\widehat{G}) \otimes E$ in $C_0(\widehat{G}, E)$. On the other hand, the dual of $C_0(\widehat{G}, E)$ is completely isomorphic to $\mathcal{L}_{E^*}^1(\widehat{G})$.

PROPOSITION 2.6. *The following is a completely isometric isomorphism:*

$$\Lambda : \mathcal{L}_{E^*}^1(\widehat{G}) \longrightarrow C_0(\widehat{G}, E)^*,$$
$$A \longmapsto \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(A^\pi \cdot).$$

Proof. Taking into account the natural isometric isomorphisms given by

$$S_n^1(\mathcal{L}_{E^*}^1(\widehat{G})) \sim \mathcal{L}_{S_n^\infty(E)^*}^1(\widehat{G}) \quad \text{and} \quad C_0(\widehat{G}, S_n^\infty(E))^* \sim S_n^1(C_0(\widehat{G}, E)^*),$$

it is enough to see that Λ is an isometric isomorphism. We prove this fact in several steps.

(1) Λ *is a contraction.* This is an obvious consequence of the duality action on the Schatten classes $S_n^p(E)$:

$$\|\Lambda(A)\|_{C_0(\widehat{G}, E)^*} \leqslant \sup_{\|B\|_{C_0(\dot{G}, E)} \leqslant 1} \sum_{\pi \in \widehat{G}} d_\pi |\mathrm{tr}(A^\pi B^\pi)|$$
$$\leqslant \sup_{\|B\|_{C_0(\dot{G}, E)} \leqslant 1} \sum_{\pi \in \widehat{G}} d_\pi \|A^\pi\|_{S_{d_\pi}^1(E^*)} \|B^\pi\|_{S_{d_\pi}^\infty(E)}$$
$$\leqslant \|A\|_{\mathcal{L}_{E^*}^1(\widehat{G})}.$$

(2) Λ *is an isometry.* Let $A \in \mathcal{L}_{E^*}^1(\widehat{G})$. For all $\varepsilon > 0$ there exists a finite set $I_{A,\varepsilon} \subset \widehat{G}$ such that

$$\sum_{\pi \notin I_{A,\varepsilon}} d_\pi \|A^\pi\|_{S_{d_\pi}^1(E^*)} < \tfrac{1}{2}\varepsilon.$$

Furthermore, for all $\pi \in \widehat{G}$ there exists $B_\varepsilon^\pi \in S_{d_\pi}^\infty(E)$ of norm 1 such that

$$\mathrm{tr}(A^\pi B_\varepsilon^\pi) > \|A^\pi\|_{S_{d_\pi}^1(E^*)} - \frac{\varepsilon/2}{|I_{A,\varepsilon}| \max_{\pi \in I_{A,\varepsilon}} d_\pi}$$

where $|I_{A,\varepsilon}|$ denotes the number of elements of $I_{A,\varepsilon}$. Let C_ε be the element of $C_0(\widehat{G}, E)$ of norm 1 defined by $C_\varepsilon^\pi = B_\varepsilon^\pi$ if $\pi \in I_{A,\varepsilon}$ and $C_\varepsilon^\pi = 0$ otherwise. This step is completed by taking ε arbitrarily small in the expression

$$\|\Lambda(A)\|_{C_0(\widehat{G},E)^*} \geqslant \left| \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi C_\varepsilon^\pi) \right| > \|A\|_{\mathcal{L}^1_{E^*}(\widehat{G})} - \varepsilon.$$

(3) Λ *is surjective.* Let $\Phi \in C_0(\widehat{G}, E)^*$. Then we define $A \in \mathcal{M}_{E^*}(\widehat{G})$ by the relation

$$A^\pi = d_\pi^{-1}(\Phi(M(\pi, j, i) \otimes \cdot)), \quad \text{for } \pi \in \widehat{G}.$$

As in Proposition 2.4, it can be shown that $\Phi = \Lambda(A)$ with $A \in \mathcal{L}^1_{E^*}(\widehat{G})$. $\square$

3. *Fourier type*

Let E be an operator space and let $1 \leqslant p \leqslant 2$. Given $f \in L^p(G)$ and $e \in E$ it is obvious that the Fourier transform of $f \otimes e$ coincides with $\widehat{f} \otimes e$. Thus, the Hausdorff–Young inequality for compact groups (see [15] or Lemma 5.1 below) provides the relation $\mathcal{F}_{G,E}(L^p(G) \otimes E) \subset \mathcal{L}^{p'}(\widehat{G}) \otimes E$. This motivates the following definition.

DEFINITION 3.1.   Let $1 \leqslant p \leqslant 2$ and let p' be the conjugate exponent of p. We say that the operator space E has *Fourier type* p with respect to the compact group G if the Fourier transform $\mathcal{F}_{G,E} : L^p(G) \otimes E \to \mathcal{L}^{p'}(\widehat{G}) \otimes E$ can be extended to a completely bounded operator

$$\Lambda^1_{G,E,p} : L^p_E(G) \longrightarrow \mathcal{L}^{p'}_E(\widehat{G}).$$

In that case, we shall denote by $\mathcal{C}^1_p(E, G)$ the cb-norm of $\Lambda^1_{G,E,p}$.

REMARK 3.2.   If the compact group G is also abelian, there already exists a notion (introduced by Milman in [17]) of Fourier type of a Banach space with respect to G. The only difference with Milman's notion is that here we require the extended operator to be completely bounded, while in the commutative setting only the boundedness of this operator is required.

The first natural question that arises after the definition of Fourier type is if the extension of $\mathcal{F}_{G,E}$ is always the natural one. That is, let us suppose that the operator space E has Fourier type p with respect to G. Then we wonder whether $\Lambda^1_{G,E,p}(f) = \mathcal{F}_{G,E}(f)$ for all $f \in L^p_E(G)$.

LEMMA 3.3.   *We have* $\|\widehat{f}\|_{\mathcal{L}^\infty_E(\widehat{G})} \leqslant \|f\|_{L^1_E(G)}$ *for all* $f \in L^1_E(G)$.

Proof.   Since E is an operator space we have $E \subset \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Hence, if $\mathbf{h} = (h_1, h_2, \dots, h_{d_\pi}) \in \ell^2_{\mathcal{H}}(d_\pi)$, we can write

$$\|\widehat{f}(\pi)\|_{S^\infty_{d_\pi}(E)} = \|\widehat{f}(\pi)\|_{\mathcal{B}(\ell^2_{\mathcal{H}}(d_\pi))}$$

$$\leqslant \sup_{\|\mathbf{h}\|_{\ell^2_{\mathcal{H}}(d_\pi)} \leqslant 1} \left(\sum_{i=1}^{d_\pi} \left[\int_G \left\| f(g) \left(\sum_{j=1}^{d_\pi} \overline{\pi_{ji}(g)} \, h_j \right) \right\|_{\mathcal{H}} d\mu(g) \right]^2 \right)^{1/2}.$$

Applying the Minkowski inequality for integrals we get

$$\|\widehat{f}(\pi)\|_{S_{d_\pi}^\infty(E)} \leqslant \sup_{\|\mathbf{h}\|_{\ell_{\mathcal{H}}^2(d_\pi)} \leqslant 1} \int_G \|f(g)\|_E \left(\sum_{i=1}^{d_\pi} \left\| \sum_{j=1}^{d_\pi} \overline{\pi_{ji}(g)} h_j \right\|_{\mathcal{H}}^2 \right)^{1/2} d\mu(g).$$

Therefore we just need to check the inequality

$$\sup_{\|\mathbf{h}\|_{\ell_{\mathcal{H}}^2(d_\pi)} \leqslant 1} \left(\sum_{i=1}^{d_\pi} \left\| \sum_{j=1}^{d_\pi} \overline{\pi_{ji}(g)} h_j \right\|_{\mathcal{H}}^2 \right)^{1/2} \leqslant 1$$

for all $\pi \in \widehat{G}$ and almost all $g \in G$. But this is a simple consequence of the unitarity of $\pi(g)$ for any $g \in G$. This completes the proof. $\qquad\square$

PROPOSITION 3.4. *Let E be an operator space having Fourier type p with respect to G. Then $\Lambda_{G,E,p}^1(f) = \mathcal{F}_{G,E}(f)$ for all $f \in L_E^p(G)$.*

Proof. Let $\{f_n\}_{n=1}^\infty \subset L^p(G) \otimes E$ be a sequence convergent to f in the norm of $L_E^p(G)$. Then, applying Lemma 3.3, we have

$$\begin{aligned}
\|\Lambda_{G,E,p}^1(f) - \mathcal{F}_{G,E}(f)\|_{\mathcal{L}_E^\infty(\widehat{G})} &\leqslant \|\Lambda_{G,E,p}^1(f - f_n)\|_{\mathcal{L}_E^\infty(\widehat{G})} + \|\mathcal{F}_{G,E}(f_n - f)\|_{\mathcal{L}_E^\infty(\widehat{G})} \\
&\leqslant \|\Lambda_{G,E,p}^1(f - f_n)\|_{\mathcal{L}_E^{p'}(\widehat{G})} + \|\mathcal{F}_{G,E}(f_n - f)\|_{\mathcal{L}_E^\infty(\widehat{G})} \\
&\leqslant \mathcal{C}_p^1(E,G)\|f - f_n\|_{L_E^p(G)} + \|f_n - f\|_{L_E^1(G)} \\
&\leqslant (\mathcal{C}_p^1(E,G) + 1)\|f - f_n\|_{L_E^p(G)}.
\end{aligned}$$

The result follows by taking the limit in n. This completes the proof. $\qquad\square$

As is well known, every Banach space has Fourier type 1 in the sense of Milman [17]. In the following result, which extends Lemma 3.3, we show that every operator space has Fourier type 1.

PROPOSITION 3.5. *We have $\mathcal{C}_1^1(E,G) = 1$ for every pair (E,G).*

Proof. Let us denote by $\max B$ the operator space which results when we impose on the Banach space B its max quantization; see [8, Chapter 3] for the details. Let E_1 and E_2 be operator spaces. Then the natural identification

$$\mathrm{cb}((\max B) \otimes^\wedge E_1, E_2) \simeq \mathcal{B}(B, \mathrm{cb}(E_1, E_2)),$$

given by $\Psi(b \otimes e_1) = \Phi(b)(e_1)$, is a completely isomorphic isomorphism. This follows by the factorization

$$\mathrm{cb}((\max B) \otimes^\wedge E_1, E_2) \simeq \mathrm{cb}(\max B, \mathrm{cb}(E_1, E_2)) \simeq \mathcal{B}(B, \mathrm{cb}(E_1, E_2))$$

which is composed of completely isometric isomorphisms; see [8, Chapters 3, 7]. Therefore, since the space $L_E^1(G)$ can be rewritten as $\max L^1(G) \otimes^\wedge E$, we get

$$\mathcal{C}_1^1(E,G) = \sup_{\|f\|_{L^1(G)} \leqslant 1} \|\widehat{f} \otimes \cdot\|_{\mathrm{cb}(E,\mathcal{L}_E^\infty(\widehat{G}))} \leqslant \sup_{\|f\|_{L^1(G)} \leqslant 1} \|\widehat{f}\|_{\mathcal{L}^\infty(\widehat{G})} = 1$$

by the Hausdorff–Young inequality on compact groups; see [15] or Lemma 5.1 below. Recall that the supremum is attained by taking f to be the constant function 1. This completes the proof. $\qquad\square$

REMARK 3.6.   There exists an alternative approach to this result using similar arguments to those employed in the proof of Proposition 4.3.

The following corollary exhibits the Fourier type as a stronger condition on the pair (E, G) as the exponent p approaches 2. Its proof follows by means of Proposition 3.5 and the complex interpolation method.

COROLLARY 3.7.   *Let* $1 \leqslant p_1 \leqslant p_2 \leqslant 2$ *and assume that the operator space* E *has Fourier type* p_2 *with respect to* G. *Then* E *has Fourier type* p_1 *with respect to* G. *Moreover we have* $\mathcal{C}^1_{p_1}(E, G) \leqslant \mathcal{C}^1_{p_2}(E, G)^{p'_2/p'_1}$.

A vector-valued version of the Riemann–Lebesgue lemma on compact groups follows easily from Proposition 3.5 and the scalar result.

COROLLARY 3.8.   *We have* $\mathcal{F}_{G,E}(L^1_E(G)) \subset C_0(\widehat{G}, E)$ *for every operator space* E.

4. Fourier cotype

If G is a locally compact abelian group, the Fourier inversion theorem asserts that any $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$ can be recovered as

$$f(g) = \widehat{\widehat{f}}(g^{-1}) \quad \text{for almost every } g \in G.$$

Furthermore, if G is compact, one can conclude that the operators $\mathcal{F}_G^{-1}$ and $\mathcal{F}_{\widehat{G}}$ are essentially the same via the topological isomorphism from G onto its bidual, given by the Pontrjagin duality theorem. In the vector-valued context this means that, in order to study the operator $\mathcal{F}_{G,E}^{-1}$, it suffices to study the Fourier transform $\mathcal{F}_{\widehat{G},E}$. For this reason we do not find the concept of Fourier cotype in commutative theory. However, for a non-commutative compact group G, the Fourier inversion theorem and the Pontrjagin duality theorem are no longer valid since the dual object $\widehat{G}$ is not even a group. These considerations explain why the study of the inverse operator $\mathcal{F}_{G,E}^{-1}$ should not be a trivial consequence of the analysis of the operator $\mathcal{F}_{G,E}$.

Let E be an operator space and $1 \leqslant p \leqslant 2$. Arguing as in §3, we can deduce the relation $\mathcal{F}_{G,E}^{-1}(\mathcal{L}^p(\widehat{G}) \otimes E) \subset L^{p'}(G) \otimes E$. This follows from Kunze's result for the inverse Fourier transform on compact groups (see [**15**] or Lemma 5.1 below). This motivates the following definition.

DEFINITION 4.1.   Let $1 \leqslant p \leqslant 2$ and let p' be the conjugate exponent of p. We say that the operator space E has *Fourier cotype* p' with respect to the compact group G if the operator $\mathcal{F}_{G,E}^{-1} : \mathcal{L}^p(\widehat{G}) \otimes E \to L^{p'}(G) \otimes E$ can be extended to a completely bounded operator

$$\Lambda^2_{G,E,p'} : \mathcal{L}^p_E(\widehat{G}) \longrightarrow L^{p'}_E(G).$$

In that case, we shall denote by $\mathcal{C}^2_{p'}(E, G)$ the cb norm of $\Lambda^2_{G,E,p'}$.

REMARK 4.2.   Now it is obvious that, for compact abelian groups, the notion of Fourier cotype is the completely bounded version of Milman's notion of Fourier type with respect to the dual group $\widehat{G}$.

Plancherel's theorem for compact groups gives an explicit formula for the action of $\mathcal{F}_G^{-1}$ on $\mathcal{L}^2(\widehat{G})$ and, by the natural embeddings, also on $\mathcal{L}^p(\widehat{G})$ for $1 \leqslant p \leqslant 2$. It is obvious that this formula remains valid if we take tensor products. Namely, given $1 \leqslant p \leqslant 2$, the action of the operator $\mathcal{F}_{G,E}^{-1}$ on $\mathcal{L}^p(\widehat{G}) \otimes E$ is given by

$$A \in \mathcal{L}^p(\widehat{G}) \otimes E \longmapsto \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \pi(\cdot)) \in L^{p'}(G) \otimes E.$$

Therefore, if we want our definition of Fourier cotype to be natural, we need affirmative answers for the following questions.

(a) *Does the operator $\Lambda_{G,E,p'}^2$ preserve the given explicit formula?* That is, if the operator space E has Fourier cotype p' with respect to G, we ask whether for all $A \in \mathcal{L}_E^p(\widehat{G})$ we have

$$\Lambda_{G,E,p'}^2(A) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \pi(\cdot)).$$

If $A \in \mathcal{L}_E^p(\widehat{G})$, it has a countable support $I_A = \{\pi_k\}_{k=1}^\infty \subset \widehat{G}$. Then we define $A_n \in \mathcal{L}^p(\widehat{G}) \otimes E$ by the relations $A_n^\pi = A^\pi$ if $\pi = \pi_k$ for $1 \leqslant k \leqslant n$ and $A_n^\pi = 0$ otherwise. Denoting

$$f = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \pi(\cdot)) \quad \text{and} \quad f_n = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_n^\pi \pi(\cdot))$$

we obtain

$$\left\| \Lambda_{G,E,p'}^2(A) - f \right\|_{L_E^{p'}(G)} \leqslant \left\| \Lambda_{G,E,p'}^2(A - A_n) \right\|_{L_E^{p'}(G)} + \left\| f - f_n \right\|_{L_E^{p'}(G)}$$

$$\leqslant C_{p'}^2(E,G) \left\| A - A_n \right\|_{\mathcal{L}_E^p(\widehat{G})} + \left\| f - f_n \right\|_{L_E^{p'}(G)}.$$

The first term of the sum is arbitrarily small as n tends to infinity. For the second term it is not difficult to check that the sequence $\{f_n\}_{n=1}^\infty$ is Cauchy. Thus, replacing this sequence if necessary by an appropriate subsequence, we can assume that $\| f_{n_2} - f_{n_1} \|_{L_E^{p'}(G)} < 2^{-m}$ for all $n_1, n_2 \geqslant m$. Hence

$$\| f - f_n \|_{L_E^{p'}(G)} \leqslant \sum_{k=n+1}^\infty \| f_k - f_{k-1} \|_{L_E^{p'}(G)} < \sum_{k=n}^\infty \frac{1}{2^k}$$

and $\Lambda_{G,E,p'}^2(A) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \pi(\cdot))$ as we wanted.

(b) *Does the operator $\Lambda_{G,E,p'}^2$ coincide with the inverse of the vector-valued Fourier transform?* That is, if the operator space E has Fourier cotype p' with respect to G, we ask whether, for all $A \in \mathcal{L}_E^p(\widehat{G})$, we have

$$\mathcal{F}_{G,E} \circ \Lambda_{G,E,p'}^2(A) = A.$$

Using the same notation as above we just need to see that $\widehat{f} = A$. Given $\pi \in \widehat{G}$ we take n_π to be the smallest positive integer satisfying $\pi \neq \pi_k$ for $k \geqslant n_\pi$. Then it is obvious that $\widehat{f}(\pi) - A^\pi = (\widehat{f} - \widehat{f_n})(\pi)$ for all $n \geqslant n_\pi$ and therefore it is enough to estimate the entries of that matrix. Namely,

$$\left\| ((\widehat{f} - \widehat{f_n})(\pi))_{ij} \right\|_E \leqslant \int_G \| (f - f_n)(g) \|_E |\pi_{ji}(g)| \, d\mu(g)$$

$$\leqslant \| f - f_n \|_{L_E^{p'}(G)} \| \pi_{ji} \|_{L^p(G)} \leqslant \mathcal{C}_{p'}^2(E,G) \| A - A_n \|_{\mathcal{L}_E^p(\widehat{G})}$$

which is arbitrarily small for large n.

PROPOSITION 4.3.   We have $\mathcal{C}_\infty^2(E, G) = 1$ for every pair (E, G).

Proof.   By Property 1 of Theorem 1.2 and a density argument, we have to see that for all $n \geqslant 1$, any family of vectors $A_{ij} \in \mathcal{L}^1(\widehat{G}) \otimes E$ and almost all $g \in G$, we have

$$\left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(g)) \right) \right\|_{S_n^\infty(E)} \leqslant \|(A_{ij})\|_{S_n^\infty(\mathcal{L}_E^1(\widehat{G}))}.$$

If we consider a vector $A \in \mathcal{L}_E^1(\widehat{G})$ as an element of $\operatorname{cb}(\mathcal{L}^\infty(\widehat{G}), E)$ by the relation

$$B \in \mathcal{L}^\infty(\widehat{G}) \longmapsto \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi B^\pi) \in E$$

then it is easy to see that, for $B_g \in \mathcal{L}^\infty(\widehat{G})$ defined by $B_g^\pi = \pi(g)$, we have

$$\left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(g)) \right) \right\|_{S_n^\infty(E)} = \left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \cdot)(B_g) \right) \right\|_{S_n^\infty(E)}$$

$$\leqslant \left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \cdot) \right) \right\|_{\operatorname{cb}(\mathcal{L}^\infty(\widehat{G}), S_n^\infty(E))}$$

$$\leqslant \|(A_{ij})\|_{S_n^\infty(\mathcal{L}^1(\widehat{G}) \otimes^\wedge E)}$$

where the last inequality follows from the complete contraction given by

$$\mathcal{L}^1(\widehat{G}) \otimes^\wedge E \longrightarrow \mathcal{L}^1(\widehat{G}) \otimes_{\min} E \longrightarrow \operatorname{cb}(\mathcal{L}^\infty(\widehat{G}), E).$$

Finally, we get the desired relation by Corollary 2.5. We have shown that $\mathcal{C}_\infty^2(E, G) \leqslant 1$. The reverse inequality follows from Corollary 5.3 below.   □

COROLLARY 4.4.   Let $1 \leqslant p_1 \leqslant p_2 \leqslant 2$ and assume that the operator space E has Fourier cotype p_2' with respect to G. Then E has Fourier cotype p_1' with respect to G. Moreover we have $\mathcal{C}_{p_1'}^2(E, G) \leqslant \mathcal{C}_{p_2'}^2(E, G)^{p_2'/p_1'}$.

5. Duality, cb-distance and some other topics

Let E be an operator space. The aim of this section is to study the Fourier type and cotype of some operator spaces related to E. We begin by stating the scalar-valued Hausdorff–Young inequality. Recall that we write $\check{A}$ to denote $\mathcal{F}_G^{-1}(A)$.

LEMMA 5.1 (Hausdorff–Young inequality).   Let $1 \leqslant p \leqslant 2$ and let p' be the conjugate exponent of p.
  1. If $f \in L^p(G)$, then $\widehat{f} \in \mathcal{L}^{p'}(\widehat{G})$ and $\|\mathcal{F}_G\|_{\operatorname{cb}(L^p(G), \mathcal{L}^{p'}(\widehat{G}))} = 1$.
  2. If $A \in \mathcal{L}^p(\widehat{G})$, then $\check{A} \in L^{p'}(G)$ and $\|\mathcal{F}_G^{-1}\|_{\operatorname{cb}(\mathcal{L}^p(\widehat{G}), L^{p'}(G))} = 1$.

Note that this statement of the inequality goes a bit further than Kunze's original result since we are asserting that the Fourier transform is not only bounded but completely bounded. The proof is straightforward; first one checks that $\mathcal{F}_G$ is a complete contraction from $L^1(G)$ into $\mathcal{L}^\infty(\widehat{G})$. But, since $L^1(G)$ is equipped with its max operator space structure, the cb-norm coincides with the operator norm; see

[**8**, Chapter 3]. The same argument works to see that the inverse Fourier transform is a complete contraction from $\mathcal{L}^1(\widehat{G})$ to $L^\infty(G)$, now that $L^\infty(G)$ is equipped with its min operator space structure. These facts can also be justified as simple consequences of Propositions 3.5 and 4.3. Second, from the Plancherel theorem for compact groups, it is easy to check that $\mathcal{F}_G$ is a complete isometric isomorphism from $L^2(G)$ onto $\mathcal{L}^2(\widehat{G})$. By complex interpolation the general case is obtained and the fact that the cb-norm of the Fourier transform is not smaller than 1 for any $1 \leqslant p \leqslant 2$ can be checked by testing with the constant function 1.

5.1. *Basic results*

We begin with the simplest case, namely, the Fourier type and cotype of the subspaces of E. The following result is a trivial consequence of Property 1 of Theorem 1.2.

PROPOSITION 5.2.   *Let F be a closed subspace of E. Then we have the estimates* $\mathcal{C}_p^1(F,G) \leqslant \mathcal{C}_p^1(E,G)$ *and* $\mathcal{C}_{q'}^2(F,G) \leqslant \mathcal{C}_{q'}^2(E,G)$ *for any* $1 \leqslant p,q \leqslant 2$.

COROLLARY 5.3.   *We have* $\mathcal{C}_p^1(E,G) \geqslant 1$ *and* $\mathcal{C}_{q'}^2(E,G) \geqslant 1$ *for any* $1 \leqslant p,q \leqslant 2$.

Now we consider complex interpolation of operator spaces. The proof of the next result is also straightforward.

PROPOSITION 5.4.   *Let $1 \leqslant p_0, p_1 \leqslant 2$ and assume that $\{E_0, E_1\}$ is compatible for complex interpolation. Then*

$$\mathcal{C}_{p_\theta}^1(E_\theta, G) \leqslant \mathcal{C}_{p_0}^1(E_0, G)^{1-\theta} \mathcal{C}_{p_1}^1(E_1, G)^\theta$$

for $p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$. A similar result holds for the Fourier cotype.

5.2. *Duality*

The following theorem can be rephrased by saying that Fourier type and cotype are dual notions.

THEOREM 5.5.   *Let E be an operator space, and let $1 \leqslant p \leqslant 2$ with p' its conjugate exponent. Then*
  (1) *E has Fourier type p with respect to a compact group G if and only if E^* has Fourier cotype p' with respect to G;*
  (2) *E has Fourier cotype p' with respect to a compact group G if and only if E^* has Fourier type p with respect to G.*
Moreover, we have $\mathcal{C}_p^1(E,G) = \mathcal{C}_{p'}^2(E^*,G)$ and $\mathcal{C}_p^1(E^*,G) = \mathcal{C}_{p'}^2(E,G)$.

Proof.   We just prove the equality $\mathcal{C}_p^1(E,G) = \mathcal{C}_{p'}^2(E^*,G)$ since the proof of the second identity is essentially the same. The case $p = 1$ follows from Propositions 3.5 and 4.3; thus we assume that $1 < p \leqslant 2$.

Step 1: $\mathcal{C}_p^1(E,G) \geqslant \mathcal{C}_{p'}^2(E^*,G)$. By a density argument and Property 1 of Theorem 1.2 we just need to check that the inequality

$$\left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(\cdot)) \right) \right\|_{S_n^{p'}(L_{E^*}^{p'}(G))} \leqslant \mathcal{C}_p^1(E,G) \|(A_{ij})\|_{S_n^{p'}(\mathcal{L}_{E^*}^p(\widehat{G}))}$$

holds for any family $A_{ij} \in \mathcal{L}^p(\widehat{G}) \otimes E^*$ (with $1 \leqslant i, j \leqslant n$) and all $n \geqslant 1$. But we have the completely isometric isomorphism $S_n^{p'}(L_{E^*}^{p'}(G)) \simeq L_{S_n^p(E)^*}^{p'}(G)$. So for all $\varepsilon > 0$ there exists $f^\varepsilon \in L_{S_n^p(E)}^p(G)$ of norm 1 such that

$$\left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(\cdot)) \right) \right\|_{S_n^{p'}(L_{E^*}^{p'}(G))}$$

$$\leqslant (1+\varepsilon) \left| \int_G \operatorname{tr} \left[\left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(g)) \right) (f_{ij}^\varepsilon(g)) \right] d\mu(g) \right|$$

and where f_{ij}^ε, the entries of f^ε, belong to $L^p(G) \otimes E$. If we denote by $\mathcal{I}$ the integral over G written above, then we would like to prove that

$$\mathcal{I} = \sum_{i,j=1}^n \sum_{\pi \in \widehat{G}} d_\pi \int_G \langle \operatorname{tr}(A_{ij}^\pi \pi(g)), f_{ji}^\varepsilon(g) \rangle \, d\mu(g).$$

Taking into account the facts that $A_{ij} \in \mathcal{L}^p(\widehat{G}) \otimes E^*$ and $f_{ij}^\varepsilon \in L^p(G) \otimes E$ it suffices to show that the expressions

$$\mathcal{I}_1 = \int_G \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^\pi \pi(g)) f(g) \, d\mu(g),$$

$$\mathcal{I}_2 = \sum_{\pi \in \widehat{G}} d_\pi \int_G \operatorname{tr}(A^\pi \pi(g)) f(g) \, d\mu(g)$$

coincide for all $A \in \mathcal{L}^p(\widehat{G})$ and all $f \in L^p(G)$. But this is an easy computation which we leave to the reader. In summary we obtain

$$\left\| \left(\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A_{ij}^\pi \pi(\cdot)) \right) \right\|_{S_n^{p'}(L_{E^*}^{p'}(G))}$$

$$\leqslant (1+\varepsilon) \left| \sum_{i,j=1}^n \sum_{\pi \in \widehat{G}} d_\pi \int_G \langle \operatorname{tr}(A_{ij}^\pi \pi(g)), f_{ji}^\varepsilon(g) \rangle \, d\mu(g) \right|$$

$$= (1+\varepsilon) \left| \sum_{i,j=1}^n \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(\langle A_{ij}^\pi, \tau(\widehat{f_{ji}^\varepsilon})(\pi) \rangle) \right|$$

$$= (1+\varepsilon) |\operatorname{tr}[(A_{ij})(\tau(\widehat{f_{ij}^\varepsilon}))]|$$

where $\tau(f)(g) = f(g^{-1})$. This step is concluded by the following inequality:

$$|\operatorname{tr}[(A_{ij})(\tau(\widehat{f_{ij}^\varepsilon}))]| \leqslant \|(A_{ij})\|_{S_n^{p'}(\mathcal{L}_{E^*}^p(\widehat{G}))} \|(\tau(\widehat{f_{ij}^\varepsilon}))\|_{S_n^p(\mathcal{L}_E^{p'}(\widehat{G}))}$$

$$\leqslant \mathcal{C}_p^1(E, G) \|(A_{ij})\|_{S_n^{p'}(\mathcal{L}_{E^*}^p(\widehat{G}))} \|f^\varepsilon\|_{S_n^p(L_E^p(G))}.$$

Step 2: $\mathcal{C}_p^1(E, G) \leqslant \mathcal{C}_{p'}^2(E^*, G)$. By the same reasons as those given in Step 1, it suffices to check that

$$\|(\widehat{f_{ij}})\|_{S_n^{p'}(\mathcal{L}_E^{p'}(\widehat{G}))} \leqslant \mathcal{C}_{p'}^2(E^*, G) \|(f_{ij})\|_{S_n^{p'}(L_E^p(G))}$$

for any family $f_{ij} \in L^p(G) \otimes E$ (with $1 \leqslant i, j \leqslant n$) and all $n \geqslant 1$. Given $\varepsilon > 0$, the complete isometry

$$S_n^{p'}(\mathcal{L}_E^{p'}(\widehat{G})) \simeq \mathcal{L}_{S_n^{p'}(E)}^{p'}(\widehat{G})$$

provides the existence of $\mathbf{A}^\varepsilon \in \mathcal{L}^p_{S^p_n(E^*)}(\widehat{G})$ of norm 1 such that

$$\|(\widehat{f}_{ij})\|_{S^{p'}_n(\mathcal{L}^{p'}_E(\widehat{G}))} \leqslant (1+\varepsilon)\left|\sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}[(A^{\varepsilon,\pi}_{ij})(\widehat{f}_{ij}(\pi))]\right|,$$

where A^ε_{ij}, the entries of $\mathbf{A}^\varepsilon$, belong to $\mathcal{L}^p(\widehat{G}) \otimes E^*$. If $\mathcal{S}$ denotes the sum written above, then we can argue as in Step 1 to obtain

$$\mathcal{S} = \sum_{i,j=1}^n \int_G \left\langle \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(A^{\varepsilon,\pi}_{ij}\pi(g)^*), f_{ji}(g) \right\rangle d\mu(g).$$

Therefore

$$\|(\widehat{f}_{ij})\|_{S^{p'}_n(\mathcal{L}^{p'}_E(\widehat{G}))} \leqslant (1+\varepsilon)\left|\sum_{i,j=1}^n \int_G \langle \mathcal{F}^{-1}_{G,E^*}(A^\varepsilon_{ij})(g), f_{ji}(g^{-1})\rangle \, d\mu(g)\right|$$

$$= (1+\varepsilon)|\operatorname{tr}[(\mathcal{F}^{-1}_{G,E^*}(A^\varepsilon_{ij}))(\tau(f_{ij}))]|$$

$$\leqslant (1+\varepsilon)\|\mathcal{F}^{-1}_{G,E^*}(\mathbf{A}^\varepsilon)\|_{S^p_n(L^p_E(G)^*)}\|(f_{ij})\|_{S^{p'}_n(L^p_E(G))}$$

$$= (1+\varepsilon)\|\mathcal{F}^{-1}_{G,E^*}(\mathbf{A}^\varepsilon)\|_{S^p_n(L^{p'}_{E^*}(G))}\|(f_{ij})\|_{S^{p'}_n(L^p_E(G))}$$

$$\leqslant (1+\varepsilon)\mathcal{C}^2_{p'}(E^*,G)\|(f_{ij})\|_{S^{p'}_n(L^p_E(G))}.$$

The proof is completed by taking ε arbitrarily small. $\qquad\square$

REMARK 5.6. There exists another possible approach to this result. Namely, E has Fourier type p if and only if the corresponding Fourier transform operator is completely bounded. But then, by [**8**, Proposition 3.2.2], the adjoint operator is also completely bounded with the same cb-norm. Moreover, it can be checked that the adjoint coincides with the inverse of the Fourier transform for functions taking values in E^*. This gives the first equality of Theorem 5.5. The second equality follows in a similar fashion.

COROLLARY 5.7. *We have*
$$\mathcal{C}^1_p(E,G) = \mathcal{C}^1_p(E^{**},G) \quad \text{and} \quad \mathcal{C}^2_{p'}(E,G) = \mathcal{C}^2_{p'}(E^{**},G).$$

5.3. The cb-distance

There exists a natural analog in the category of operator spaces of the Banach–Mazur distance, due to Pisier. It is called the cb-distance and it is defined by

$$d_{\mathrm{cb}}(E_1,E_2) = \inf\{\|u\|_{\mathrm{cb}(E_1,E_2)}\|u^{-1}\|_{\mathrm{cb}(E_2,E_1)}\}$$

where the infimum runs over all complete isomorphisms $u : E_1 \to E_2$.

THEOREM 5.8. *Let E_1 and E_2 be operator spaces and let G be a compact group. Then the following inequalities hold for $1 \leqslant p \leqslant 2$:*

$$\mathcal{C}^1_p(E_2,G) \leqslant d_{\mathrm{cb}}(E_1,E_2)\mathcal{C}^1_p(E_1,G),$$

$$\mathcal{C}^2_{p'}(E_2,G) \leqslant d_{\mathrm{cb}}(E_1,E_2)\mathcal{C}^2_{p'}(E_1,G),$$

$$\mathcal{C}^1_p(E_2,G) \leqslant d_{\mathrm{cb}}(E_1,E_2^*)\mathcal{C}^2_{p'}(E_1,G),$$

$$\mathcal{C}^2_{p'}(E_2,G) \leqslant d_{\mathrm{cb}}(E_1,E_2^*)\mathcal{C}^1_p(E_1,G).$$

Proof. The last two inequalities follow from the first two plus duality. Let us assume that the first inequality holds; then the second inequality is also an immediate consequence of Theorem 5.5:

$$\mathcal{C}_{p'}^2(E_2, G) = \mathcal{C}_p^1(E_2^*, G) \leqslant d_{\mathrm{cb}}(E_1^*, E_2^*)\mathcal{C}_p^1(E_1^*, G)$$
$$= d_{\mathrm{cb}}(E_1^*, E_2^*)\mathcal{C}_{p'}^2(E_1, G) = d_{\mathrm{cb}}(E_1, E_2)\mathcal{C}_{p'}^2(E_1, G)$$

where we have applied the identity $\|u\|_{\mathrm{cb}(E_1, E_2)} = \|u^*\|_{\mathrm{cb}(E_2^*, E_1^*)}$ to justify the equality $d_{\mathrm{cb}}(E_1^*, E_2^*) = d_{\mathrm{cb}}(E_1, E_2)$; see [**8**] for details. Therefore we will have finished if we prove the validity of the first inequality. For that it suffices to see that

$$\|(\widehat{f}_{ij})\|_{S_n^{p'}(\mathcal{L}_{E_2}^{p'}(\widehat{G}))} \leqslant \|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}}\mathcal{C}_p^1(E_1, G)\|(f_{ij})\|_{S_n^{p'}(L_{E_2}^p(G))}$$

for any family $f_{ij} \in L^p(G) \otimes E_2$ (with $1 \leqslant i, j \leqslant n$), any complete isomorphism $u : E_1 \to E_2$ and all $n \geqslant 1$. But

$$\|(\widehat{f}_{ij})\|_{S_n^{p'}(\mathcal{L}_{E_2}^{p'}(\widehat{G}))} = \left(\sum_{\pi \in \widehat{G}} d_\pi \|(\widehat{f}_{ij}(\pi))\|_{S_{d_\pi n}^{p'}(E_2)}^{p'}\right)^{1/p'}$$

$$\leqslant \|u\|_{\mathrm{cb}}\left(\sum_{\pi \in \widehat{G}} d_\pi \|(I_{M_{d_\pi n}} \otimes u^{-1})(\widehat{f}_{ij}(\pi))\|_{S_{d_\pi n}^{p'}(E_1)}^{p'}\right)^{1/p'}$$

$$= \|u\|_{\mathrm{cb}}\|(\mathcal{F}_{G, E_1}((I_{L^p(G)} \otimes u^{-1})(f_{ij})))\|_{\mathcal{L}_{S_n^{p'}(E_1)}^{p'}(\widehat{G})}$$

$$\leqslant \|u\|_{\mathrm{cb}}\mathcal{C}_p^1(E_1, G)\|((I_{L^p(G)} \otimes u^{-1})(f_{ij}))\|_{S_n^{p'}(L_{E_1}^p(G))}$$

$$\leqslant \|u\|_{\mathrm{cb}}\mathcal{C}_p^1(E_1, G)\|I_{L^p(G)} \otimes u^{-1}\|_{\mathrm{cb}}\|(f_{ij})\|_{S_n^{p'}(L_{E_2}^p(G))}$$

$$= \|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}}\mathcal{C}_p^1(E_1, G)\|(f_{ij})\|_{S_n^{p'}(L_{E_2}^p(G))}.$$

This completes the proof.   □

We recall here that, if $\mathcal{OS}_n$ denotes the class of all n-dimensional operator spaces, Pisier proved the estimate $d_{\mathrm{cb}}(E, \mathrm{OH}_n) \leqslant \sqrt{n}$ for any operator space $E \in \mathcal{OS}_n$. Here OH_n denotes the n-dimensional operator Hilbert space OH; see [**20**]. Therefore, by taking $E_1 = l^2(n)$ in Theorem 5.8 and invoking the results of the next section, we get the following result.

COROLLARY 5.9.   *We have* $\mathcal{C}_2^1(E, G), \mathcal{C}_2^2(E, G) \leqslant \sqrt{n}$ *for any* $E \in \mathcal{OS}_n$.

6. *Basic examples*

We study here the Fourier type and cotype of Lebesgue spaces, Schatten classes and their vector-valued versions. We start by stating some inequalities of Minkowski type in the operator space setting.

If $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and the measure spaces $(\Omega_1, \mathcal{M}_1, \nu_1)$ and $(\Omega_2, \mathcal{M}_2, \nu_2)$ are σ-finite, then the classical Minkowski inequality for integrals asserts that the natural map

$$L_{L^{p_2}(\Omega_2)}^{p_1}(\Omega_1) \longrightarrow L_{L^{p_1}(\Omega_1)}^{p_2}(\Omega_2)$$

is contractive. The same happens if our functions $f : \Omega_1 \times \Omega_2 \to E$ take values in a Banach space E. We are interested in the complete boundedness of this operator

and some others in which we replace the Lebesgue spaces $L^p(\Omega)$ by the Schatten classes S_n^p. For this purpose, by complex interpolation, it suffices to check the cases $p_1 = p_2$ and $(p_1, p_2) = (1, \infty)$. The first case follows from the Fubini type results stated in Theorem 1.2. The second case reduces to seeing that the natural map

$$E_1 \otimes^\wedge (E_2 \otimes_{\min} E_3) \longrightarrow (E_1 \otimes^\wedge E_2) \otimes_{\min} E_3$$

is a complete contraction. The proof of this result can be found in [**8**, Theorem 8.1.10]. In summary we can state the following results.

THEOREM 6.1 (Quantized Minkowski inequalities). *Let us consider an operator space E and let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$.*

1. Lebesgue spaces. *Let $(\Omega_1, \mathcal{M}_1, \nu_1)$ and $(\Omega_2, \mathcal{M}_2, \nu_2)$ be σ-finite measure spaces. Then the following natural map is a complete contraction:*

$$L^{p_1}_{L^{p_2}_E(\Omega_2)}(\Omega_1) \longrightarrow L^{p_2}_{L^{p_1}_E(\Omega_1)}(\Omega_2).$$

2. Schatten classes. *Let $k_1, k_2 \geqslant 1$. Then the following natural map is a complete contraction:*

$$S^{p_1}_{k_1}(S^{p_2}_{k_2}(E)) \longrightarrow S^{p_2}_{k_2}(S^{p_1}_{k_1}(E)).$$

3. Combined results. *Let $(\Omega, \mathcal{M}, \nu)$ be a measure space and $k \geqslant 1$. Then the following natural maps are complete contractions:*

$$S^{p_1}_k(L^{p_2}_E(\Omega)) \longrightarrow L^{p_2}_{S^{p_1}_k(E)}(\Omega) \quad \text{and} \quad L^{p_1}_{S^{p_2}_k(E)}(\Omega) \longrightarrow S^{p_2}_k(L^{p_1}_E(\Omega)).$$

REMARK 6.2. The arguments sketched above in order to prove Theorem 6.1 need extra hypotheses. A different proof, without those unnecessary hypotheses, can be found in the thesis [**18**] of the second-named author.

In the study of the Fourier type of a Banach space with respect to a locally compact abelian group, Andersson [**1**] gave the following version of Minkowski inequality for regular measures. We recall that our notion of regular measure is the same as the one given in [**5**].

PROPOSITION 6.3 (Andersson). *Let $1 \leqslant p_1 \leqslant p_2 < \infty$ and assume that $(\Omega_1, \mathcal{M}_1, \nu_1)$ and $(\Omega_2, \mathcal{M}_2, \nu_2)$ are regular measure spaces. Let us denote by H the space of functions $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ such that $|f|$ is bounded lower semicontinuous and $\|f_{\omega_2}\|_{L^{p_1}(\Omega_1)}$ is bounded in Ω_2. Then the following natural map is contractive:*

$$L^{p_1}_{L^{p_2}(\Omega_2)}(\Omega_1) \cap H \longrightarrow L^{p_2}_{L^{p_1}(\Omega_1)}(\Omega_2) \cap H.$$

Let us note that if $(\Omega, \mathcal{M}, \nu)$ denotes a regular measure space and we take $\Omega_1 = G$ and $\Omega_2 = \Omega$ in Proposition 6.3 then the space $C_c(G \times \Omega)$ of continuous functions with compact support, defined on $G \times \Omega$ and with values in $\mathbb{C}$, is contained in the space H. Hence, by the density of $C_c(G \times \Omega)$ in $L^{p_1}_{L^{p_2}(\Omega)}(G)$ and $L^{p_2}_{L^{p_1}(G)}(\Omega)$, we deduce that the natural map

$$L^{p_1}_{L^{p_2}(\Omega)}(G) \longrightarrow L^{p_2}_{L^{p_1}(G)}(\Omega)$$

is a contraction whenever $1 \leqslant p_1 \leqslant p_2 < \infty$. Then, by the same arguments as we gave in the proof of Theorem 6.1, we conclude that we have in fact a complete contraction. Furthermore, the same happens if we take $\Omega_1 = \Omega$ and $\Omega_2 = G$.

Therefore we have shown the validity of the following result, which we enunciate for vector-valued functions since its proof is analogous.

LEMMA 6.4. *Let* $1 \leqslant p_1 \leqslant p_2 < \infty$ *and assume that* E *is an operator space,* G *is a compact group, and* $(\Omega, \mathcal{M}, \nu)$ *is a regular measure space. Then the following natural maps are complete contractions:*

$$L^{p_1}_{L^{p_2}_E(\Omega)}(G) \longrightarrow L^{p_2}_{L^{p_1}_E(G)}(\Omega) \quad \text{and} \quad L^{p_1}_{L^{p_2}_E(G)}(\Omega) \longrightarrow L^{p_2}_{L^{p_1}_E(\Omega)}(G).$$

THEOREM 6.5. *Let* $1 \leqslant p, q \leqslant 2$ *and assume that* E *is an operator space having Fourier type* p *and Fourier cotype* q' *with respect to a compact group* G. *Let* $(\Omega, \mathcal{M}, \nu)$ *be a regular or* σ-*finite measure space. Then*
  (1) $L^r_E(\Omega)$ *has Fourier type* p *with respect to* G *for all* $p \leqslant r \leqslant p'$,
  (2) $L^s_E(\Omega)$ *has Fourier cotype* q' *with respect to* G *for all* $q \leqslant s \leqslant q'$.
Moreover, $\mathcal{C}^1_p(L^r_E(\Omega), G) = \mathcal{C}^1_p(E, G)$ *and* $\mathcal{C}^2_{q'}(L^s_E(\Omega), G) = \mathcal{C}^2_{q'}(E, G)$.

Proof. We start by proving the relation $\mathcal{C}^1_p(L^r_E(\Omega), G) = \mathcal{C}^1_p(E, G)$. For $p = 1$ we just need to apply Proposition 3.5. Thus we assume that $1 < p \leqslant 2$. The inequality $\mathcal{C}^1_p(L^r_E(\Omega), G) \geqslant \mathcal{C}^1_p(E, G)$ follows from Proposition 5.2 and, by complex interpolation, it then suffices to see that $\mathcal{C}^1_p(L^r_E(\Omega), G) \leqslant \mathcal{C}^1_p(E, G)$ for $r = p$ and $r = p'$. For $r = p$ we observe that the natural map

$$L^p_{\mathcal{L}^{p'}_E(\widehat{G})}(\Omega) \longrightarrow \mathcal{L}^{p'}_{L^p_E(\Omega)}(\widehat{G})$$

is a complete contraction. The proof of this fact is similar to that of Theorem 6.1. In particular, we have

$$\|(\widehat{f}_{ij})\|_{S^p_n(\mathcal{L}^{p'}_{L^p_E(\Omega)}(\widehat{G}))} \leqslant \|(\widehat{f}_{ij})\|_{L^p_{S^p_n(\mathcal{L}^{p'}_E(\widehat{G}))}(\Omega)} \leqslant \mathcal{C}^1_p(E, G)\|(f_{ij})\|_{S^p_n(L^p_{L^p_E(\Omega)}(G))}.$$

For $r = p'$ we use Theorem 6.1 or Lemma 6.4, depending on the measure space $(\Omega, \mathcal{M}, \nu)$, to get the desired relation:

$$\|(\widehat{f}_{ij})\|_{S^{p'}_n(\mathcal{L}^{p'}_{L^{p'}_E(\Omega)}(\widehat{G}))} \leqslant \mathcal{C}^1_p(E, G)\|(f_{ij})\|_{L^{p'}_{S^{p'}_n(L^p_E(G))}(\Omega)}$$

$$\leqslant \mathcal{C}^1_p(E, G)\|(f_{ij})\|_{S^{p'}_n(L^{p'}_{L^p_E(\Omega)}(G))}.$$

The proof of the inequality $\mathcal{C}^2_{q'}(L^s_E(\Omega), G) \leqslant \mathcal{C}^2_{q'}(E, G)$ is analogous. $\square$

REMARK 6.6. The proof of Theorem 6.5 for scalar-valued Lebesgue spaces is much simpler. Namely, one only has to see that $L^2(\Omega)$ has Fourier type 2 and then the result follows by duality and complex interpolation with the trivial cases $p = 1$ and $p = \infty$. But the case $p = 2$ is a simple consequence of Plancherel's theorem on compact groups.

It is well known that the dual of $L^p_E(\Omega)$ is not in general $L^{p'}_{E^*}(\Omega)$. However it is so when the dual E^* possesses the Radon–Nikodym property RNP. In [**21**], Pisier developed an operator space version of the Radon–Nikodym property which he called ORNP. The following corollary, which is a very simple consequence of Theorems 5.5 and 6.5, shows that both spaces have the same Fourier type and cotype even if E^* does not satisfy the ORNP.

COROLLARY 6.7. *Let* $1 \leqslant p, q \leqslant 2$ *and assume that* E *is an operator space having Fourier type* p *and Fourier cotype* q' *with respect to a compact group* G. *Let* $(\Omega, \mathcal{M}, \nu)$ *be a regular or* σ-*finite measure space. Then we have*
 (1) $\mathcal{C}_q^1(L_E^s(\Omega)^*, G) = \mathcal{C}_q^1(L_{E^*}^{s'}(\Omega), G)$ *for all* $q \leqslant s \leqslant q'$,
 (2) $\mathcal{C}_{p'}^2(L_E^r(\Omega)^*, G) = \mathcal{C}_{p'}^2(L_{E^*}^{r'}(\Omega), G)$ *for all* $p \leqslant r \leqslant p'$.

We now study the Fourier type and cotype of Schatten classes. We will denote by S^p the infinite-dimensional Schatten class of exponent p. The definition and properties of the vector-valued version of S^p are similar to the finite-dimensional case; see [**21**, Chapter 1]. We omit the proof of the following result since the arguments needed can be found in the proof of Theorem 6.5.

THEOREM 6.8. *Let* $1 \leqslant p, q \leqslant 2$ *and assume that* E *is an operator space having Fourier type* p *and Fourier cotype* q' *with respect to a compact group* G. *Then*
 (1) $S^r(E)$ *has Fourier type* p *with respect to* G *for all* $p \leqslant r \leqslant p'$,
 (2) $S^s(E)$ *has Fourier cotype* q' *with respect to* G *for all* $q \leqslant s \leqslant q'$.
Moreover, $\mathcal{C}_p^1(S^r(E), G) = \mathcal{C}_p^1(E, G)$ *and* $\mathcal{C}_{q'}^2(S^s(E), G) = \mathcal{C}_{q'}^2(E, G)$.

REMARK 6.9. We already know that the Fourier type and cotype become stronger conditions on the pair (E, G) as p and p' approach 2. This gives rise to the notions of sharp Fourier type and cotype exponents. The problem of finding the sharp exponents of a given operator space is highly non-trivial even for the simplest case of Lebesgue spaces or Schatten classes. Part of this problem is solved in [**12**]. Namely, if $1 \leqslant p \leqslant 2$ and $(\Omega, \mathcal{M}, \nu)$ is not the union of finitely many ν-atoms, then we show that $L^p(\Omega)$ has sharp Fourier type p with respect to any compact semisimple Lie group. By duality we also see that $L^{p'}(\Omega)$ has sharp Fourier cotype p' for those groups. By the nature of Ω and Proposition 5.2, we have

$$\mathcal{C}_q^1(L^p(\Omega), G) \geqslant \lim_{n \to \infty} \mathcal{C}_q^1(l^p(n), G)$$

for $1 \leqslant p < q \leqslant 2$. Moreover, Theorem 5.8 gives $\mathcal{C}_q^1(l^p(n), G) \leqslant n^{1/p - 1/q}$. The main result of [**12**] asserts that there exists a positive constant $\mathcal{K}(G, q)$, such that

$$\mathcal{K}(G, q) n^{1/p - 1/q} \leqslant \mathcal{C}_q^1(l^p(n), G) \leqslant n^{1/p - 1/q}$$

for all $n \geqslant 1$ and any compact semisimple G. The constant $\mathcal{K}(G, q)$ can be defined as

$$\mathcal{K}(G, q) = \inf_{n \geqslant 1} \sup \left\{ \frac{\|\hat{f}\|_{\mathcal{L}^{q'}(\hat{G})}}{\|f\|_{L^q(G)}} : f \text{ central}, \ f \in L^q(G), \ \text{supp}(f) \subset \mathcal{U}_n \right\}$$

where $\{\mathcal{U}_n : n \geqslant 1\}$ denotes a neighborhood basis at the identity of G. The interesting point lies in the inequality $\mathcal{K}(G, q) > 0$ which constitutes a local variant of the Hausdorff–Young inequality on G with parameter q. The proof obtained for this local inequality is based upon the semisimplicity of G since it uses the very well-developed theory of representations on such groups. The need to use these algebraic techniques forced us to present the proof of this result in a separate work; see [**12**].

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