Wavelet characterization of weighted spaces

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Abstract

We give a characterization of weighted Hardy spaces $H^p(w)$, valid for a rather large collection of wavelets, $0 < p \leq 1$, and weights w in the Muckenhoupt class A_∞. We improve the previously known results and adopt a systematic point of view based upon the theory of vector-valued Calderón-Zygmund operators. Some consequences of this characterization are also given, like the criterion for a wavelet to give an unconditional basis and a criterion for membership into the space from the size of the wavelet coefficients.

1 Introduction.

Although many of the results that we obtain in this paper can be proved in higher dimensions, we shall always work in $\mathbb{R}$ for simplicity. Our weights will belong to the class A_∞, which is the union of the classes A_q, $1 \leq q < \infty$. A weight $w \geq 0$ belongs to the class A_q for a given q, $1 < q < \infty$, if

$$(A_q) \qquad \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{q-1}}\,dx\right)^{q-1} \leq C,$$

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with C finite independent of the interval I. The class A_1 is defined by letting $q \longrightarrow 1$, that is,

$$(A_1) \qquad \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\|w^{-1}\|_{L^\infty(I)} \le C,$$

with C finite independent of the interval I.

These classes were introduced by Muckenhoupt in [Muc] and their theory was further developed in [CF] (see also [GR]). We shall always assume $w \in A_\infty$ and define

$$q_w = \inf\{q > 1 : w \in A_q\},$$

the *critical index* of w.

We shall be concerned with the weighted Hardy spaces $H^p(w)$, $0 < p \le 1$, which we define by means of their atomic characterization as follows.

Definition 1.1 *Given a weight w in A_∞, $0 < p \le 1$ and $1 < q \le \infty$, a (p,q)-atom with respect to w will be a function a satisfying the following three conditions:*

i) $\operatorname{supp} a \subset I$*, where I is a bounded interval in $\mathbb{R}$.*

ii)

$$\|a\|_{L^q(w)} \le \begin{cases} w(I)^{\frac{1}{q}-\frac{1}{p}} & \text{if} \quad q < \infty, \\ w(I)^{-\frac{1}{p}} & \text{if} \quad q = \infty, \end{cases} \qquad \text{where } w(I) = \int_I w(x)\,dx.$$

iii) $\displaystyle\int_{\mathbb{R}} x^k\,a(x)\,dx = 0, \quad 0 \le k \le N_p(w) = \left[\frac{q_w}{p}\right] - 1.$

We will simply speak of p-atoms when $q = \infty$. These p-atoms are the basic building blocks of the Hardy spaces.

Definition 1.2 *Let $w \in A_\infty$ be a weight and let $0 < p \le 1$. A tempered distribution $f \in \mathcal{S}'$ belongs to $H^p(w)$ if and only if f can be written as a series*

(1) $$f = \sum_j \lambda_j\,a_j \quad \text{converging in } \mathcal{S}',$$

where each a_j is a p-atom with respect to w and

$$\sum_j |\lambda_j|^p < \infty. \tag{2}$$

Moreover, by setting $\|f\|^p_{H^p(w)}$ to be the infimum of the sums (2) *over all decompositions* (1)*, one obtains the p-norm for this space.*

Remark 1.3 *These spaces can be defined in terms of (p, q)-atoms obtaining equivalent p-norms* (*see* [Gar] *and* [ST]).

Given φ, we define the square function

$$\mathcal{G}(f)(x) = \Big\{ \sum_{j\in\mathbb{Z}} |\varphi_{2^{-j}} * f(x)|^2 \Big\}^{\frac{1}{2}},$$

where $\varphi_{2^{-j}}(x) = 2^j\,\varphi(2^j\,x)$. We are interested in the boundedness of this operator between $H^p(w)$ and $L^p(w)$, $0 < p \leq 1$. By using the ideas of [BCP], [RRT] and also [GR], this square function fits into the theory of vector-valued Calderón-Zygmund operators. In Section 2 we shall put into a vectorial context the results about Calderón-Zygmund operators that appear in [GK1]. Then, we apply these results to the square function $\mathcal{G}$ to conclude the boundedness between these spaces (Section 3).

An orthonormal wavelet shall be a function $\psi \in L^2(\mathbb{R})$ such that the system

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\,\psi(2^j\,x - k), \quad j, k \in \mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$. Next, we define the operator

$$\begin{aligned} \mathcal{W}_\psi f &= \Big\{ \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\, 2^j\, \chi_{[2^{-j}\,k, 2^{-j}\,(k+1)]} \Big\}^{\frac{1}{2}} \\ &= \Big\{ \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\, |I_{j,k}|^{-1}\, \chi_{I_{j,k}} \Big\}^{\frac{1}{2}}, \end{aligned}$$

where $I_{j,k} = [2^{-j}\,k, 2^{-j}\,(k+1)]$. Denoting by $\mathcal{D}$ the set of all dyadic intervals $I_{j,k}$, with $j, k \in \mathbb{Z}$, and letting $\psi_{I_{j,k}} = \psi_{j,k}$, we can also write

$$\mathcal{W}_\psi f = \Big\{ \sum_{I\in\mathcal{D}} |\langle f, \psi_I\rangle|^2\, |I|^{-1}\, \chi_I \Big\}^{\frac{1}{2}}.$$

We devote Section 4 to prove that, for $0 < p \leq 1$, $w \in A_\infty$ and ψ an orthonormal wavelet in certain regularity class —depending on the value of p and the critical index of w—, the following inequalities hold

$$c\,\|f\|^p_{H^p(w)} \leq \|\mathcal{W}_\psi f\|^p_{L^p(w)} \leq C\,\|f\|^p_{H^p(w)},$$

for all $f \in H^p(w)$.

When the wavelet has compact support and comes from an MRA, the characterization was obtained in [Wu] for A_∞ weights. In that paper, the author deals with a maximal operator, related with the scale function of the MRA, and uses good-λ inequalities to compare it with $\mathcal{W}_\psi$, what leads to the desired characterization. In this work, we not only improve the result of Wu, extending the class of the wavelets for which the characterization holds, but also the result is obtained by using a different approach, based on vector-valued analysis. These ideas come from [HW], where a similar characterization of the space $H^1(\mathbb{R})$ is given.

Finally, in Section 5, some consequences of the characterization are given. We study when the wavelets give unconditional bases for the weighted Hardy spaces. The history of this topic goes as follows. B. Maurey [Mau] proved that H^1 has an unconditional basis and L. Carleson [Car] gave an explicit unconditional basis. The fact that the Franklin system is an unconditional basis for this space was proved by P. Wojtaszczyk [Woj]. These constructions can be found in [Mey]. The spline bases in the unweighted Hardy spaces are treated in [Str], [CC] and [SS], after the previous works [Ci1], [Ci2], [Boc]. While [GK1] and [GK1] handle the weighted case.

Besides, we show how the operator $\mathcal{W}_\psi$ provides a criterion for membership into these spaces from the size of the wavelet coefficients.

2 Vector-valued operators on weighted Hardy spaces.

[BCP] and [RRT] are the main sources of the vector-valued Calderón-Zygmund theory. Another reference is [GR]. During this section we want to put the results of [GK1], about the boundedness of the Calderón-Zygmund operators in weighted Hardy spaces, into the framework of the vector-valued theory.

A, B will be Banach spaces and $\mathcal{L}(A,B)$ will denote the set of bounded linear operators from A to B. As in the scalar context, we can also define weighted Hardy spaces of Banach-space-valued distributions. In this case we shall only use p-atoms.

Definition 2.1 *Given a weight w in A_∞ and $0 < p \le 1$, an A-valued p-atom with respect to w will be a measurable function*

$$a : \mathbb{R} \longrightarrow A,$$

that verifies:

i) $\operatorname{supp} a \subset I$, *where I is a bounded interval in $\mathbb{R}$.*

ii) $\|a\|_{L^\infty_A} \le w(I)^{-\frac{1}{p}}$.

iii) $\displaystyle\int_{\mathbb{R}} x^k\, a(x)\, dx = 0, \quad 0 \le k \le N_p(w) = \left[\frac{q_w}{p}\right] - 1.$

Definition 2.2 *Let $w \in A_\infty$ be a weight and let $0 < p \le 1$. We shall say that $f \in \mathcal{L}(\mathcal{S}, A)$ belongs to $H^p_A(w)$ if and only if f can be written as a series*

$$f = \sum_j \lambda_j\, a_j \quad \textit{converging in } \mathcal{L}(\mathcal{S}, A), \tag{3}$$

where each a_j is an A-valued p-atom with respect to w and

$$\sum_j |\lambda_j|^p < \infty. \tag{4}$$

We shall denote by $\|f\|^p_{H^p_A(w)}$ the infimum of the sums (4) *over all decompositions* (3)*. This infimum will be the p-norm for this space.*

Definition 2.3 *Let T be a linear operator mapping every function in L^∞_A with compact support to a measurable B-valued function and let*

$$K : \mathbb{R} \setminus \{0\} \longrightarrow \mathcal{L}(A,B)$$

be measurable and locally integrable outside the origin. Suppose that the following two conditions hold:

i) For each $f \in L^\infty_A$ *with compact support, and for almost every* $x \notin \operatorname{supp} f$

$$Tf(x) = \int_{\mathbb{R}} K(x-y) f(y)\, dy.$$

ii) T *extends to a bounded operator from* L^r_A *to* L^r_B*, for some* $1 < r \leq \infty$.

Then, we shall say that T *is a* **vector-valued Calderón-Zygmund operator (V.V.C-Z.O)** *with kernel* K.

Definition 2.4 *Let* K *be the kernel of a V.V.C-Z.O Given* $\gamma \in \mathbb{R}$, $\gamma > 0$, *we shall say that* K *is a* γ*-regular vector-valued kernel if the following two conditions are satisfied:*

$$\|K(x)\|_{\mathcal{L}(A,B)} \leq \frac{C}{|x|}, \quad \forall\, x \neq 0 \tag{5}$$

and for every j, $0 \leq j < \gamma$, *and every* $x \neq 0$, *there exist* $a_m(x) \in \mathcal{L}(A,B)$, $0 \leq m \leq j$, *such that*

$$P_x(y) = \sum_{m=0}^{j} a_m(x)\, y^m$$

verifies

$$\|K(x-y) - P_x(y)\|_{\mathcal{L}(A,B)} \leq C\, \frac{|y|^{\min\{\gamma, j+1\}}}{|x|^{1+\min\{\gamma, j+1\}}}, \quad |x| > 2\,|y|. \tag{6}$$

Remark 2.5 *In the above definition the "coefficient"* $a_0(x)$ *must be* $K(x)$. *So, the condition* (6) *when* $j = 0$ *will be*

$$\|K(x-y) - K(x)\|_{\mathcal{L}(A,B)} \leq C\, \frac{|y|^{\min\{\gamma,1\}}}{|x|^{1+\min\{\gamma,1\}}}, \quad |x| > 2\,|y|.$$

Thus, if T *is a V.V.C-Z.O with kernel* K γ*-regular, for some* $\gamma > 0$, *we will have that* T *is bounded from* $L^q_A(w)$ *to* $L^q_B(w)$, *for* $1 < q < \infty$ *and* $w \in A_q$ *(see* [RRT] *or* [GR]*).*

Remark 2.6 *If* $0 < \gamma < \gamma'$ *and* K *is* γ'*-regular, then* K *is also* γ*-regular.*

We shall denote by $[\gamma]^-$ the unique integer such that $\gamma - 1 \leq [\gamma]^- < \gamma$. Actually, $[\gamma]^-$ coincides with the integer part function $[\gamma]$ except at the integers, where we have modified it to make it continuous from the left.

We want to study when a V.V.C-Z.O with regular kernel is bounded from $H^p_A(w)$ to $L^p_B(w)$. The polynomials appearing in the definition of γ-regularity will play the role of Taylor's polynomials. This fact is the main tool in the next theorem.

Theorem 2.7 *Let T be a V.V.C-Z.O with kernel K γ-regular. Let f be an L^∞_A function supported in the bounded interval I with centre x_0 such that*

$$\int_{\mathbb{R}} x^k f(x)\, dx = 0 \quad 0 \leq k < \gamma.$$

Then, for almost every $x \notin I^2$, the 2-dilate of I with the same centre, we have that

$$\|Tf(x)\|_B \leq C \left(\frac{|I|}{|x - x_0|} \right)^{1+\gamma} \|f\|_{L^\infty_A}.$$

Proof. For almost every $x \notin I^2$, we have

$$Tf(x) = \int_I K(x-y) f(y)\, dy = \int_I (K(x-y) - P_{x-x_0}(y - x_0)) f(y)\, dy,$$

where P_{x-x_0} is the polynomial in Definition 2.4 with $j = [\gamma]^-$, which has degree less or equal than $[\gamma]^-$ and so the integral of P_{x-x_0} times f is 0. By using (6) we get

$$\|Tf(x)\|_B \leq C \left(\frac{|I|}{|x - x_0|} \right)^{1+\gamma} \|f\|_{L^\infty_A}.$$

□

Corollary 2.8 *Let T be a V.V.C-Z.O with kernel K γ-regular. Let $f \in L^\infty_A$ supported in the interval I with centre x_0 and with moments vanishing up to order N. Then, for almost every $x \notin I^2$,*

$$\|Tf(x)\|_B \leq C \left(\frac{|I|}{|x - x_0|} \right)^{1+\min\{N+1,\gamma\}} \|f\|_{L^\infty_A}.$$

Proof. When $\min\{\gamma, N+1\} = \gamma$, f has moments vanishing up to order strictly less than γ and the above theorem provides the result. In the other case, $\min\{\gamma, N+1\} = N+1$, K is $N+1$-regular and the conclusion of the previous theorem holds with $N+1$ in place of γ. □

Proposition 2.9 *Let T be a V.V.C-Z.O with kernel K γ-regular. Let w be a weight in A_∞ and f an A-valued p-atom with respect to w supported in the interval I. Suppose that $0 < p \leq 1$ is such that $\frac{q_w}{p} < 1 + \gamma$. Then*

$$\int_{\mathbb{R}\setminus I^2} \|Tf(x)\|_B^p \, w(x)\, dx \leq C,$$

with C independent of f.

Proof. Let x_0 be the centre of the interval I and $N = N_p(w) = \left[\frac{q_w}{p}\right] - 1$. By hypothesis, we know that $\frac{q_w}{p} < 1 + \gamma$. So, $1 + \min\{N+1, \gamma\} > \frac{q_w}{p}$. By using the previous corollary, we have

$$\|Tf(x)\|_B \leq C \left(\frac{|I|}{|x - x_0|} \right)^{1+\min\{N+1,\gamma\}} \|f\|_{L_A^\infty}, \quad \text{for } a.e. x \notin I^2.$$

Setting $q = p\,(1 + \min\{N+1, \gamma\}) > q_w$, we obtain

$$\begin{aligned} \int_{\mathbb{R}\setminus I^2} \|Tf(x)\|_B^p \, w(x)\, dx &\leq C\, |I|^q \, \|f\|_{L_A^\infty}^p \int_{\mathbb{R}\setminus I^2} \frac{1}{|x - x_0|^q} \, w(x)\, dx \\ &\leq C\, w(I)\, \|f\|_{L_A^\infty}^p, \end{aligned}$$

where the last inequality holds because $w \in A_q$. Since f is an A-valued p-atom, we obtain the desired inequality. □

Theorem 2.10 *Let T be a V.V.C-Z.O with kernel K γ-regular. If $0 < p \leq 1$ and w is a weight in the class A_∞, with critical index q_w verifying $\frac{q_w}{p} < 1 + \gamma$, then T is bounded from $H_A^p(w)$ to $L_B^p(w)$.*

Proof. Let f be an A-valued p-atom with respect to w. It is enough to prove that we have

$$\int_{\mathbb{R}} \|Tf(x)\|_B^p \, w(x)\, dx \leq C,$$

with C finite independent of f. If f is supported in the interval I, by using Proposition 2.9, we know that the last inequality holds when we integrate outside of I^2. So we need a similar estimate over I^2. Letting $q > q_w \geq 1$, $w \in A_q$ and T is bounded from $L^q_A(w)$ to $L^q_B(w)$ (see Remark 2.5). Since $\frac{q}{p} \geq q > 1$, we can use Hölder's inequality to get

$$\begin{aligned}
\int_{I^2} \|Tf(x)\|_B^p \, w(x)\, dx &= \int_{I^2} \|Tf(x)\|_B^p \, w(x)^{\frac{p}{q}} \, w(x)^{1-\frac{p}{q}} \, dx \\
&\leq \left\{ \int_{\mathbb{R}} \|Tf(x)\|_B^q \, w(x)\, dx \right\}^{\frac{p}{q}} w(I^2)^{1-\frac{p}{q}} \\
&\leq C \left\{ \int_I \|f(x)\|_A^q \, w(x)\, dx \right\}^{\frac{p}{q}} w(I)^{1-\frac{p}{q}} \\
&\leq C \, \|f\|^p_{L^\infty_A} \, w(I)^{\frac{p}{q}} \, w(I)^{1-\frac{p}{q}} \leq C.
\end{aligned}$$

□

3 Application: The square function.

We consider the square function

$$\mathcal{G}(f)(x) = \Big\{ \sum_{j\in\mathbb{Z}} |\varphi_{2^{-j}} * f(x)|^2 \Big\}^{\frac{1}{2}},$$

where $\varphi_{2^{-j}}(x) = 2^j \, \varphi(2^j \, x)$. The vectorial point of view will allow us to apply the results of the previous section in order to establish the boundedness of $\mathcal{G}$ in weighted Hardy spaces. As in [GR] —pages 505-506— and in [RRT], the $\ell^2(\mathbb{Z})$-valued operator associated is $Tf = \{\varphi_{2^{-j}} * f\}_{j\in\mathbb{Z}}$, with kernel $K(x) = \{\varphi_{2^{-j}}(x)\}_{j\in\mathbb{Z}}$ as an element of $\ell^2(\mathbb{Z})$ (there exists an isometry between $\mathcal{L}(\mathbb{C}, \ell^2(\mathbb{Z}))$ and $\ell^2(\mathbb{Z})$). Let us see what hypothesis we must require about φ so that the kernel will be regular.

First we want that $\varphi \in L^1(\mathbb{R})$ and

(7) $$\sum_{j\in\mathbb{Z}} |\widehat{\varphi}(2^{-j}\,\xi)|^2 \leq C, \quad \text{for a.e.} \xi \in \mathbb{R}.$$

Under these conditions, we get the continuity of T from $L^2(\mathbb{R})$ to $L^2_{\ell^2(\mathbb{Z})}(\mathbb{R})$, or equivalently, that $\mathcal{G}$ is bounded in $L^2(\mathbb{R})$. The following lemma, that is essentially proved in [GR] —pages 505-507—, guarantees that (7) holds.

Lemma 3.1 *Let $\varphi \in L^1(\mathbb{R})$ with*

$$\widehat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)\, dx = 0.$$

Suppose that there exist $\alpha, \beta > 0$ such that

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{1+\alpha}}, \quad x \in \mathbb{R} \tag{8}$$

and

$$\int_{\mathbb{R}} |\varphi(x+h) - \varphi(x)|\, dx \le C\, |h|^{\beta}, \quad h \in \mathbb{R}. \tag{9}$$

Then (7) holds.

We want to set some conditions about φ in order to conclude the regularity of the associated kernel.

Definition 3.2 *Let $\gamma > 0$, we will say that $\varphi \in L^1(\mathbb{R})$ is a γ-regular function when*

$$\widehat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)\, dx = 0;$$

and there exists $\alpha > 0$ such that, for every $x \in \mathbb{R}$,

$$|D^j \varphi(x)| \le \frac{C}{(1+|x|)^{1+j+\alpha}}, \quad 0 \le j < \gamma. \tag{10}$$

Moreover, if $\gamma - 1 \le j < \gamma$, it also satisfies

$$|D^j \varphi(x-y) - D^j \varphi(x)| \le C\, \frac{|y|^{\gamma - j}}{(1+|x|)^{1+\gamma+\alpha}}, \quad |x| > 2\,|y|. \tag{11}$$

Remark 3.3 *If $0 < \gamma < \gamma'$ and φ is γ'-regular, then φ is also γ-regular.*

With this definition one can prove that every regular function gives an associated regular kernel.

Proposition 3.4 *Let $\gamma > 0$ and $\varphi \in L^1(\mathbb{R})$ be a γ-regular function. Then $K = \{\varphi_{2^{-j}}\}_{j \in \mathbb{Z}}$ is a γ-regular vector-valued kernel and (7) holds.*

Proof. We want to apply Lemma 3.1. The condition (8) is the same as (10) with $j = 0$. Because $\varphi \in L^1(\mathbb{R})$, the inequality (9) is trivial for $|h| \geq 1$. On the other hand, when $|h| \leq 1$, setting $\delta = \min\{1, \gamma\}$, we know that φ is δ-regular, so that

$$|\varphi(x - y) - \varphi(x)| \leq C \, \frac{|y|^\delta}{(1 + |x|)^{1+\delta+\alpha}}, \quad |x| > 2\,|y|.$$

This implies that

$$\int_{\mathbb{R}} |\varphi(x + h) - \varphi(x)| \, dx = \int_{|x|>2\,|h|} + \int_{|x|\leq 2\,|h|} \leq C \left(|h|^\delta + |h|\right) \leq C\,|h|^{\min\{1,\delta\}},$$

where we estimate the first integral by using the above condition and the second observing that φ is bounded. By applying now Lemma 3.1, we obtain (7). To prove the condition (5), by homogeneity, it is enough to deal with the case $1 < |x| \leq 2$ and to establish

$$\sum_{m\in\mathbb{Z}} |x|^2 \, |\varphi_{2^{-m}}(x)|^2 = \sum_{m\in\mathbb{Z}} |2^m \, x|^2 \, |\varphi(2^m \, x)|^2 \leq C.$$

The terms corresponding to $m \geq 0$ are estimated by using (10) with $j = 0$ and the remaining ones by using the boundedness of φ. For $0 \leq j < \gamma$ we consider the operators $\{a_m(x)\}_{m=0}^{j} \in \mathcal{L}(\mathbb{C}, \ell^2(\mathbb{Z})) \equiv \ell^2(\mathbb{Z})$, defined by

$$a_m(x) = \left\{ \frac{1}{m!} \, 2^n \, D^m \varphi(2^n \, x) \, (-2^n)^m \right\}_{n\in\mathbb{Z}},$$

which give

$$P_x(y) = \sum_{m=0}^{j} a_m(x) \, y^m = \left\{ \sum_{m=0}^{j} \frac{1}{m!} \, D^m \varphi_{2^{-n}}(x) \, (-y)^m \right\}_{n\in\mathbb{Z}}.$$

When $j = [\gamma]^-$, $1 < |x| \leq 2$ and $|x| > 2\,|y|$, by using (11) and Taylor's theorem, we get

$$\left| \varphi_{2^{-n}}(x - y) - \sum_{m=0}^{j} \frac{1}{m!} \, D^m \varphi_{2^{-n}}(x) \, (-y)^m \right| \leq C \, |y|^\gamma \, \frac{2^{n\,(\gamma+1)}}{(1 + 2^n)^{1+\gamma+\alpha}}.$$

Then

$$\begin{aligned}\|K(x-y)-P_x(y)\|^2_{\ell^2(\mathbb{Z})} &\leq C\,|y|^{2\gamma}\sum_{n\in\mathbb{Z}}\frac{2^{2\,n\,(\gamma+1)}}{(1+2^n)^{2\,(1+\gamma+\alpha)}} \leq C\,|y|^{2\gamma}\\ &\leq C\left(\frac{|y|^\gamma}{|x|^{1+\gamma}}\right)^2.\end{aligned}$$

In the general case, $|x|>2\,|y|$, we can put $x=2^k\,x'$, $k\in\mathbb{Z}$, with $1<|x'|\leq 2$. So, by setting $y=2^k\,y'$, we have

$$\begin{aligned}\|K(x-y)-P_x(y)\|^2_{\ell^2(\mathbb{Z})} &= 2^{-2\,k}\,\|K(x'-y')-P_{x'}(y')\|^2_{\ell^2(\mathbb{Z})}\\ &\leq 2^{-2\,k}\,C\left(\frac{|y'|^\gamma}{|x'|^{1+\gamma}}\right)^2 = C\left(\frac{|y|^\gamma}{|x|^{1+\gamma}}\right)^2.\end{aligned}$$

Thus, we have obtained (6) for this $j=[\gamma]^-$. When $0\leq j<\gamma-1$, φ will be $(j+1)$-regular and the previous inequality holds with $j+1$ instead of γ, that is, we have (6). □

With the above result, one can conclude the boundedness of the square function $\mathcal{G}$ in the weighted Hardy spaces.

Corollary 3.5 *Let $\varphi\in L^1(\mathbb{R})$ be a γ-regular function. If $0<p\leq 1$ and w is a weight in the class A_∞ with critical index q_w, $\frac{q_w}{p}<1+\gamma$, then*

$$\mathcal{G}(f)=\left\{\sum_{j\in\mathbb{Z}}|\varphi_{2^{-j}}*f|^2\right\}^{\frac{1}{2}}$$

is bounded from $H^p(w)$ to $L^p(w)$.

Proof. Proposition 3.4 says that T is a V.V.C-Z.O with kernel K γ-regular. Then, by using Theorem 2.10 we know that T is continuous between $H^p_{\mathbb{C}}(w)$ and $L^p_{\ell^2(\mathbb{Z})}(w)$, or equivalently, $\mathcal{G}$ is bounded from $H^p(w)$ to $L^p(w)$. □

Remark 3.6 *If $\varphi\in L^1(\mathbb{R})$ is γ-regular, for some $\gamma>0$, the square function $\mathcal{G}$ will be bounded in $L^q(w)$ for $1<q<\infty$ and $w\in A_q$ (see Remark 2.5).*

4 Characterization.

First we define the regularity class with which we obtain the desired characterization.

Definition 4.1 *Let $\alpha \geq 1$. We shall say that φ belongs to the regularity class $\mathcal{R}^\alpha$ if $\varphi \in C^{[\alpha]}$ and there exist $C, r, \varepsilon > 0$ such that*

$$(i) \qquad \int_{\mathbb{R}} x^n \, \varphi(x)\, dx = 0 \quad \textit{for all } 0 \leq n \leq [\alpha] - 1.$$

$$(ii) \qquad |\varphi(x)| \leq \frac{C}{(1+|x|)^{1+[\alpha]+r}} \quad \textit{for all } x \in \mathbb{R}.$$

$$(iii) \qquad |D^n \varphi(x)| \leq \frac{C}{(1+|x|)^{\alpha+\varepsilon}} \quad \textit{for all } x \in \mathbb{R} \textit{ and } 0 \leq n \leq [\alpha].$$

We will prove the following theorem.

Theorem 4.2 *Let $\psi \in \mathcal{R}^\alpha$ be an orthonormal wavelet, $\alpha \geq 1$. If $0 < p \leq 1$ and w is a weight in A_∞ with critical index q_w verifying $\frac{q_w}{p} \leq \alpha$, then there exist two constants $0 < c \leq C < \infty$ such that*

$$(12) \qquad c\, \|f\|^p_{H^p(w)} \leq \|\mathcal{W}_\psi f\|^p_{L^p(w)} \leq C\, \|f\|^p_{H^p(w)},$$

for all $f \in H^p(w)$. In other words, the mapping

$$\begin{array}{rcl} H^p(w) & \longrightarrow & [0, \infty) \\ f & \longmapsto & \|\mathcal{W}_\psi f\|^p_{L^p(w)} \end{array}$$

is a p-norm equivalent to $\|\cdot\|^p_{H^p(w)}$.

Remark 4.3 *In* [Wu]*, by using different methods, the previous characterization was proved for compactly supported wavelets coming from an MRA.*

We prove the characterization in three steps.

Theorem 4.4 *For $0 < p \leq 1$, $\psi \in L^1(\mathbb{R})$ a band-limited γ-regular function and $w \in A_\infty$ a weight with critical index q_w, $\frac{q_w}{p} < 1 + \gamma$, there exists a constant $0 < C < \infty$ such that*

$$\|\mathcal{W}_\psi f\|^p_{L^p(w)} \leq C\, \|f\|^p_{H^p(w)}, \quad \textit{for all } f \in H^p(w) \cap L^2(\mathbb{R}).$$

Theorem 4.5 *Let $\alpha \geq 1$ and suppose that $\varphi, \psi \in \mathcal{R}^\alpha$ with φ an orthonormal wavelet. If $0 < p \leq 1$ and $w \in A_\infty$ has critical index q_w, with $\frac{q_w}{p} \leq \alpha$, then there exists a constant $0 < C < \infty$ such that*

$$\|\mathcal{W}_\psi f\|^p_{L^p(w)} \leq C\, \|\mathcal{W}_\varphi f\|^p_{L^p(w)}, \quad \text{for all } f \in L^2(\mathbb{R}).$$

Theorem 4.6 *Let $w \in A_\infty$ be a weight with critical index q_w. If $0 < p \leq 1$ and $\psi \in C^1$ is a compactly supported orthonormal wavelet which satisfies*

$$\int_{\mathbb{R}} x^n\, \psi(x)\, dx = 0, \quad 0 \leq n \leq N_p(w) = \left[\frac{q_w}{p}\right] - 1.$$

Then, there exists a constant C, $0 < C < \infty$, such that for every $f \in L^2(\mathbb{R})$ with $\mathcal{W}_\psi f \in L^p(w)$, it follows that $f \in H^p(w)$ and

$$\|f\|^p_{H^p(w)} \leq C\, \|\mathcal{W}_\psi f\|^p_{L^p(w)}.$$

Proof of Theorem 4.2. By density, it is enough to prove (12) for functions f in $H^p(w) \cap L^2(\mathbb{R})$. By choosing $\varphi_1 \in \mathcal{R}^\alpha$ a band-limited γ-regular wavelet (for some γ such that $1 + \gamma > \frac{q_w}{p}$), we can use Theorems 4.4 and 4.5 concluding

$$\|\mathcal{W}_\psi f\|^p_{L^p(w)} \leq C\, \|\mathcal{W}_{\varphi_1} f\|^p_{L^p(w)} \leq C\, \|f\|^p_{H^p(w)}.$$

Finally, we take a compactly supported orthonormal wavelet $\varphi_2 \in \mathcal{R}^\alpha$ and by using Theorems 4.5, 4.6 we obtain

$$\|f\|^p_{H^p(w)} \leq C\, \|\mathcal{W}_{\varphi_2} f\|^p_{L^p(w)} \leq C\, \|\mathcal{W}_\psi f\|^p_{L^p(w)}.$$

$\square$

Remark 4.7 *Following the same ideas, we can extend Theorem 4.6 for generic orthonormal wavelets in $\mathcal{R}^\alpha$, $\frac{q_w}{p} \leq \alpha$. We should only take a compactly supported wavelet in $\mathcal{R}^\alpha$ and apply Theorems 4.5 and 4.6.*

4.1 Proof of Theorem 4.4.

We need the following proposition:

Proposition 4.8 *Let $0 < p \leq 1$, $\varphi \in L^1(\mathbb{R})$ be a band-limited γ-regular function and $w \in A_\infty$ be a weight with critical index q_w that verifies $\frac{q_w}{p} < 1 + \gamma$. Given $\lambda > \frac{q_w}{p}$, there exists $0 < C_\lambda < \infty$ such that*

$$\left\| \left\{ \sum_{j \in \mathbb{Z}} |\varphi^{**}_{j,\lambda} f|^2 \right\}^{\frac{1}{2}} \right\|^p_{L^p(w)} \leq C_\lambda \, \|f\|^p_{H^p(w)}$$

for all $f \in H^p(w) \cap L^2(\mathbb{R})$, where

$$(\varphi^{**}_{j,\lambda} f)(x) = \sup_{y \in \mathbb{R}} \frac{|(\varphi_{2^{-j}} * f)(x-y)|}{(1 + 2^j\,|y|)^\lambda}.$$

Proof. Because of Corollary 3.5, the above conditions guarantee that $\|\mathcal{G}(f)\|^p_{L^p(w)} \leq C\,\|f\|^p_{H^p(w)}$. Also $\mathcal{G}$ is bounded in $L^2(\mathbb{R})$. So, for every $j \in \mathbb{Z}$, we have $\varphi_{2^{-j}} * f \in L^2(\mathbb{R})$. If $\mathcal{M}$ denotes the Hardy-Littlewood maximal function and we use a result in [HW] —page 271—, we obtain

$$(\varphi^{**}_{j,\lambda} f)(x) \leq C_\lambda \left[\mathcal{M}(|\varphi_{2^{-j}} * f|^{\frac{1}{\lambda}})(x) \right]^\lambda,$$

for all $\lambda > 0$ and, in particular, for $\lambda > \frac{q_w}{p}$. In this case, $\lambda > 1$, $\lambda\,p > 1$, $w \in A_{\lambda p}$. So, the weighted vectorial inequality for the Hardy-Littlewood maximal function, proved in [AJ], is used to obtain

$$\begin{aligned} \left\| \left\{ \sum_{j \in \mathbb{Z}} |\varphi^{**}_{j,\lambda} f|^2 \right\}^{\frac{1}{2}} \right\|^p_{L^p(w)} &\leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[\mathcal{M}(|\varphi_{2^{-j}} * f|^{\frac{1}{\lambda}}) \right]^{2\lambda} \right\}^{\frac{1}{2\lambda}} \right\|^{\lambda p}_{L^{\lambda p}(w)} \\ &\leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} |\varphi_{2^{-j}} * f|^2 \right\}^{\frac{1}{2\lambda}} \right\|^{\lambda p}_{L^{\lambda p}(w)} \\ &= C\,\|\mathcal{G}(f)\|^p_{L^p(w)} \leq C\,\|f\|^p_{H^p(w)}. \end{aligned}$$

□

Now we can prove Theorem 4.4. If $f \in H^p(w) \cap L^2(\mathbb{R})$ we have

$$\begin{aligned} \left|\langle f, \psi_{j,k} \rangle\right| &= 2^{\frac{j}{2}} \left| \int_{\mathbb{R}} f(x)\,\overline{\psi(2^j\,x - k)}\,dx \right| = 2^{-\frac{j}{2}} \left| \big(\widetilde{\psi}_{2^{-j}} * f\big)(2^{-j}\,k) \right| \\ &\leq 2^{-\frac{j}{2}} \sup_{y \in I_{j,k}} \left| \big(\widetilde{\psi}_{2^{-j}} * f\big)(y) \right|, \end{aligned}$$

where $\widetilde{\psi}(x) = \overline{\psi(-x)}$ and $I_{j,k} = [2^{-j}\,k, 2^{-j}(k+1)]$. Fixing $j \in \mathbb{Z}$, for almost every $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\, 2^j\, \chi_{I_{j,k}}(x) &\leq \sum_{k\in\mathbb{Z}} \Big\{ \sup_{y\in I_{j,k}} |(\widetilde{\psi}_{2^{-j}} * f)(y)| \Big\}^2 \chi_{I_{j,k}}(x) \\
&\leq \sum_{k\in\mathbb{Z}} \Big\{ \sup_{|z|\leq 2^{-j}} |(\widetilde{\psi}_{2^{-j}} * f)(x-z)| \Big\}^2 \chi_{I_{j,k}}(x) \\
&= \Big\{ \sup_{|z|\leq 2^{-j}} \frac{|(\widetilde{\psi}_{2^{-j}} * f)(x-z)|}{(1+2^j|z|)^\lambda} (1+2^j|z|)^\lambda \Big\}^2 \\
&\leq 2^{2\,\lambda} \Big[(\widetilde{\psi}^{**}_{j,\lambda} f)(x) \Big]^2,
\end{aligned}$$

for every $\lambda > 0$. Choosing $\lambda > \frac{q_w}{p}$, the previous proposition leads to

$$\begin{aligned}
\|\mathcal{W}_\psi f\|^p_{L^p(w)} &= \Big\| \Big\{ \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\, 2^j \chi_{I_{j,k}} \Big\}^{\frac{1}{2}} \Big\|^p_{L^p(w)} \\
&\leq C \Big\| \Big\{ \sum_{j\in\mathbb{Z}} |\widetilde{\psi}^{**}_{j,\lambda} f|^2 \Big\}^{\frac{1}{2}} \Big\|^p_{L^p(w)} \leq C\, \|f\|^p_{H^p(w)}.
\end{aligned}$$

□

4.2 Proof of Theorem 4.5.

The following lemma establishes a result in the direction of those obtained in [HW] —pages 272-276—.

Lemma 4.9 *Let $r \geq \varepsilon > 0$ and $\alpha \geq 1$. Suppose that g and h satisfy*

(i) $$|D^n g(x)| \leq \frac{C}{(1+|x|)^{\alpha+\varepsilon}} \quad \text{for all } x \in \mathbb{R} \text{ and } 0 \leq n \leq [\alpha];$$

(ii) $$\int_{\mathbb{R}} x^n\, h(x)\, dx = 0 \quad \text{for all } 0 \leq n \leq [\alpha]-1;$$

(iii) $$|h(x)| \leq \frac{C}{(1+|x|)^{1+[\alpha]+r}} \quad \text{for all } x \in \mathbb{R}.$$

Then, for every $j, k, l, m \in \mathbb{Z}$ with $j \leq l$, we have

$$|(g_{j,k} * h_{l,m})(x)| \leq \frac{C\, 2^{(j-l)\,([\alpha]+\frac{1}{2})}}{(1+2^j\,|x-2^{-j}\,k-2^{-l}\,m|)^{\alpha+\varepsilon}} \quad \forall\, x \in \mathbb{R}.$$

Proof. Since $(g_{j,k} * h_{l,m})(x) = (g_{0,0} * h_{l-j,0})(2^j\, x - k - 2^{j-l}\, m)$, it is enough to consider the case $j = k = m = 0$ and $l \geq 0$. Set $N = [\alpha] - 1$ and

$$\begin{aligned} E_1 &= \{y \in \mathbb{R} : |y - x| \leq 3\}, \\ E_2 &= \{y \in \mathbb{R} : |y - x| > 3 \quad \text{and} \quad |y| \leq \frac{1}{2}\,|x|\}, \\ E_3 &= \{y \in \mathbb{R} : |y - x| > 3 \quad \text{and} \quad |y| > \frac{1}{2}\,|x|\}. \end{aligned}$$

By (ii), we can subtract the N-degree Taylor's polynomial. So, we get

$$|(g_{0,0} * h_{l,0})(x)| = \left| \int_{\mathbb{R}} \left\{ g(y) - \sum_{n=0}^{N} \frac{1}{n!}\, D^n g(x)\, (y - x)^n \right\} h_{l,0}(x - y)\, dy \right|$$

$$\begin{aligned} &\leq \left\{ \int_{E_1} + \int_{E_2} + \int_{E_3} \right\} \left\{ \left| g(y) - \sum_{n=0}^{N} \frac{1}{n!}\, D^n g(x)\, (y - x)^n \right| \cdot |h_{l,0}(x - y)| \right\} dy \\ &= I + II + III. \end{aligned}$$

For $y \in E_1$, we use (i) with $n = N + 1$,

$$\left| g(y) - \sum_{n=0}^{N} \frac{1}{n!}\, D^n g(x)\, (y - x)^n \right| \leq \frac{C}{(1 + |z|)^{\alpha+\varepsilon}}\, |y - x|^{N+1},$$

where z lies between x and y. Since $|x - y| < 3$,

$$1 + |x| \leq 1 + |x - z| + |z| \leq 4 + |z| \leq 4\,(1 + |z|).$$

Thus, by using (iii)

$$\begin{aligned} I &\leq C\, \frac{2^{\frac{l}{2}}}{(1 + |x|)^{\alpha+\varepsilon}} \int_{E_1} |y - x|^{N+1}\, \frac{1}{(1 + 2^l\, |x - y|)^{2+N+r}}\, dy \\ &\leq C\, \frac{2^{-l\,(\frac{1}{2}+N+1)}}{(1 + |x|)^{\alpha+\varepsilon}} \int_0^\infty \frac{t^{N+1}}{(1 + t)^{2+N+r}}\, dt \leq C\, \frac{2^{-l\,(\frac{1}{2}+[\alpha])}}{(1 + |x|)^{\alpha+\varepsilon}}. \end{aligned}$$

If $y \in E_2$, then $\frac{1}{2}\,|x| \leq |x - y| \leq \frac{3}{2}\,|x|$ and

$$4\,|x - y| = |x - y| + 3\,|x - y| > 3 + \frac{3}{2}\,|x| > 1 + |x|.$$

Thus, $1 + 2^l\,|x-y| \geq 2^l\,|x-y| \geq 2^{l-2}\,(1+|x|)$. By (i) and (iii) we obtain

$$\begin{aligned} II &\leq C \int_{E_2} \Big\{ \frac{1}{(1+|y|)^{\alpha+\varepsilon}} + \sum_{n=0}^{N} \frac{|x-y|^n}{(1+|x|)^{\alpha+\varepsilon}} \Big\} \frac{2^{\frac{l}{2}}}{(1+2^l\,|x-y|)^{2+N+r}}\, dy \\ &\leq C\, \frac{2^{-l\,(\frac{3}{2}+N+r)}}{(1+|x|)^{2+N+r}} \Big\{ \int_{E_2} \frac{1}{(1+|y|)^{\alpha+\varepsilon}}\, dy + \frac{|x|^N}{(1+|x|)^{\alpha+\varepsilon}} \int_{|y|\leq \frac{1}{2}\,|x|} 1\, dy \Big\} \\ &\leq C\, \frac{2^{-l\,(\frac{3}{2}+N+r)}}{(1+|x|)^{2+N+r}} \Big\{ 1 + \frac{|x|^{N+1}}{(1+|x|)^{\alpha+\varepsilon}} \Big\} \leq C\, \frac{2^{-l\,(\frac{1}{2}+N+1)}}{(1+|x|)^{\alpha+r}} \\ &\leq C\, \frac{2^{-l\,(\frac{1}{2}+[\alpha])}}{(1+|x|)^{\alpha+\varepsilon}}, \end{aligned}$$

where we have needed that $l \geq 0$ and $r \geq \varepsilon > 0$.

Finally, if $y \in E_3$, $1+|x| \leq 1 + 2\,|y| \leq 2\,(1+|y|)$. Hence,

$$\begin{aligned} III &\leq C \int_{E_3} \Big\{ \frac{1}{(1+|y|)^{\alpha+\varepsilon}} + \sum_{n=0}^{N} \frac{|x-y|^n}{(1+|x|)^{\alpha+\varepsilon}} \Big\} \frac{2^{\frac{l}{2}}}{(1+2^l\,|x-y|)^{2+N+r}}\, dy \\ &\leq C\, \frac{2^{\frac{l}{2}}}{(1+|x|)^{\alpha+\varepsilon}} \int_{3\cdot 2^l}^{\infty} \frac{(2^{-l}\,t)^N}{(1+t)^{2+N+r}}\, 2^{-l}\, dt \leq C\, \frac{2^{-l\,(\frac{1}{2}+N+1)}}{(1+|x|)^{\alpha+\varepsilon}}\, 2^{-l\,r} \\ &\leq C\, \frac{2^{-l\,(\frac{1}{2}+[\alpha])}}{(1+|x|)^{\alpha+\varepsilon}}, \end{aligned}$$

where we have used (i), (iii) and the fact that $r, l \geq 0$. □

By using this lemma an the relation between the dot product (in $L^2(\mathbb{R})$) and the convolution of functions, we can prove the following estimates.

Lemma 4.10 *Let* $\psi, \varphi \in \mathcal{R}^\alpha$, $\alpha \geq 1$. *There exists* $\varepsilon > 0$ *such that, if* $j, k, l, m \in \mathbb{Z}$, *we have*

(i) $$|\langle \psi_{j,k}, \varphi_{l,m} \rangle| \leq C\, \frac{2^{(l-j)\,([\alpha]+\frac{1}{2})}}{(1+2^l|2^{-j}\,k - 2^{-l}\,m|)^{\alpha+\varepsilon}} \quad \textit{for } l \leq j,$$

(ii) $$|\langle \psi_{j,k}, \varphi_{l,m} \rangle| \leq C\, \frac{2^{(j-l)\,([\alpha]+\frac{1}{2})}}{(1+2^j|2^{-l}\,m - 2^{-j}\,k|)^{\alpha+\varepsilon}} \quad \textit{for } l \geq j.$$

The following lemma is implicitly proved in [HW] —pages 277-279—.

Lemma 4.11 *Given $\alpha \geq 1$, $\varepsilon > 0$ and $1 \leq r < \alpha+\varepsilon$, there exists a constant C such that, for all sequences $\{s_{j,k} : j,k \in \mathbb{Z}\}$ of complex numbers and all $x \in I_{j,k} = [2^{-j}\, k, 2^{-j}\,(k+1)]$,*

$$(i) \quad \sum_{m\in\mathbb{Z}} \frac{|s_{l,m}|}{(1+2^l\,|2^{-j}\,k - 2^{-l}\,m|)^{\alpha+\varepsilon}} \leq C \left\{\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |s_{l,m}|^{\frac{1}{r}}\,\chi_{I_{l,m}}\Big)(x)\right\}^r$$

if $l \leq j$.

$$(ii) \quad \sum_{m\in\mathbb{Z}} \frac{|s_{l,m}|}{(1+2^j\,|2^{-l}\,m - 2^{-j}\,k|)^{\alpha+\varepsilon}} \leq C\, 2^{(l-j)\,r} \left\{\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |s_{l,m}|^{\frac{1}{r}}\,\chi_{I_{l,m}}\Big)(x)\right\}^r$$

if $l \geq j$.

We shall prove the comparison result, Theorem 4.5. Since φ is an orthonormal wavelet, we can write

$$\psi_{j,k} = \sum_{l,m\in\mathbb{Z}} \langle \psi_{j,k}, \varphi_{l,m}\rangle\, \varphi_{l,m} \quad \text{in } L^2(\mathbb{R}).$$

Hence,

$$(13)\quad \mathcal{W}_\psi f(x) = \Big\{\sum_{j,k\in\mathbb{Z}} \Big|\sum_{l,m\in\mathbb{Z}} \langle f, \varphi_{l,m}\rangle\, \overline{\langle \psi_{j,k}, \varphi_{l,m}\rangle}\Big|^2\, 2^j\, \chi_{I_{j,k}}(x)\Big\}^{\frac{1}{2}}$$

$$\leq \Big\{\sum_{j,k\in\mathbb{Z}} |B_1(j,k)|^2\, 2^j \chi_{I_{j,k}}(x)\Big\}^{\frac{1}{2}} + \Big\{\sum_{j,k\in\mathbb{Z}} |B_2(j,k)|^2\, 2^j \chi_{I_{j,k}}(x)\Big\}^{\frac{1}{2}},$$

where

$$B_1(j,k) = \sum_{l\leq j}\sum_{m\in\mathbb{Z}} \ldots \quad \text{and} \quad B_2(j,k) = \sum_{l>j}\sum_{m\in\mathbb{Z}} \ldots$$

We set $N = [\alpha] - 1$. For B_1, by using Lemma 4.10 part (i) and Lemma 4.11 part (i) with r such that $1 \leq \frac{q_w}{p} < r < \alpha + \varepsilon$, we obtain, for all $x \in I_{j,k}$,

$$\begin{aligned} |B_1(j,k)| &\leq C \sum_{l\leq j}\sum_{m\in\mathbb{Z}} |\langle f, \varphi_{l,m}\rangle|\, \frac{2^{(l-j)\,(N+\frac{3}{2})}}{(1+|2^{l-j}\,k - m|)^{\alpha+\varepsilon}} \\ &\leq C \sum_{l\leq j} 2^{(l-j)\,(N+\frac{3}{2})} \Big[\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |\langle f, \varphi_{l,m}\rangle|^{\frac{1}{r}}\,\chi_{I_{l,m}}\Big)(x)\Big]^r. \end{aligned}$$

But we know that $\{I_{j,k}\}_{k\in\mathbb{Z}}$ is an almost partition of $\mathbb{R}$ and therefore

$$\begin{aligned}
&\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |B_1(j,k)|^2\, 2^j\, \chi_{I_{j,k}}\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)}\\
&\leq\ C\,\Big\|\Big\{\sum_{j\in\mathbb{Z}}\Big\{\sum_{l\leq j} 2^{(l-j)\,(N+1)}\,\Big[\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |\langle f,\varphi_{l,m}\rangle|^{\frac{1}{r}}\, 2^{\frac{l}{2r}}\,\chi_{I_{l,m}}\Big)\Big]^r\Big\}^2\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)}\\
&=\ C\,\Big\|\Big\{\sum_{j\in\mathbb{Z}}\Big|\sum_{l\in\mathbb{Z}} a_{j-l}\,b_l\Big|^2\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)},
\end{aligned}$$

when we set

$$a_j = \begin{cases} 2^{-j\,(N+1)} & \text{if} \quad j\geq 0,\\ 0 & \text{if} \quad j<0,\end{cases} \quad \text{and } b_l = \Big[\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |\langle f,\varphi_{l,m}\rangle|^{\frac{1}{r}}\, 2^{\frac{l}{2r}}\,\chi_{I_{l,m}}\Big)(x)\Big]^r.$$

By using Young's inequality for convolutions,

$$\Big\{\sum_{j\in\mathbb{Z}}\Big|\sum_{l\in\mathbb{Z}} a_{j-l}\,b_l\Big|^2\Big\}^{\frac{1}{2}} = \Big\|\{a_j\}*\{b_l\}\Big\|_{\ell^2} \leq \Big\|\{a_j\}\Big\|_{\ell^1}\,\Big\|\{b_l\}\Big\|_{\ell^2}.$$

But $\|\{a_j\}\|_{\ell^1}\leq C$ because $N+1>0$. Thus, the weighted vectorial inequality for the Hardy-Littlewood maximal function ($2r>1$, $p\,r>q_w\geq 1$ and so $w\in A_{p\,r}$) is used to get

$$\begin{aligned}
(14)\quad &\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |B_1(j,k)|^2\, 2^j\, \chi_{I_{j,k}}\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)}\\
&\leq C\,\Big\|\Big\{\sum_{l\in\mathbb{Z}}\Big[\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |\langle f,\varphi_{l,m}\rangle|^{\frac{1}{r}}\, 2^{\frac{l}{2r}}\,\chi_{I_{l,m}}\Big)\Big]^{2\,r}\Big\}^{\frac{1}{2\,r}}\Big\|^{p\,r}_{L^{p\,r}(w)}\\
&\leq C\,\Big\|\Big\{\sum_{l\in\mathbb{Z}}\Big[\sum_{m\in\mathbb{Z}} |\langle f,\varphi_{l,m}\rangle|^{\frac{1}{r}}\, 2^{\frac{l}{2r}}\,\chi_{I_{l,m}}\Big]^{2\,r}\Big\}^{\frac{1}{2\,r}}\Big\|^{p\,r}_{L^{p\,r}(w)} = C\,\|\mathcal{W}_\varphi f\|^p_{L^p(w)}.
\end{aligned}$$

For B_2 we follow the steps of B_1. By using part (ii) of Lemmas 4.10, 4.11 with r such that

$$1\leq \frac{q_w}{p} < r < \min\{[\alpha]+1,\alpha+\varepsilon\} = \min\{N+2,\alpha+\varepsilon\}.$$

we obtain

$$\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |B_2(j,k)|^2\, 2^j\, \chi_{I_{j,k}}\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)} \leq C\,\Big\|\Big\{\sum_{j\in\mathbb{Z}}\Big|\sum_{l\in\mathbb{Z}} a_{j-l}\,b_l\Big|^2\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)},$$

where

$$a_j = \begin{cases} 2^{j\,(N+2-r)} & \text{if} \quad j < 0, \\ 0 & \text{if} \quad j \geq 0, \end{cases} \text{ and } b_l = \Big[\mathcal{M}\Big(\sum_{m\in\mathbb{Z}} |\langle f, \varphi_{l,m}\rangle|^{\frac{1}{r}}\, 2^{\frac{l}{2\,r}}\, \chi_{I_{l,m}}\Big)(x)\Big]^r.$$

We use again Young's inequality for convolutions, but $\|\{a_j\}\|_{\ell^1} \leq C$ because $N+2-r > 0$. As in the case of B_1 we can apply the result of [AJ] obtaining

$$\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |B_2(j,k)|^2\, 2^j\, \chi_{I_{j,k}}\Big\}^{\frac{1}{2}}\Big\|^p_{L^p(w)} \leq C\, \|\mathcal{W}_\varphi f\|^p_{L^p(w)}. \tag{15}$$

Finally, collecting (13), (14), (15) and using the fact that $\|\cdot\|^p_{L^p(w)}$ is a p-norm, we conclude the desired inequality. □

4.3 Proof of Theorem 4.6.

In order to prove the left-hand side of the inequality (12), we need to obtain a characterization result on $L^q(w)$ for q large enough. This fact will allow us to establish that some atoms, very close to the wavelets, are actually $H^p(w)$-atoms. The proof of the new characterization is a suitable modification of the case of weighted Hardy spaces.

Proposition 4.12 *If $1 < q < \infty$, $w \in A_q$, $\gamma > 0$, $\varphi \in L^1(\mathbb{R})$ is a band-limited γ-regular function and $\lambda \geq 1$, there exists $0 < C_\lambda < \infty$ such that*

$$\Big\|\Big\{\sum_{j\in\mathbb{Z}} |\varphi^{**}_{j,\lambda} f|^2\Big\}^{\frac{1}{2}}\Big\|_{L^q(w)} \leq C_\lambda\, \|f\|_{L^q(w)}$$

for all $f \in L^q(w) \cap L^2(\mathbb{R})$.

Proof. We follow the steps of Proposition 4.8. Since $\lambda \geq 1$, we have $\lambda\, q \geq q$ and so, $w \in A_{\lambda\, q}$. This fact allow us to use the weighted vectorial inequality for Hardy-Littlewood maximal function. To finish the proof we should observe that the square function is bounded in $L^q(w)$ (see Remark 3.6). □

Now, we can obtain the right-hand side inequality in $L^q(w)$ repeating the steps of Theorem 4.4.

Theorem 4.13 *For $1 < q < \infty$, $w \in A_q$, $\gamma > 0$ and $\psi \in L^1(\mathbb{R})$ a band-limited γ-regular function, there exists a constant C, $0 < C < \infty$, such that*

$$\|\mathcal{W}_\psi f\|_{L^q(w)} \leq C\,\|f\|_{L^q(w)}, \quad \text{for all } f \in L^q(w) \cap L^2(\mathbb{R}).$$

Besides, we have a comparison result.

Theorem 4.14 *Let $\varphi, \psi \in \mathcal{R}^1$ with φ an orthonormal wavelet. For $1 < q < \infty$ and w a weight in A_q we have*

$$\|\mathcal{W}_\psi f\|_{L^q(w)} \leq C\,\|\mathcal{W}_\varphi f\|_{L^q(w)}, \quad \text{for all } f \in L^2(\mathbb{R}).$$

Proof. We follow the proof of the Hardy spaces case putting q in place of p. Here, N will be 0 and in both estimates of B_1, B_2 we will take $r = 1$. We can apply the result of [AJ], because $w \in A_q$, and the desired inequality is obtained. □

By using the ideas of the proof of Theorem 4.2 we can get the next result.

Theorem 4.15 *Let $\psi \in \mathcal{R}^1$ be an orthonormal wavelet. If $1 < q < \infty$ and w is a weight in A_q, we have*

$$\|\mathcal{W}_\psi f\|_{L^q(w)} \leq C\,\|f\|_{L^q(w)}, \quad \text{for all } f \in L^q(w).$$

The reverse inequality can be proved by using a duality argument.

Theorem 4.16 *Let $\psi \in \mathcal{R}^1$ be an orthonormal wavelet. If $1 < q < \infty$ and w is a weight in A_q we have*

$$c\,\|f\|_{L^q(w)} \leq \|\mathcal{W}_\psi f\|_{L^q(w)} \leq C\,\|f\|_{L^q(w)},$$

for all $f \in L^q(w)$. In other words, the mapping

$$\begin{array}{rcl} L^q(w) & \longrightarrow & [0,\infty) \\ f & \longmapsto & \|\mathcal{W}_\psi f\|_{L^q(w)} \end{array}$$

is a norm equivalent to $\|\cdot\|_{L^q(w)}$.

Proof. It is enough to obtain the left-hand side inequality for a dense class of functions. For $f \in L^2(\mathbb{R})$, we define the vector-valued operator

$$T_\psi f = \left\{ \langle f, \psi_{j,k} \rangle \, 2^{\frac{j}{2}} \, \chi_{I_{j,k}} \right\}_{j,k \in \mathbb{Z}}.$$

Then $\mathcal{W}_\psi f = \Big(T_\psi f \cdot T_\psi f \Big)^{\frac{1}{2}}$, where $\cdot$ denotes the dot product in $\ell^2(\mathbb{Z} \times \mathbb{Z})$. Since ψ is an orthonormal wavelet we have

$$\int_{\mathbb{R}} T_\psi f(x) \cdot T_\psi f(x) \, dx = \|\mathcal{W}_\psi f\|^2_{L^2(\mathbb{R})} = \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 = \|f\|^2_{L^2(\mathbb{R})}.$$

By using polarization and Cauchy-Schwarz's inequality, we obtain that for all $f, g \in L^2(\mathbb{R})$,

$$\Big| \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, dx \Big| \le \int_{\mathbb{R}} |T_\psi f(x) \cdot T_\psi g(x)| \, dx \le \int_{\mathbb{R}} \mathcal{W}_\psi f(x) \, \mathcal{W}_\psi g(x) \, dx.$$

By taking $f \in L^q(w) \cap L^2(\mathbb{R})$ and using Hölder's inequality we prove

$$\begin{aligned} \|f\|_{L^q(w)} &= \sup \left\{ \Big| \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, dx \Big| : g \in L^2(\mathbb{R}), \|g\|_{L^{q'}(w^{-\frac{q'}{q}})} \le 1 \right\} \\ &\le \ \sup \left\{ \|\mathcal{W}_\psi f\|_{L^q(w)} \, \|\mathcal{W}_\psi g\|_{L^{q'}(w^{-\frac{q'}{q}})} : g \in L^2(\mathbb{R}), \|g\|_{L^{q'}(w^{-\frac{q'}{q}})} \le 1 \right\}. \end{aligned}$$

But $w \in A_q$ if and only if $w^{-\frac{q'}{q}} \in A_{q'}$. So, by Theorem 4.15 we obtain

$$\|\mathcal{W}_\psi g\|_{L^{q'}(w^{-\frac{q'}{q}})} \le C \, \|g\|_{L^{q'}(w^{-\frac{q'}{q}})} \le C.$$

□

Once we have the characterization for the weighted Lebesgue spaces, we can define some special atoms. We will work with compactly supported wavelets.

Definition 4.17 *Let $0 < p \le 1$, $w \in A_\infty$ and $\psi \in C^1$ be a compactly supported orthonormal wavelet with moments vanishing up to order $N_p(w) =$*

$\left[\frac{q_w}{p}\right] - 1$. *For $q > 1$, we shall say that $a \in L^2(\mathbb{R})$ is a $(p, q; \psi)$-atom with respect to w if there exists a dyadic interval R such that*

$$a = \sum_{\substack{I \subset R \\ I \in \mathcal{D}}} a_I \, \psi_I \quad \text{and} \quad \|\mathcal{W}_\psi a\|_{L^q(w)} = \left\| \Big\{ \sum_{I \subset R} |a_I|^2 \, |I|^{-1} \, \chi_I \Big\}^{\frac{1}{2}} \right\|_{L^q(w)} \leq w(R)^{\frac{1}{q} - \frac{1}{p}}.$$

It is at this point where the characterization of the weighted Lebesgue spaces plays an important role.

Lemma 4.18 *In the above conditions, if a is a $(p, q; \psi)$-atom with respect to w and $q > q_w \geq 1$, there exists a constant $0 < \sigma < \infty$, independent of a, such that a is σ times a (p, q)-atom with respect to w.*

Proof. The interval R will have the form $R = [2^{-j_0} \, k_0, 2^{-j_0} \, (k_0 + 1)]$, for some $j_0, k_0 \in \mathbb{Z}$. Assume $\operatorname{supp} \psi \subset [a, b]$. For $I = I_{j,k}$ dyadic we set

$$I[a, b] = I_{j,k}[a, b] = [2^{-j} \, (k + a), 2^{-j} \, (k + b)].$$

Then, $\operatorname{supp} \psi_I \subset I[a, b] \subset \tilde{R} = R[-|a|, 1 + |b|]$ for all $I \subset R$, and so $\operatorname{supp} a \subset \tilde{R}$. As $q > q_w$ then $w \in A_q$. Besides, $\psi \in \mathcal{R}^1$ and Theorem 4.16 can be used to obtain

$$\int_{\mathbb{R}} |a(x)|^q \, w(x) \, dx \leq C_1 \, \|\mathcal{W}_\psi a\|^q_{L^q(w)} \leq C_1 \, w(R)^{1 - \frac{q}{p}}.$$

But $w \in A_q$ and $R \subset \tilde{R}$, so we know that $w(R) \geq (C_w \, (1 + |a| + |b|))^{-q} \, w(\tilde{R})$. Therefore,

$$\int_{\mathbb{R}} |a(x)|^q \, w(x) \, dx \leq \frac{C_1}{(C_w \, (1 + |a| + |b|))^{q \, (1 - \frac{q}{p})}} \, w(\tilde{R})^{1 - \frac{q}{p}} = \sigma^q \, w(\tilde{R})^{1 - \frac{q}{p}}.$$

Finally, a has vanishing moments up to order $N_p(w)$, because ψ does. Setting $\widetilde{a} = a/\sigma$ we conclude that $\widetilde{a}$ is a (p, q)-atom with respect to w. □

Now, we can prove Theorem 4.6. For every $k \in \mathbb{Z}$ set $\Omega_k = \{x \in \mathbb{R} : \mathcal{W}_\psi f(x) > 2^k\}$. As $p > 0$, we have

$$\|\mathcal{W}_\psi f\|^p_{L^p(w)} = p \int_0^\infty \lambda^{p-1} w(\{x \in \mathbb{R} : \mathcal{W}_\psi f(x) > \lambda\}) \, d\lambda.$$

The integrand is a non-increasing function of λ. By taking the partition of $(0,\infty)$ based on the intervals $[2^k, 2^{k+1})$, with $k \in \mathbb{Z}$, and considering the lower Riemann sum we obtain

$$\|\mathcal{W}_\psi f\|^p_{L^p(w)} \geq 2^{-1}\, p \sum_{k\in\mathbb{Z}} 2^{k\,p}\, w(\Omega_k). \tag{16}$$

Let

$$\mathcal{B}_k = \Big\{ I \in \mathcal{D} : w(I \cap \Omega_k) \geq \frac{1}{2}\, w(I),\ w(I \cap \Omega_{k+1}) < \frac{1}{2}\, w(I) \Big\} \ \text{ and } \widetilde{\mathcal{D}} = \bigcup_{k\in\mathbb{Z}} \mathcal{B}_k.$$

Then, for every $I \in \widetilde{\mathcal{D}}$, there exists a unique $k \in \mathbb{Z}$ such that $I \in \mathcal{B}_k$. Also, due to the nesting property of dyadic intervals, for every $I \in \mathcal{B}_k$ there exists a unique maximal interval $\tilde{I} \in \mathcal{B}_k$ such that $I \subset \tilde{I}$. Let $\{\tilde{I}^i_k : i \in \Delta_k\}$ be the collection of all such maximal dyadic intervals in $\mathcal{B}_k$. We have obtained a partition of $\widetilde{\mathcal{D}}$,

$$\widetilde{\mathcal{D}} = \bigcup_{k\in\mathbb{Z}} \mathcal{B}_k = \bigcup_k \bigcup_{i\in\Delta_k} \{I : I \subset \tilde{I}^i_k, I \in \mathcal{B}_k\}.$$

Let us analyze those intervals that do not belong to $\widetilde{\mathcal{D}}$. If $\langle f, \psi_I\rangle \neq 0$, there exists $k_0 \in \mathbb{Z}$ such that $|\langle f, \psi_I\rangle|\,|I|^{-\frac{1}{2}} \geq 2^{k_0}$. For $x \in I$, $\mathcal{W}_\psi f(x) \geq 2^{k_0}$. Then $I \subset \Omega_{k_0}$ and so $w(I \cap \Omega_{k_0}) = w(I) \geq w(I)/2$. Also, $\mathcal{W}_\psi f \in L^p(w)$ and then $w(\Omega_k) \searrow 0$ as $k \longrightarrow \infty$. By using these facts and the nesting property of the level sets Ω_k, we can conclude that there is an integer k_1 such that $I \in \mathcal{B}_{k_1} \subset \widetilde{\mathcal{D}}$. That is, the coefficients of the intervals $I \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$ are zero. Moreover, as $f \in L^2(\mathbb{R})$, the representation

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\, \psi_{j,k} = \sum_{I\in\mathcal{D}} \langle f, \psi_I\rangle\, \psi_I = \sum_{I\in\widetilde{\mathcal{D}}} \langle f, \psi_I\rangle\, \psi_I$$

converges in $L^2(\mathbb{R})$ and so in the distribution sense. Then it can be expressed as

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} \Big\{ \sum_{\substack{I \subset \tilde{I}^i_k \\ I \in \mathcal{B}_k}} \langle f, \psi_I\rangle\, \psi_I \Big\}, \tag{17}$$

in the distribution sense. We want to prove that this series can be rearranged to obtain an atomic decomposition. Letting $q > q_w$, we define

$$\lambda(k,i) = w(\tilde{I}^i_k)^{\frac{1}{p}-\frac{1}{q}} \Big\| \Big\{ \sum_{\substack{I \subset \tilde{I}^i_k \\ I \in \mathcal{B}_k}} |\langle f, \psi_I\rangle|^2\, |I|^{-1}\, \chi_I \Big\}^{\frac{1}{2}} \Big\|_{L^q(w)},$$

and

$$a_{(k,i)} = \begin{cases} \dfrac{1}{\lambda(k,i)} \displaystyle\sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} \langle f, \psi_I \rangle \, \psi_I & \text{if} \quad \lambda(k,i) \neq 0, \\ 0 & \text{if} \quad \lambda(k,i) = 0. \end{cases}$$

Since $\tilde{I}_k^i$ is a dyadic interval and

$$\|\mathcal{W}_\psi a_{(k,i)}\|_{L^q(w)} = w(\tilde{I}_k^i)^{\frac{1}{q}-\frac{1}{p}},$$

every $a_{(k,i)}$ will be a $(p,q;\psi)$-atom with respect to w. As $q > q_w$, Lemma 4.18 guarantees that every $a_{(k,i)}$ will be a multiple of a (p,q)-atom with respect to w. Then (17) can be written as

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} \lambda(k,i)\, a_{(k,i)} \quad \text{in } \mathcal{S}'. \tag{18}$$

We want to estimate the sum $\sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} |\lambda(k,i)|^p$. Let $n \in \mathbb{Z}$ such that $2\,n \geq q$. By using Hölder's inequality for $\frac{2\,n}{q}$, it follows that

$$\begin{aligned} &\Bigg\| \Big\{ \sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} |\langle f, \psi_I \rangle|^2 \, |I|^{-1} \, \chi_I \Big\}^{\frac{1}{2}} \Bigg\|_{L^q(w)}^q \\ &\leq \Bigg(\int \Big\{ \sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} |\langle f, \psi_I \rangle|^2 \, |I|^{-1} \, \chi_I(x) \Big\}^n w(x)\, dx \Bigg)^{\frac{q}{2n}} w(\tilde{I}_k^i)^{1-\frac{q}{2\,n}}. \end{aligned}$$

Then,

$$\int \Big\{ \sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} \ldots \Big\}^n w(x)\, dx = \sum_{\substack{I_1,\ldots,I_n \subset \tilde{I}_k^i \\ I_1,\ldots,I_n \in \mathcal{B}_k}} \Big(\prod_{j=1}^n |\langle f, \psi_{I_j} \rangle|^2 \, |I_j|^{-1} \Big) \, w(I_1 \cap \ldots \cap I_n).$$

But the dyadic intervals have the property that their intersection is, either the empty set, or else some of them. Moreover, if $I \in \mathcal{B}_k$, $w(I \setminus \Omega_{k+1}) = w(I) - w(I \cap \Omega_{k+1}) > \frac{1}{2}\, w(I)$. Since all the I_j's belong to $\mathcal{B}_k$, we obtain

$$w(I_1 \cap \ldots \cap I_n) \leq 2\, w\Big((I_1 \cap \ldots \cap I_n) \setminus \Omega_{k+1} \Big).$$

Therefore,

$$\int \Big\{ \sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} \dots \Big\}^n w(x)\, dx \le 2 \int_{\tilde{I}_k^i \setminus \Omega_{k+1}} \Big\{ \sum_{\substack{I \subset \tilde{I}_k^i \\ I \in \mathcal{B}_k}} \dots \Big\}^n w(x)\, dx$$

$$\le \; 2 \int_{\tilde{I}_k^i \setminus \Omega_{k+1}} [\mathcal{W}_\psi f(x)]^{2\,n}\, w(x)\, dx \le 2 \cdot (2^{k+1})^{2\,n}\, w(\tilde{I}_k^i),$$

because outside of Ω_{k+1} we have $\mathcal{W}_\psi f(x) \le 2^{k+1}$. Then,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} |\lambda(k,i)|^p \;&\le\; \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} w(\tilde{I}_k^i)^{1-\frac{p}{q}} \left(2^{q\,\frac{2\,n+1}{2\,n}}\, 2^{k\,q}\, w(\tilde{I}_k^i) \right)^{\frac{p}{q}} \\ &\le\; 2^{p\,\frac{2\,n+1}{2\,n}} \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} 2^{k\,p}\, w(\tilde{I}_k^i). \end{aligned}$$

Since $\tilde{I}_k^i \in \mathcal{B}_k$, $w(\tilde{I}_k^i) \le 2\, w(\tilde{I}_k^i \cap \Omega_k)$ and the $\tilde{I}_k^i$ are disjoint. It follows that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} |\lambda(k,i)|^p \;&\le\; 2^{p\,\frac{2\,n+1}{2\,n}}\, 2 \sum_{k\in\mathbb{Z}} 2^{k\,p} \sum_{i\in\Delta_k} w(\tilde{I}_k^i \cap \Omega_k) \\ &\le\; 2^{p\,\frac{2\,n+1}{2\,n}+1} \sum_{k\in\mathbb{Z}} 2^{k\,p}\, w(\Omega_k) \le C\, \|\mathcal{W}_\psi f\|_{L^p(w)}^p, \end{aligned}$$

where the last inequality is consequence of (16). That way has proved that (18) is an atomic decomposition of f in terms of (p,q)-atoms with respect to w, so $f \in H^p(w)$ and

$$\|f\|_{H^p(w)}^p \le C \sum_{k\in\mathbb{Z}} \sum_{i\in\Delta_k} |\lambda(k,i)|^p \le C\, \|\mathcal{W}_\psi f\|_{L^p(w)}^p.$$

□

5 Further results.

Some further consequences are obtained from the characterizations.

5.1 Unconditional Bases.

Lemma 5.1 *Let $\psi \in \mathcal{R}^\alpha$ be an orthonormal wavelet, $\alpha \ge 1$, $0 < p \le 1$ and $w \in A_\infty$ with critical index satisfying $\frac{q_w}{p} \le \alpha$. Then, for all $I \in \mathcal{D}$ we have* *i)* $\psi_I \in H^p(w)$.

ii) The functional ψ_I^* *defined by* $\psi_I^*(f) = \langle f, \psi_I \rangle$ *belongs to* $(H^p(w))^*$.

Proof. By taking $f = \psi_I \in L^2(\mathbb{R})$, we have that $\mathcal{W}_\psi(\psi_I) \in L^p(w)$ and, according with Remark 4.7, this guarantees that $\psi_I \in H^p(w)$. The second part is proved by using Theorem 4.2 and observing that

$$|\langle f, \psi_I \rangle| \leq \frac{|I|^{\frac{1}{2}}}{w(I)^{\frac{1}{p}}} \|\mathcal{W}_\psi f\|_{L^p(w)} \leq C \frac{|I|^{\frac{1}{2}}}{w(I)^{\frac{1}{p}}} \|f\|_{H^p(w)}.$$

□

Theorem 5.2 *If* $0 < p \leq 1$, ψ *is an orthonormal wavelet in* $\mathcal{R}^\alpha$, $\alpha \geq 1$, *and* $w \in A_\infty$ *has critical index* q_w, $\frac{q_w}{p} \leq \alpha$, *then the system*

$$\mathcal{B} = \{\psi_I : I \in \mathcal{D}\}$$

is an unconditional basis for $H^p(w)$.

Proof. The previous lemma says that $\mathcal{B} \subset H^p(w)$ and, since ψ is an orthonormal wavelet, the system

$$\mathcal{B}^* = \{\psi_I^* : I \in \mathcal{D}\} \subset (H^p(w))^*$$

is a biorthogonal system of $\mathcal{B}$. Also, for $\Omega \subset \mathcal{D}$ finite and $\theta = \{\theta_I\}_{I \in \mathcal{D}}$ a sequence of ± 1's we have

$$\begin{aligned} \|S_{\Omega,\theta} f\|_{H^p(w)}^p &= \Big\| \sum_{I \in \Omega} \theta_I \langle f, \psi_I \rangle \psi_I \Big\|_{H^p(w)}^p \leq C \|\mathcal{W}_\psi (S_{\Omega,\theta} f)\|_{L^p(w)}^p \\ &= C \left\| \Big\{ \sum_{I \in \Omega} |\theta_I \langle f, \psi_I \rangle|^2 |I|^{-1} \chi_I \Big\}^{\frac{1}{2}} \right\|_{L^p(w)}^p \leq C \|\mathcal{W}_\psi f\|_{L^p(w)}^p \leq C \|f\|_{H^p(w)}^p. \end{aligned}$$

So the partial sum operators and the modified partial sum operators with any sequence of signs are uniformly bounded in $H^p(w)$. Thus, we should prove that, for every $f \in H^p(w)$, its wavelet expansion converges. But since $\mathcal{W}_\psi f \in L^p(w)$, the series appearing in the operator converges almost everywhere, because all the terms are positive. By using the dominated convergence theorem, we obtain that $\mathcal{W}_\psi(f - S_\Omega f) \longrightarrow 0$ as $\Omega \nearrow \mathcal{D}$ in $L^p(w)$. Theorem 4.2 leads to

$$f = \sum_{I \in \mathcal{D}} \langle f, \psi_I \rangle \, \psi_I \quad \text{in } H^p(w).$$

□

By using the characterization of weighted Lebesgue spaces, we can prove, very much as in the previous result, the following theorem.

Theorem 5.3 *Let $\psi \in \mathcal{R}^1$ be an orthonormal wavelet, $1 < q < \infty$ and $w \in A_q$. Then, the system*

$$\mathcal{B} = \{\psi_I : I \in \mathcal{D}\}$$

is an unconditional basis for $L^q(w)$.

Remark 5.4 *The last two results appear in* [GK1] *when the wavelets are splines. In fact, an m-spline wavelet belongs to $\mathcal{R}^\alpha$ for $m \le \alpha < m + 1$. Related topics can be found in* [GK2].

5.2 Membership criterion.

We have proved that $\mathcal{W}_\psi$ gives a p-norm for $H^p(w)$. We ask if this new p-norm provides a criterion for membership in this space. As in [Mey] —page 144— if we take the distribution $f \equiv 1$, we obtain that $\mathcal{W}_\psi f \equiv 0$ (the wavelet has integral 0), but $f \notin H^p(w)$. This fact leads to restrict the class of distributions in which we can apply the criterion.

Theorem 5.5 *For $0 < p \le 1$, $\psi \in \mathcal{R}^\alpha$ an orthonormal wavelet, $\alpha \ge 1$, and $w \in A_\infty$ with critical index such that $\frac{q_w}{p} \le \alpha$, the following properties hold:*

(a) If $\beta = \{\beta_I\}_{I \in \mathcal{D}}$ is a sequence of complex numbers with

$$\mathcal{W}_\psi \beta = \Big\{ \sum_{I \in \mathcal{D}} |\beta_I|^2 \, |I|^{-1} \, \chi_I \Big\}^{\frac{1}{2}} \in L^p(w),$$

there exists $f \in H^p(w)$ such that

$$f = \sum_{I \in \mathcal{D}} \beta_I \, \psi_I \quad \textit{in } H^p(w), \quad \textit{and} \quad \beta_I = \langle f, \psi_I \rangle.$$

(b) If $f \in \mathcal{S}'$ with $\mathcal{W}_\psi f \in L^p(w)$ and

$$f = \sum_{I \in \mathcal{D}} \langle f, \psi_I \rangle \, \psi_I \quad \textit{in } \mathcal{S}',$$

then $f \in H^p(w)$ and the wavelet expansion converges in $H^p(w)$.

Proof. The part (b) is a consequence of (a): it is only necessary to use the uniqueness of limits and the fact that the convergence in $H^p(w)$ is stronger than the convergence in the distribution sense. For (a), we have $\mathcal{W}_\psi \beta \in L^p(w)$ and all the terms appear in the series are positive, so

$$\{\mathcal{W}_\psi \beta(x)\}^2 = \sum_{I \in \mathcal{D}} |\beta_I|^2 \, |I|^{-1} \, \chi_I(x) \quad \text{a.e.} x \in \mathbb{R}.$$

By using the dominated convergence theorem, we can obtain that the tail converges to 0 in $L^{\frac{p}{2}}(w)$. Via Theorem 4.2, the partial sum operator (of the wavelet expansion) is a Cauchy sequence in $H^p(w)$. But $H^p(w)$ is complete, so there exists $f \in H^p(w)$ such that

$$f = \sum_{I \in \mathcal{D}} \beta_I \, \psi_I \quad \text{en } H^p(w).$$

On the other hand, $\psi_I \in (H^p(w))^*$ and it follows that $\beta_I = \langle f, \psi_I \rangle$. □

Again, we can obtain a similar result in $L^q(w)$, $1 < q < \infty$.

Theorem 5.6 *Let $\psi \in \mathcal{R}^1$ be an orthonormal wavelet, $1 < q < \infty$ and $w \in A_q$. Then,*

(a) For $\beta = \{\beta_I\}_{I \in \mathcal{D}}$ a sequence of complex numbers with

$$\mathcal{W}_\psi \beta = \Big\{ \sum_{I \in \mathcal{D}} |\beta_I|^2 \, |I|^{-1} \, \chi_I \Big\}^{\frac{1}{2}} \in L^q(w),$$

there exists $f \in L^q(w)$ such that

$$f = \sum_{I \in \mathcal{D}} \beta_I \, \psi_I \quad \textit{in } L^q(w), \quad \textit{and} \quad \beta_I = \langle f, \psi_I \rangle.$$

(b) If $f \in \mathcal{S}'$ with $\mathcal{W}_\psi f \in L^q(w)$ and

$$f = \sum_{I \in \mathcal{D}} \langle f, \psi_I \rangle \, \psi_I \quad \textit{in } \mathcal{S}',$$

then $f \in L^q(w)$ and the wavelet expansion converges in $L^q(w)$.

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