On the existence of principal values for the Cauchy integral on weighted Lebesgue spaces for non-doubling measures.

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Abstract

Let T be a Calderón-Zygmund operator in a "non-homogeneous" space $(\mathbb{X}, d, \mu)$, where, in particular, the measure μ may be non-doubling. Much of the classical theory of singular integrals has been recently extended to this context by F. Nazarov, S. Treil and A. Volberg and, independently by X. Tolsa. In the present work we study some weighted inequalities for $T_\star$, which is the supremum of the truncated operators associated with T. Specifically, for $1 < p < \infty$, we obtain sufficient conditions for the weight in one side, which guarantee that another weight exists in the other side, so that the corresponding L^p weighted inequality holds for $T_\star$. The main tool to deal with this problem is the theory of vector-valued inequalities for $T_\star$ and some related operators. We discuss it first by showing how these operators are connected to the general theory of vector-valued Calderón-Zygmund operators in non-homogeneous spaces, developed in our previous paper [GM]. For the Cauchy integral operator $\mathcal{C}$, which is the main example, we apply the two-weight inequalities for $\mathcal{C}_\star$ to characterize the existence of principal values for functions in weighted L^p.

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1 Introduction.

A non-homogeneous space $(\mathbb{X}, d)$ will be a separable metric space endowed with a non-negative "n-dimensional" Borel measure μ, that is,

$$\mu(B(x,r)) \leq r^n, \qquad \text{for all } x \in \mathbb{X}, r > 0,$$

where $B(x,r) = \{y \in \mathbb{X} : d(x,y) \leq r\}$ and n is a fixed positive number (not necessarily an integer).

Definition 1.1 *A bounded linear operator T on $L^2(\mu)$ is said to be a Calderón-Zygmund operator with "n-dimensional" kernel K if for every $f \in L^2(\mu)$,*

$$Tf(x) = \int_{\mathbb{X}} K(x,y)\, f(y)\, d\mu(y), \qquad \textit{for } \mu\textit{-almost every } x \in \mathbb{X} \setminus \operatorname{supp} f,$$

where, for some $A > 0$, $\delta > 0$, $K : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{C}$ satisfies

(*i*) $|K(x,y)| \leq \dfrac{A}{d(x,y)^n}$, *for all $x \neq y$;*

(*ii*) $|K(x,y) - K(x',y)|$, $|K(y,x) - K(y,x')| \leq A \dfrac{d(x,x')^\delta}{d(x,y)^{n+\delta}}$ *whenever x, x', $y \in \mathbb{X}$ and $d(x,y) \geq 2\, d(x,x')$.*

For such a T, we define the maximal operator associated with it as follows:

$$T_\star f(x) = \sup_{r>0} |T_r f(x)| = \sup_{r>0} \Big| \int_{\mathbb{X}\setminus B(x,r)} K(x,y)\, f(y)\, d\mu(y) \Big|.$$

This kind of operators have been introduced by [NTV2], where strong and weak type estimates for them as well as for the corresponding maximal operators have been obtained. The main example is the Cauchy integral $\mathcal{C}$ given by the "1-dimensional" kernel $K(z,\xi) = \frac{1}{z-\xi}$; where the metric space is $\mathbb{C}$ endowed with some measure which has linear growth (that is, $n = 1$) and such that this operator is bounded in $L^2(\mu)$. This kind of measures has been characterized in [To1]. The existence of the principal value is treated in [To2]. Many references about the development of the Cauchy integral and other topics related with it can be found in [Da2], [Da1], [Chr], [Mur], [MMV]. About the boundedness on $L^2(\mu)$ of these Calderón-Zygmund operators, the reader is also referred to [NTV1] and for the special case of the Cauchy integral to [Ver].

In our previous work [GM], we were concerned with the following problem: for $1 < p < \infty$, find conditions on $0 \le v < \infty$ μ-a.e. (resp. $u > 0$ μ-a.e.) such that

$$\int_{\mathbb{X}} |Tf(x)|^p\, u(x)\, d\mu(x) \le \int_{\mathbb{X}} |f(x)|^p\, v(x)\, d\mu(x), \qquad \text{for } f \in L^p(v) = L^p(v\, d\mu),$$

holds for some $u > 0$ μ-a.e. (resp. $0 \le v < \infty$ μ-a.e.).

We introduced there the classes of weights D_p and Z_p, $1 < p < \infty$, which were defined by

$$\begin{aligned} D_p &= \Big\{0 \le w < \infty \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x)^{1-p'}\, (1 + d(x, x_0))^{-n\,p'}\, d\mu(x) < \infty\Big\} \\ Z_p &= \Big\{w > 0 \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x)\, (1 + d(x, x_0))^{-n\,p}\, d\mu(x) < \infty\Big\}. \end{aligned}$$

for some $x_0 \in \mathbb{X}$. Note that these classes of weights do not depend on the point x_0 and that this definition becomes simpler for finite diameter spaces. By using vector-valued methods we obtained in that work, that the answer to this problem is $v \in D_p$ (resp. $u \in Z_p$). For the case of Cauchy integral operator, we even obtained the necessity of these classes.

The aim of the present work is to extend these results to the maximal operator associated with T. We shall prove that these classes are sufficient to have the desired inequality when we replace T by $T_\star$. As in [GM], the main tool will be the vector-valued theory developed there. We shall see that, after some modifications, the maximal operator fits into that theory. This will allow us to obtain some vector-valued inequalities which will be the key to cope with the weighted inequalities.

For the case of the Cauchy integral operator further consequences will be obtained. We shall be able to prove some results about the existence of principal values on weighted Lebesgue spaces. Namely, we obtain the equivalence of the existence of principal values, the finiteness almost everywhere of the supremum of the truncated Cauchy integrals, some two-weight inequalities and the fact that one of the weights belongs to the corresponding class. The main result, which is proved in Section 4, is:

Theorem 1.2 *Let μ be a "1-dimensional" measure in $\mathbb{C}$ such that the Cauchy integral operator $\mathcal{C}$ is bounded on $L^2(\mu)$. Take $1 < p < \infty$ and let v be a μ-a.e. positive measurable function such that $v \in L^1_{loc}(\mu)$. Then the following statements are equivalent:*

(a) $v \in D_p$.

(b) *There exists a μ-a.e. positive measurable function u, such that,*

$$\int_{\mathbb{C}} |\mathcal{C}f(z)|^p\, u(z)\, d\mu(z) \le C \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z), \quad \text{for any } f \in L^p(v).$$

(c) *There exists a μ-a.e. positive measurable function u, such that,*

$$\int_{\{z\in\mathbb{C}:|\mathcal{C}f(z)|>\lambda\}} u(z)\, d\mu(z) \le \frac{C}{\lambda^p} \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z),$$

for any $f \in L^p(v), \lambda > 0$.

(d) *There exists a μ-a.e. positive measurable function u, such that,*

$$\int_{\mathbb{C}} (\mathcal{C}^\star f(z))^p\, u(z)\, d\mu(z) \le C \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z), \quad \text{for any } f \in L^p(v).$$

(e) *There exists a μ-a.e. positive measurable function u, such that,*

$$\int_{\{z\in\mathbb{C}:\mathcal{C}^\star f(z)>\lambda\}} u(z)\, d\mu(z) \le \frac{C}{\lambda^p} \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z),$$

for any $f \in L^p(v), \lambda > 0$.

(f) *If $f \in L^p(v)$, the principal value*

$$\lim_{\varepsilon\to 0} \mathcal{C}^\varepsilon f(z) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|z-\xi|<\frac{1}{\varepsilon}} \frac{f(\xi)}{z-\xi}\, d\mu(\xi)$$

exists for μ-a.e. $z \in \mathbb{C}$.

(g) *If $f \in L^p(v)$, then $\mathcal{C}^\star f < \infty$ μ-a.e..*

Note that we have introduced here a slightly different type of truncated integrals $\mathcal{C}^\varepsilon$ and we have denoted by $\mathcal{C}^\star$ the supremum of them when $0 < \varepsilon < 1$. The same equivalences for $\mathcal{C}_\star$ will be also obtained.

The plan of the paper is the following. Section 2 is devoted to prove vector-valued inequalities for the maximal operator. To do this we shall handle some "smooth" version of it which will be a vector-valued Calderón-Zygmund operator. In Section 3 we obtain the sufficiency of the classes of weights for the problem we are dealing with. Finally, in Section 4 we prove the main result.

2 Vector-valued inequalities for maximal operators.

Let $\mathbb{A}$, $\mathbb{B}$ be a couple of Banach spaces, and denote by $\mathcal{L}(\mathbb{A},\mathbb{B})$ the set of bounded operators from $\mathbb{A}$ to $\mathbb{B}$. Roughly speaking, $K : \mathbb{X}\times\mathbb{X} \longrightarrow \mathcal{L}(\mathbb{A},\mathbb{B})$ is a vector-valued "n-dimensional" Calderón-Zygmund kernel if it verifies (i) and (ii) (of Definition 1.1) but with the $\mathcal{L}(\mathbb{A},\mathbb{B})$-norm instead of the complex modulus. In the same manner, T is a vector-valued Calderón-Zygmund operator if it is given by one of these kernels away from the support of the functions and if it extends to a bounded operator between $L^2_{\mathbb{A}}(\mu)$ and $L^2_{\mathbb{B}}(\mu)$. For the precise definition and more details, the reader is referred to [GM]. The classical theory for these operators can be found in [BCP], [RRT] and [GR].

In [GM], we proved that these operators satisfy the same inequalities as the "scalar" ones treated in [NTV2]. That is, they are bounded from $L^1_{\mathbb{A}}(\mu)$ to $L^{1,\infty}_{\mathbb{B}}(\mu)$ and from $L^p_{\mathbb{A}}(\mu)$ to $L^p_{\mathbb{B}}(\mu)$, for $1<p<\infty$. Besides, that result had an immediate self-improvement: with no extra hypothesis the operator can be extended to sequence-valued functions. The concrete result we obtained there, is as follows:

Theorem 2.1 *Let T be a vector-valued Calderón-Zygmund operator and take q, $1<q<\infty$. Then*

(i) T is bounded from $L^1_{\ell^q_{\mathbb{A}}}(\mu)$ to $L^{1,\infty}_{\ell^q_{\mathbb{B}}}(\mu)$, that is,

$$\mu\Big\{x : \Big\{\sum_j \|Tf_j(x)\|^q_{\mathbb{B}}\Big\}^{\frac{1}{q}} > \lambda\Big\} \le \frac{C}{\lambda}\int_{\mathbb{X}}\Big\{\sum_j \|f_j(x)\|^q_{\mathbb{A}}\Big\}^{\frac{1}{q}}\,d\mu(x).$$

(ii) T is bounded from $L^p_{\ell^q_{\mathbb{A}}}(\mu)$ to $L^p_{\ell^q_{\mathbb{B}}}(\mu)$, for $1<p<\infty$, that is,

$$\Big\|\Big\{\sum_j \|Tf_j\|^q_{\mathbb{B}}\Big\}^{\frac{1}{q}}\Big\|_{L^p(\mu)} \le C\,\Big\|\Big\{\sum_j \|f_j\|^q_{\mathbb{A}}\Big\}^{\frac{1}{q}}\Big\|_{L^p(\mu)}.$$

In what follows T will be a (scalar) Calderón-Zygmund operator with "n-dimensional" kernel K (see Definition 1.1). For such a T recall the definition of the maximal operator associated with it,

$$T_\star f(x) = \sup_{r>0}|T_r f(x)| = \sup_{r>0}\Big|\int_{\mathbb{X}\setminus B(x,r)} K(x,y)\,f(y)\,d\mu(y)\Big|,$$

which can be viewed as a vector-valued operator. First observe that it is enough to take the supremum just over $\mathbb{Q}^+$, the set of positive rationals.

Next consider the $\ell^\infty(\mathbb{Q}^+)$-valued operator defined by $\widetilde{T}f(x) = \{T_r f(x)\}_{r\in\mathbb{Q}^+}$. That is, we choose $\mathbb{A} = \mathbb{C}$ and $\mathbb{B} = \ell^\infty(\mathbb{Q}^+)$. The kernel will be $\widetilde{K}(x,y) = \{\chi_{\mathbb{X}\setminus B(x,r)}(y)\, K(x,y)\}_{r\in\mathbb{Q}^+}$ taken like an element of $\mathcal{L}(\mathbb{C}, \ell^\infty(\mathbb{Q}^+)) = \ell^\infty(\mathbb{Q}^+)$. However it will not be a Calderón-Zygmund kernel, since in general condition (ii) is not satisfied. This happens because we are cutting off with characteristic functions which give us a "rough kernel". Then we can not obtain, in a straightforward way, vector-valued inequalities —like in Theorem 2.1— for the operator $T_\star$. We avoid this problem by introducing a smooth approximation to the characteristic function. When we will estimate the difference between the smooth operator and the previous one, the following version of the Hardy-Littlewood maximal function will be needed:

$$\widehat{\mathcal{M}}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

For this operator we observe the same behaviour as for $T_\star$, although we can look at this operator under a vector-valued point of view, the kernel does not verify, in general, condition (ii). So, it will not be a Calderón-Zygmund operator and we have to get a smooth version of it.

To be more precise, take $\varphi, \psi \in C^1([0,\infty))$, such that,

$$\chi_{[0,1]}(t) \le \varphi(t) \le \chi_{[0,2]}(t) \qquad \text{and} \qquad \chi_{[1,\infty)}(t) \le \psi(t) \le \chi_{[\frac{1}{2},\infty)}(t).$$

Let us consider the following "smooth" version of the maximal function $\widehat{\mathcal{M}}$:

$$\mathcal{M}_\varphi f(x) = \sup_{r>0} \frac{1}{r^n} \int_{\mathbb{X}} \varphi\Big(\frac{d(x,y)}{r}\Big)\, |f(y)|\, d\mu(y).$$

Then, $\widehat{\mathcal{M}}f(x) \le \mathcal{M}_\varphi f(x) \le 2^n\, \widehat{\mathcal{M}}f(x)$. On the other hand it is also clear that $\widehat{\mathcal{M}}f(x) \le 3^n\, \widetilde{\mathcal{M}}f(x)$, where

$$\widetilde{\mathcal{M}}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,3\,r))} \int_{B(x,r)} |f|\, d\mu$$

is the maximal operator considered in [NTV2]. The boundedness of $\widetilde{\mathcal{M}}$ (which is proved there, as an easy consequence of Vitali covering theorem) allows us to obtain that the operators $\widehat{\mathcal{M}}$ and $\mathcal{M}_\varphi$ are of weak type $(1,1)$ and strong type (p,p), for $1 < p \le \infty$. For the maximal operator associated with T, we perform the regularization given by

$$T_{\psi,\star}f(x) = \sup_{r>0} \Big| \int_{\mathbb{X}} \psi\Big(\frac{d(x,y)}{r}\Big)\, K(x,y)\, f(y)\, d\mu(y)\Big|.$$

Hence, by using the properties of K,

$$|T_{\psi,\star}f(x) - T_\star f(x)| \le 2^n\, A\, \widehat{\mathcal{M}} f(x) \le 2^n\, A\, \mathcal{M}_\varphi f(x). \tag{1}$$

This estimate gives the boundedness of $T_{\psi,\star}$ in $L^2(\mu)$, because $\mathcal{M}_\varphi$ is bounded and, as it is proved in [NTV2], $T_\star$ is also continuous in this space. On the other hand, (1) leads to

$$T_\star f(x) \le T_{\psi,\star} f(x) + 2^n\, A\, \mathcal{M}_\varphi f(x), \tag{2}$$

which will be useful to get vector-valued inequalities for $T_\star$, once we have proved these inequalities for the "smooth" operators $T_{\psi,\star}$ and $\mathcal{M}_\varphi$.

The next step is to prove that these two operators are in fact vector-valued Calderón-Zygmund operators. Then, after using the self-improvement (Theorem 2.1) we can extend them to sequence spaces.

Theorem 2.2 *Under the previous assumptions, $T_{\psi,\star}$ and $\mathcal{M}_\varphi$ can be viewed as vector-valued Calderón-Zygmund operators. Therefore these operators, and, consequently, also $T_\star$ will be of weak type $(1,1)$ and strong type (p,p), for $1 < p < \infty$ —for $T_\star$ these estimates were proved in* [NTV2]*—. Furthermore, if $1 < q < \infty$, the following vector-valued inequalities hold*

(i) $T_{\psi,\star}$, $\mathcal{M}_\varphi$ and $T_\star$ are bounded from $L^1_{\ell^q}(\mu)$ to $L^{1,\infty}_{\ell^q}(\mu)$.

(ii) $T_{\psi,\star}$, $\mathcal{M}_\varphi$ and $T_\star$ are bounded from $L^p_{\ell^q}(\mu)$ to $L^p_{\ell^q}(\mu)$, for $1 < p < \infty$.

Proof. Note that by (2) we only have to get these inequalities for $T_{\psi,\star}$ and $\mathcal{M}_\varphi$. For the first one, by the monotone convergence theorem, it is enough to consider the $\ell^\infty(J)$-valued operator

$$T^J_\psi f(x) = \{T_{\psi,r} f(x)\}_{r\in J} = \Big\{ \int_{\mathbb{X}} \psi\Big(\frac{d(x,y)}{r}\Big)\, K(x,y)\, f(y)\, d\mu(y)\Big\}_{r\in J},$$

with $J \subset \mathbb{Q}^+$ finite, and to obtain boundedness properties independently of J. We choose $\mathbb{A} = \mathbb{C}$ and $\mathbb{B} = \ell^\infty(J)$. The *a priori* estimate of the operator is obtained by using that $T_{\psi,\star}$ is continuous in $L^2(\mu)$,

$$\|T^J_\psi f\|_{L^2_{\ell^\infty(J)}(\mu)} \le \|T_{\psi,\star} f\|_{L^2(\mu)} \le C\, \|f\|_{L^2(\mu)},$$

where C is independent of J. This operator is given by the kernel

$$K^J_\psi(x,y) = \Big\{\psi\Big(\frac{d(x,y)}{r}\Big)\, K(x,y)\Big\}_{r\in J} \in \mathcal{L}(\mathbb{C}, \ell^\infty(J)) = \ell^\infty(J).$$

The properties of K and the assumptions on ψ yield that K_ψ^J is a Calderón-Zygmund kernel with constants which do not depend on J. Thus, T_ψ^J is a vector-valued Calderón-Zygmund operator such that every constant involved is independent of the chosen subset J. So, by the observations above and by the vector-valued Theorem obtained in [GM], we conclude that T_ψ is continuous from $L^1(\mu)$ to $L^{1,\infty}_{\ell^\infty}(\mu)$ and from $L^p(\mu)$ to $L^p_{\ell^\infty}(\mu)$, $1 < p < \infty$. Or equivalently, that $T_{\psi,\star}$ is of weak type $(1,1)$ and strong type (p,p) for $1 < p < \infty$. Furthermore, by using the self-improvement (Theorem 2.1), for $1 < q < \infty$ we have

$$\begin{aligned} &T_\psi : L^1_{\ell^q}(\mu) \longrightarrow L^{1,\infty}_{\ell^q_{\ell^\infty}}(\mu), \\ &T_\psi : L^p_{\ell^q}(\mu) \longrightarrow L^p_{\ell^q_{\ell^\infty}}(\mu), \quad 1 < p < \infty, \end{aligned}$$

which, indeed, are the desired inequalities for $T_{\psi,\star}$.

For $\mathcal{M}_\varphi$, we can also take the supremum over $\mathbb{Q}^+$. For some finite $J \subset \mathbb{Q}^+$, define the $\ell^\infty(J)$-valued operator

$$T_\varphi^J f(x) = \{T_{\varphi,r} f(x)\}_{r\in J} = \Big\{ \frac{1}{r^n} \int_{\mathbb{X}} \varphi\Big(\frac{d(x,y)}{r}\Big) f(y)\, d\mu(y) \Big\}_{r\in J}.$$

Take the Banach spaces $\mathbb{A} = \mathbb{C}$ and $\mathbb{B} = \ell^\infty(J)$. Observe that

$$\|T_\varphi^J f\|_{L^2_{\ell^\infty(J)}(\mu)} \le \|\mathcal{M}_\varphi f\|_{L^2(\mu)} \le C\, \|f\|_{L^2(\mu)},$$

where the constant C does not depend on J. The kernel of T_φ^J is

$$K_\varphi^J(x,y) = \Big\{ \frac{1}{r^n} \varphi\Big(\frac{d(x,y)}{r}\Big) \Big\}_{r\in J} \in \mathcal{L}(\mathbb{C}, \ell^\infty(J)) = \ell^\infty(J).$$

The properties of φ imply that it is a Calderón-Zygmund kernel. Thus, T_φ^J is a vector-valued Calderón-Zygmund operator and, moreover, every constant involved is independent of J. Just as before, if we evaluate this operator on $|f|$ and we use the monotone convergence theorem, we obtain the desired inequalities for $\mathcal{M}_\varphi$. $\square$

Consider the $\ell^\infty(J)$-valued operator defined by $\widetilde{T}^J f(x) = \{T_r f(x)\}_{r\in J}$, for $J \subset \mathbb{Q}^+$ finite. Theorem 2.2 guarantees that $\widetilde{T}^J$ is continuous from $L^1(\mu)$ to $L^{1,\infty}_{\ell^\infty(J)}(\mu)$, and from $L^p(\mu)$ to $L^p_{\ell^\infty(J)}(\mu)$, $1 < p < \infty$. Moreover, when these spaces are ℓ^q-valued $(1 < q < \infty)$, the corresponding vector-valued inequalities hold (see Theorem 2.2). In fact, each estimate is uniform in J.

The adjoint operator of T is T^*, whose kernel is $K^*(x,y) = K(y,x)$. For $f = \{f_r\}_{r\in J}$, we define

$$\widehat{T}^J f(x) = \sum_{r\in J} T_r^* f_r(x) = \sum_{r\in J} \int_{\mathbb{X}\setminus B(x,r)} K(y,x)\, f_r(y)\, d\mu(y).$$

Then, it is clear that the adjoint of $\widehat{T}^J$ is $\widetilde{T}^J$ and that $\widehat{T}^J$ is bounded between $L^2_{\ell^1(J)}(\mu)$ and $L^2(\mu)$ with constant independent of J. It will not be a Calderón-Zygmund operator, because the kernel does not satisfy the "smoothness" condition (ii). Then, by using the functions ψ, φ, define

$$\widehat{T}^J_\psi f(x) = \sum_{r\in J} T^*_{\psi,r} f_r(x) = \sum_{r\in J} \int_{\mathbb{X}} \psi\Big(\frac{d(x,y)}{r}\Big)\, K(y,x)\, f_r(y)\, d\mu(y).$$

Just as we did in (1), we can estimate

$$|\widehat{T}^J_\psi f(x) - \widehat{T}^J f(x)| \leq 2^n\, A\, \widehat{T}^J_\varphi(|f|)(x), \tag{3}$$

where, for $g = \{g_r\}_{r\in J}$, $|g| = \{|g_r|\}_{r\in J}$ and

$$\widehat{T}^J_\varphi g(x) = \sum_{r\in J} T_{\varphi,r} g_r(x) = \sum_{r\in J} \frac{1}{r^n} \int_{\mathbb{X}} \varphi\Big(\frac{d(x,y)}{r}\Big) g_r(y)\, d\mu(y).$$

Observe that the adjoint operator of $\widehat{T}^J_\varphi$ is T^J_φ. Then, we obtain that $\widehat{T}^J_\varphi$ is bounded from $L^2_{\ell^1(J)}(\mu)$ to $L^2(\mu)$ independently of J. On the other hand, by means of (3) we can see that $\widehat{T}^J_\psi$ is continuous from $L^2_{\ell^1(J)}(\mu)$ to $L^2(\mu)$ uniformly in J. But, it is also true that

$$|\widehat{T}^J f(x)| \leq |\widehat{T}^J_\psi f(x)| + 2^n\, A\, \widehat{T}^J_\varphi(|f|)(x), \tag{4}$$

and this inequality allows us to get vector-valued estimates for $\widehat{T}^J$, once we have proved them for the other two operators. The next result will be a consequence of Theorem 2.2.

Corollary 2.3 *Under the previous hypotheses $\widehat{T}^J_\psi$ and $\widehat{T}^J_\varphi$ are vector-valued Calderón-Zygmund operators. Consequently for $1 < p, q < \infty$, we obtain that these two operators together with $\widehat{T}^J$ are continuous between the following spaces:*

(i) $L^1_{\ell^1(J)}(\mu) \longrightarrow L^{1,\infty}(\mu)$; $L^p_{\ell^1(J)}(\mu) \longrightarrow L^p(\mu)$.

(ii) $L^1_{\ell^q_{\ell^1(J)}}(\mu) \longrightarrow L^{1,\infty}_{\ell^q}(\mu)$; $L^p_{\ell^q_{\ell^1(J)}}(\mu) \longrightarrow L^p_{\ell^q}(\mu)$.

Furthermore, every boundedness does not depend on J.

Proof. By the vector-valued results in [GM] and Theorem 2.1, it is enough to check that $\widehat{T}^J_\psi$ and $\widehat{T}^J_\varphi$ are vector-valued Calderón-Zygmund operators.

For the first one, the kernel is $\widehat{K}_\psi^J(x,y) = K_\psi^J(y,x) \in \mathcal{L}(\ell^1(J),\mathbb{C}) = \ell^\infty(J)$. Since, we proved in Theorem 2.2 that K_ψ^J, satisfies (i), (ii), then $\widehat{K}_\psi^J$ also does with the Banach spaces $\mathbb{A} = \ell^1(J)$, $\mathbb{B} = \mathbb{C}$. For the other operator the kernel is $\widehat{K}_\varphi^J(x,y) = K_\varphi^J(x,y) \in \mathcal{L}(\ell^1(J),\mathbb{C}) = \ell^\infty(J)$. We showed in Theorem 2.2 that this kernel verifies the required hypotheses; as a consequence, $\widehat{T}_\varphi^J$ will be a vector-valued Calderón-Zygmund operator (here we also have the previous Banach spaces). $\square$

Remark 2.4 *In the last result, there is no dependence on J, so by taking finite subsets $J \nearrow \mathbb{Q}^+$, the limit operator $\widehat{T}$ does not depend on this sequence. This new operator acts continuously between the previous spaces with $\ell^1(\mathbb{Q}^+)$ instead of $\ell^1(J)$. Moreover, the restriction of $\widehat{T}$ to a $\ell^1(\mathbb{Q}^+)$-valued sequence with a finite number of non-zero coordinates can be written as $\widehat{T}^J$ for some J.*

3 Two-weight inequalities for $T_\star$.

If $1 < p < \infty$, consider the two-weight inequality for $T_\star$:

$$\int_{\mathbb{X}} (T_\star f(x))^p \, u(x) \, d\mu(x) \le \int_{\mathbb{X}} |f(x)|^p \, v(x) \, d\mu(x), \tag{5}$$

for $f \in L^p(v) = L^p(v\,d\mu)$ and u, v μ-a.e. positive functions. Throughout this section we shall be concerned with the following problem:

> *Find conditions on $0 \le v < \infty$ μ-a.e. (resp. $u > 0$ μ-a.e.) such that (5) is satisfied by some $u > 0$ μ-a.e. (resp. $0 \le v < \infty$ μ-a.e.).*

This problem has been previously studied in [GM] for T. We would like to follow those ideas to conclude similar results for $T_\star$. As there, we shall need the following theorem, proved in [FT], which establishes the relationship between vector-valued inequalities and weights (for closely related results see [GR] pp. 549–554).

Theorem 3.1 *Let $(\mathbb{Y}, d\nu)$ be a measure space; $\mathbb{F}$, $\mathbb{G}$ Banach spaces, and $\{A_k\}_{k\in\mathbb{Z}}$ a sequence of pairwise disjoint measurable subsets of $\mathbb{Y}$ such that $\mathbb{Y} = \bigcup_k A_k$. Consider $0 < s < p < \infty$ and T a sublinear operator which satisfies the following vector-valued inequality*

$$\left\| \Big\{ \sum_j \|Tf_j\|_{\mathbb{G}}^p \Big\}^{\frac{1}{p}} \right\|_{L^s(A_k, d\nu)} \le C_k \Big\{ \sum_j \|f_j\|_{\mathbb{F}}^p \Big\}^{\frac{1}{p}}, \quad k \in \mathbb{Z}, \tag{6}$$

where, for every $k \in \mathbb{Z}$, C_k only depends on $\mathbb{F}$, $\mathbb{G}$, p and s. Then, there exists a positive function $u(x)$ on $\mathbb{Y}$ such that

$$\Big\{ \int_{\mathbb{Y}} \|Tf(x)\|_{\mathbb{G}}^p \, u(x) \, d\nu(x) \Big\}^{\frac{1}{p}} \leq C \|f\|_{\mathbb{F}},$$

where C depends on $\mathbb{F}$, $\mathbb{G}$, p and s. Moreover, given a sequence of positive numbers $\{a_k\}_{k\in\mathbb{Z}}$ with $\sum_k a_k^p < \infty$, and $\sigma = \left(\frac{p}{s}\right)'$, $u(x)$ can be found in such a way that $\|u^{-1} \chi_{A_k}\|_{L^{\sigma-1}(A_k, d\mu)} \leq (a_k^{-1} C_k)^p$.

In our context $(\mathbb{Y}, d\nu) = (\mathbb{X}, d\mu)$ which is a σ-finite measure space. Then, a simple argument shows that the weight u can be also taken so that $u < \infty$ μ-a.e..

Given $1 < p < \infty$ and some $x_0 \in \mathbb{X}$, remember the definition of the classes of weights in $\mathbb{X}$:

$$\begin{aligned} D_p &= \Big\{ 0 \leq w < \infty \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x)^{1-p'} \, (1 + d(x, x_0))^{-n\,p'} \, d\mu(x) < \infty \Big\} \\ Z_p &= \Big\{ w > 0 \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x) \, (1 + d(x, x_0))^{-n\,p} \, d\mu(x) < \infty \Big\}. \end{aligned}$$

Note that these classes of weights do not depend on the point x_0.

Remark 3.2 *In the case that the diameter of the space is finite, there exists R such that $\mathbb{X} \subset B(x_0, R)$ and so $\mu(\mathbb{X}) \leq R^n < \infty$. Thus, the previous classes can be given by the equivalent definition:*

$$\begin{aligned} D_p &= \Big\{ 0 \leq w < \infty \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x)^{1-p'} \, d\mu(x) < \infty \Big\} \\ Z_p &= \Big\{ w > 0 \ \mu\text{-a.e.} : \int_{\mathbb{X}} w(x) \, d\mu(x) < \infty \Big\}. \end{aligned}$$

If the support of the measure is a bounded set, by restricting the whole space to this set, we are in the previous case.

In the next result we prove the vector-valued inequalities which will be needed to apply theorem 3.1.

Proposition 3.3 *Take $0 < s < 1 < p < \infty$ and $v \in D_p$. Then, if the diameter of $\mathbb{X}$ is infinite, we have*

$$\text{(i)} \quad \Big\| \Big\{ \sum_j (T_\star f_j)^p \Big\}^{\frac{1}{p}} \Big\|_{L^s(S_k, d\,\mu)} \leq C_{s,p} \, 2^{\frac{k\,n}{s}} \, \Big\{ \sum_j \|f_j\|_{L^p(v\,d\mu)}^p \Big\}^{\frac{1}{p}},$$

(ii) $\displaystyle \Big\|\Big\{\sum_j |\widehat{T}f_j|^p\Big\}^{\frac{1}{p}}\Big\|_{L^s(S_k,d\,\mu)} \le C_{s,p}\, 2^{\frac{k\,n}{s}} \Big\{\sum_j \|f_j\|^p_{L^p_{\ell^1(\mathbb{Q}^+)}(v\,d\mu)}\Big\}^{\frac{1}{p}},$

for $k = 0, 1, \ldots,$ *where* $S_0 = \{x : d(x, x_0) \le 1\}$ *and* $S_k = \{x : 2^{k-1} < d(x, x_0) \le 2^k\}$*, for* $k = 1, 2, \ldots$. *Otherwise,*

(i)′ $\displaystyle \Big\|\Big\{\sum_j (T_\star f_j)^p\Big\}^{\frac{1}{p}}\Big\|_{L^s(\mu)} \le C_{s,p} \Big\{\sum_j \|f_j\|^p_{L^p(v\,d\mu)}\Big\}^{\frac{1}{p}},$

(ii)′ $\displaystyle \Big\|\Big\{\sum_j |\widehat{T}f_j|^p\Big\}^{\frac{1}{p}}\Big\|_{L^s(\mu)} \le C_{s,p} \Big\{\sum_j \|f_j\|^p_{L^p_{\ell^1(\mathbb{Q}^+)}(v\,d\mu)}\Big\}^{\frac{1}{p}}.$

Proof. The proof is similar to what we did for T in [GM]. First, we work in the case when $\mathbb{X}$ has infinite diameter. Fix $k \ge 0$ and set $B_{k+1} = B(x_0, 2^{k+1})$. We decompose every function $f = f' + f'' = f\,\chi_{B_{k+1}} + f\,\chi_{\mathbb{X}\setminus B_{k+1}}$. Then, for $x \in S_k$, since $v \in D_p$, it follows

$$\begin{aligned} T_\star f''(x) &\le \int_{\mathbb{X}\setminus B_{k+1}} \frac{A}{d(x,y)^n}\,|f(y)|\,d\mu(y) \\ &\le 4^n A \int_{\mathbb{X}} (1 + d(y,x_0))^{-n}\,|f(y)|\,v(y)^{\frac{1}{p}}\,v(y)^{-\frac{1}{p}}\,d\mu(y) \\ &\le 4^n A \Big\{\int_{\mathbb{X}} |f(y)|^p\,v(y)\,d\mu(y)\Big\}^{\frac{1}{p}} \Big\{\int_{\mathbb{X}} \frac{v(y)^{1-p'}}{(1+d(y,x_0))^{n\,p'}}\,d\mu(y)\Big\}^{\frac{1}{p'}} \\ &\le C\,\|f\|_{L^p(v\,d\mu)}. \end{aligned}$$

Due to the fact that $\mu(S_k) \le \mu(B_k) \le 2^{k\,n}$, we prove

$$\Big\|\Big\{\sum_j (T_\star f_j'')^p\Big\}^{\frac{1}{p}}\Big\|_{L^s(S_k,d\mu)} \le C\,2^{\frac{k\,n}{s}} \Big\{\sum_j \|f_j\|^p_{L^p(v\,d\mu)}\Big\}^{\frac{1}{p}}.$$

On the other hand, since $0 < s < 1$, by Kolmogorov inequality (see [GR] p. 485) and Theorem 2.2, we can obtain

$$\begin{aligned} &\Big\|\Big\{\sum_j (T_\star f_j')^p\Big\}^{\frac{1}{p}}\Big\|_{L^s(S_k,d\mu)} \le C_s\,\mu(S_k)^{\frac{1}{s}-1} \Big\|\Big\{\sum_j (T_\star f_j')^p\Big\}^{\frac{1}{p}}\Big\|_{L^{1,\infty}(S_k,d\mu)} \\ &\le C\,\mu(S_k)^{\frac{1}{s}-1} \int_{B_{k+1}} \Big\{\sum_j |f_j(x)|^p\Big\}^{\frac{1}{p}} v(x)^{\frac{1}{p}}\,v(x)^{-\frac{1}{p}}\,d\mu(x) \\ &\le C\,\mu(S_k)^{\frac{1}{s}-1} \Big\{\sum_j \|f_j\|^p_{L^p(v\,d\mu)}\Big\}^{\frac{1}{p}} \Big\{\int_{B_{k+1}} v(x)^{1-p'}\,d\mu(x)\Big\}^{\frac{1}{p'}} \\ &\le C\,2^{\frac{k\,n}{s}} \Big\{\sum_j \|f_j\|^p_{L^p(v\,d\mu)}\Big\}^{\frac{1}{p}}. \end{aligned}$$

The last inequality holds because $\frac{1}{s} - 1 > 0$ and $v \in D_p$. Thus, collecting these estimates we finish (i).

For the other operator we use the same approach, but according to Remark 2.4, it is enough to prove that, for any finite $J \subset \mathbb{Q}^+$, the inequality holds for $\widehat{T}^J$ uniformly in J. If $f = \{f_r\}_{r \in J}$, for $x \in S_k$, we observe that

$$|\widehat{T}^J f''(x)| \leq \sum_{r \in J} |T_r^* f_r''(x)| \leq \int_{\mathbb{X} \setminus B_{k+1}} \frac{A}{d(y,x)^n} \, \|f(y)\|_{\ell^1(J)} \, d\mu(y),$$

and it can be proved just as before

$$\left\| \Big\{ \sum_j |\widehat{T}^J f_j''|^p \Big\}^{\frac{1}{p}} \right\|_{L^s(S_k, d\mu)} \leq C \, 2^{\frac{k\,n}{s}} \, \Big\{ \sum_j \|f_j\|^p_{L^p_{\ell^1(J)}(v\,d\mu)} \Big\}^{\frac{1}{p}}.$$

On the other hand, since $0 < s < 1$ and repeating the previous computations,

$$\begin{aligned} \left\| \Big\{ \sum_j |\widehat{T}^J f_j'|^p \Big\}^{\frac{1}{p}} \right\|_{L^s(S_k, d\mu)} &\leq C_s \, \mu(S_k)^{\frac{1}{s}-1} \left\| \Big\{ \sum_j |\widehat{T}^J f_j'|^p \Big\}^{\frac{1}{p}} \right\|_{L^{1,\infty}(S_k, d\mu)} \\ &\leq \; C \, \mu(S_k)^{\frac{1}{s}-1} \int_{B_{k+1}} \Big\{ \sum_j \|f_j(x)\|^p_{\ell^1(J)} \Big\}^{\frac{1}{p}} v(x)^{\frac{1}{p}} \, v(x)^{-\frac{1}{p}} \, d\mu(x) \\ &\leq \; C \, 2^{\frac{k\,n}{s}} \Big\{ \sum_j \|f_j\|^p_{L^p_{\ell^1(J)}(v\,d\mu)} \Big\}^{\frac{1}{p}} \end{aligned}$$

by Kolmogorov inequality (see [GR] p. 485) and Corollary 2.3. Then collecting both estimates we conclude (ii). In order to deal with the finite diameter case, since the measure of the space is finite, we do not have to decompose the functions, we only follow the ideas we have used for the functions f_j' (for a similar reasoning and more details see [GM]). $\square$

With these vector-valued inequalities, we are able to prove the following result.

Theorem 3.4 *Take p, $1 < p < \infty$. If $u \in Z_p$ (resp. $v \in D_p$), then there exists some weight $0 < v < \infty$ μ-a.e. (reps. $0 < u < \infty$ μ-a.e.) such that* (5) *holds. Moreover, v (resp. u) can be found in such a way that $v^\alpha \in Z_p$ (resp. $u^\alpha \in D_p$), provided that $0 < \alpha < 1$.*

Proof. First, we shall prove the case $v \in D_p$ for infinite diameter spaces. Take $0 < \alpha < 1$ and $q = 1 + \alpha\,(p' - 1)$. Then $1 < q < p'$ and we can find s, $0 < s < 1$, such that $\sigma = \left(\frac{p}{s}\right)' > q$. We use Theorem 3.1 with

$(\mathbb{Y}, d\nu) = (\mathbb{X}, d\mu)$, $\mathbb{F} = L^p(v\, d\mu)$, $\mathbb{G} = \mathbb{C}$, $\{A_k\}_k = \{S_k\}_{k=0}^{\infty}$, $C_k = C\, 2^{\frac{k\,n}{s}}$ and with the sublinear operator $T_\star$. Part (i) of Proposition 3.3 leads to the vector-valued inequality (6). Then, there exists a weight u such that (5) holds. Furthermore, u can be taken so that $\|u^{-1}\|_{L^{\sigma-1}(S_k, d\mu)} \leq C\, (a_k^{-1}\, 2^{\frac{k\,n}{s}})^p$, with $a_k > 0$ and $\sum a_k^p < \infty$. From this point, we only have to repeat the corresponding proof in [GM] to obtain that $u^\alpha \in D_p$. When the space we are concerned with, has finite diameter, it is easier because we do not have to decompose the space. For more details see that reference.

To get the other case we proceed as follows. Call $\widetilde{u} = u^{1-p'}$ and $r = p'$. Since $u \in Z^p$, then $\widetilde{u} \in D_{p'} = D_r$. For a fixed $0 < \alpha < 1$, we have $q = 1 + \alpha\,(r' - 1)$ and thus $1 < q < r'$. We find $0 < s < 1$, such that $\sigma = \left(\frac{r}{s}\right)' > q$. If the space has finite diameter, we use Theorem 3.1 with $(\mathbb{Y}, d\nu) = (\mathbb{X}, d\mu)$, $\mathbb{F} = L^r_{\ell^1(\mathbb{Q}^+)}(\widetilde{u}\, d\mu)$, $\mathbb{G} = \mathbb{C}$, $\{A_k\}_k = \{S_k\}_{k=0}^{\infty}$, $C_k = C\, 2^{\frac{k\,n}{s}}$ and with the operator $\widehat{T}$. The vector-valued inequality (6) arises from part (ii) of Proposition 3.3. Therefore, we know that there exists some weight $\widetilde{v}$ such that

$$\int_{\mathbb{X}} |\widehat{T} f(x)|^r\, \widetilde{v}(x)\, d\mu(x) \leq C \int_{\mathbb{X}} \|f(x)\|^r_{\ell^1(\mathbb{Q}^+)}\, \widetilde{u}(x)\, d\mu(x). \tag{7}$$

Moreover, like in [GM], $\widetilde{v}$ can be found such that $\widetilde{v}^\alpha \in D_r$. Take v so that $\widetilde{v} = v^{1-p'}$, then $v^\alpha \in Z_p$. For the finite diameter case we proceed analogously and again it is easier because we do not need to decompose the space.

We want to come back to $T_\star$. We restrict $\ell^1(\mathbb{Q}^+)$ with the aim of having only a finite number of non-zero coordinates. Take some finite set $J \subset \mathbb{Q}^+$. For $f = \{f_r\}_{r \in J}$ define $\widetilde{f} = \{\widetilde{f}_r\}_{r \in \mathbb{Q}^+}$, where $\widetilde{f}_r = f_r$ if $r \in J$ and 0 otherwise. Inequality (7) applied to these sequences and Remark 2.4 allow us to observe that

$$\begin{aligned}
\int_{\mathbb{X}} |\widehat{T}^J f(x)|^r\, \widetilde{v}(x)\, d\mu(x) &= \int_{\mathbb{X}} |\widehat{T} \widetilde{f}(x)|^r\, \widetilde{v}(x)\, d\mu(x) \\
&\leq \; C \int_{\mathbb{X}} \|\widetilde{f}(x)\|^r_{\ell^1(\mathbb{Q}^+)}\, \widetilde{u}(x)\, d\mu(x) = C \int_{\mathbb{X}} \|f(x)\|^r_{\ell^1(J)}\, \widetilde{u}(x)\, d\mu(x).
\end{aligned}$$

By a duality argument and by recalling that $r = p'$, this is equivalent to

$$\int_{\mathbb{X}} \|\widetilde{T}^J f(x)\|^p_{\ell^\infty(J)}\, u(x)\, d\mu(x) \leq C \int_{\mathbb{X}} |f(x)|^p\, v(x)\, d\mu(x).$$

Besides $\|\widetilde{T}^J f(x)\|_{\ell^\infty(J)} = T^J_\star f(x)$ and the monotone convergence theorem leads to (5). $\square$

4 Cauchy integral operator.

For a non-negative Borel measure μ in the complex plane $\mathbb{C}$, the Cauchy integral operator of a compactly supported function $f \in L^p(\mu)$, $1 \le p \le \infty$, is defined as

$$\mathcal{C}f(z) = \mathcal{C}_\mu f(z) = \int_{\mathbb{C}} \frac{f(\xi)}{z-\xi}\, d\mu(\xi), \qquad \text{for } \mu\text{-a.e. } z \in \mathbb{C} \setminus \operatorname{supp} f.$$

Note that, in general, this definition makes no sense when $z \in \operatorname{supp} f$. This fact leads to consider the truncated Cauchy integrals $\mathcal{C}_\varepsilon$ and try to find the boundedness of these operators uniformly in ε. In [To1], necessary and sufficient conditions on the measure μ are given to ensure that the truncated Cauchy integrals are uniformly bounded in $L^2(\mu)$. About the existence of the principal value, in [To2] it is obtained that the boundedness of the Cauchy integral operator in $L^2(\mu)$ is a sufficient condition in order to have that for compactly supported functions in C^1, $\mathcal{C}_\varepsilon f(z)$ converges for μ-a.e. $z \in \mathbb{C}$ as ε goes to zero. By using that this space is densely contained in $L^p(\mu)$, $1 \le p < \infty$ and the continuity in these spaces of $\mathcal{C}_\star$ which is obtained in [NTV2] and in [To2], it follows that for $f \in L^p(\mu)$, $1 \le p < \infty$, the principal value

$$\lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon f(z) = \lim_{\varepsilon \to 0} \int_{|z-\xi|>\varepsilon} \frac{f(\xi)}{z-\xi}\, d\mu(\xi)$$

exist for μ-a.e. $z \in \mathbb{C}$. Then we can define, at least almost everywhere, the Cauchy integral operator. It is clear that this "new" definition agrees with the definition away from the support.

We have a metric space $\mathbb{C}$ with the euclidean metric and μ any "1-dimensional" measure for which the Cauchy integral operator is bounded in $L^2(\mu)$. First, we can observe that this operator falls within the framework of theory developed in [NTV2]: the Cauchy integral operator is defined for compactly supported functions in $L^2(\mu)$ by means of the "1-dimensional" Calderón-Zygmund kernel $K(z,\xi) = \frac{1}{z-\xi}$. A bounded extension to the whole $L^2(\mu)$ arises from the existence of the principal value and the boundedness of $\mathcal{C}_\star$. Then we can apply the results we have obtained to get vector-valued inequalities for $\mathcal{C}_\star$. By Theorem 2.2, the following result is established.

Theorem 4.1 *Under the above assumptions and for $1 < p, q < \infty$ we have*

(i) $$\mu\Big\{ z \in \mathbb{C} : \Big\{ \sum_j (\mathcal{C}_\star f_j(z))^q \Big\}^{\frac{1}{q}} > \lambda \Big\} \le \frac{C}{\lambda} \int_{\mathbb{C}} \Big\{ \sum_j |f_j(z)|^q \Big\}^{\frac{1}{q}}\, d\mu(z).$$

(ii) $$\Big\| \Big\{ \sum_j (\mathcal{C}_\star f_j)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p(\mu)} \le C \Big\| \Big\{ \sum_j |f_j|^q \Big\}^{\frac{1}{q}} \Big\|_{L^p(\mu)}.$$

In this framework, for $1 < p < \infty$, the classes of weights will be

$$
\begin{aligned}
D_p &= \left\{0 \le w < \infty \ \mu\text{-a.e.} : \int_{\mathbb{C}} w(z)^{1-p'} (1+|z|)^{-p'} \, d\mu(z) < \infty\right\} \\
Z_p &= \left\{w > 0 \ \mu\text{-a.e.} : \int_{\mathbb{C}} w(z) (1+|z|)^{-p} \, d\mu(z) < \infty\right\}.
\end{aligned}
$$

If the measure has bounded support, these classes admit the equivalent definition given in Remark 3.2. In fact, several results will be easier when this happens. For $w \ge 0$ μ-a.e. we denote $w(A) = \int_A w(z)\, d\mu(z)$, for any measurable set $A \subset \mathbb{C}$.

As a consequence of Theorem 3.4 we obtain:

Corollary 4.2 *Let $1 < p < \infty$. If $u \in Z_p$ (resp. $v \in D_p$), then there exists a weight v (resp. u) such that*

$$
\int_{\mathbb{C}} |\mathcal{C}_\star f(z)|^p \, u(z)\, d\mu(z) \le C \int_{\mathbb{C}} |f(z)|^p \, v(z)\, d\mu(z), \quad \textit{for any } f \in L^p(v\, d\mu).
$$

Moreover, v (resp. u) can be taken in such a way that $v^\alpha \in Z_p$ (resp. $u^\alpha \in D_p$), provided that $0 < \alpha < 1$.

For a fixed $v \in D_p$, with $v \in L^1_{loc}(\mu)$, it follows that there exists another weight u such that the previous inequality holds. The density of continuous functions with compact support in the space $L^p(v)$ and standard arguments yield the existence of the principal value

$$
\lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon f(z) = \lim_{\varepsilon \to 0} \int_{|z-\xi|>\varepsilon} \frac{f(\xi)}{z-\xi} \, d\mu(\xi)
$$

μ-almost everywhere for any $f \in L^p(v)$. Besides it is clear that $\mathcal{C}_\star f$ is μ-a.e. finite for each $f \in L^p(v)$. So, the existence of this principal value and the finiteness of this maximal operator follow from the conditions on v. A natural question is whether a converse result can be obtained. The answer is affirmative, that is, the existence of the principal value or the finiteness almost everywhere of $\mathcal{C}_\star$ for any function of this weighted space imply that v is in D_p. We shall introduce some notation before proving these equivalence.

Let us define a new maximal operator

$$
\mathcal{C}^\star f(z) = \sup_{0<\varepsilon<1} |\mathcal{C}^\varepsilon f(z)| = \sup_{0<\varepsilon<1} \left| \int_{\varepsilon<|z-\xi|<\frac{1}{\varepsilon}} \frac{f(\xi)}{z-\xi} \, d\mu(\xi) \right|.
$$

If f is a compactly supported continuous function,

$$
|\mathcal{C}^\varepsilon f(z) - \mathcal{C}_\varepsilon f(z)| \le \left| \int_{|z-\xi|>\frac{1}{\varepsilon}} \frac{f(\xi)}{z-\xi} \, d\mu(\xi) \right| \le \varepsilon \, \|f\|_{L^1(\mu)} \longrightarrow 0, \quad \text{as } \varepsilon \to 0.
$$

Then,

$$\exists \lim_{\varepsilon\to 0} \mathcal{C}^{\varepsilon} f(z) \Longleftrightarrow \exists \lim_{\varepsilon\to 0} \mathcal{C}_{\varepsilon} f(z), \tag{8}$$

and, if one of them exists, we get that both limits are equal. On the other hand, if f is such that $\mathcal{C}_{\star} f(z) < \infty$, then for $0 < \varepsilon < 1$,

$$|\mathcal{C}^{\varepsilon} f(z)| \leq |\mathcal{C}_{\varepsilon} f(z)| + |\mathcal{C}_{\frac{1}{\varepsilon}} f(z)| \leq 2\, \mathcal{C}_{\star} f(z),$$

and therefore

$$\mathcal{C}^{\star} f(z) \leq 2\, \mathcal{C}_{\star} f(z). \tag{9}$$

Lemma 4.3 *Take $1 < p < \infty$ and $0 < v < \infty$ μ-a.e. a measurable function.*

(i) If $v^{1-p'} \in L^1_{loc}(\mu)$ and $f \in L^p(v) = L^p(v\, d\mu)$, then for $0 < \varepsilon < 1$, $\mathcal{C}^{\varepsilon} f$ makes sense because it is defined by means of an absolutely convergent integral.

(ii) Assume that for any $f \in L^p(v)$ and for any z outside of a set of μ-measure zero, we have some $0 < \varepsilon_z < 1$ in such a way that $\mathcal{C}^{\varepsilon} f(z)$ exists for $0 < \varepsilon < \varepsilon_z$, that is,

$$\int_{\varepsilon < |z-\xi| < \frac{1}{\varepsilon}} \frac{|f(\xi)|}{|z-\xi|}\, d\mu(\xi) < \infty, \quad \text{for } 0 < \varepsilon < \varepsilon_z.$$

Then, $v^{1-p'} \in L^1_{loc}(\mu)$.

Proof. For (i) it is enough to observe that

$$\begin{aligned} \int_{\varepsilon < |z-\xi| < \frac{1}{\varepsilon}} \frac{|f(\xi)|}{|z-\xi|}\, d\mu(\xi) &\leq \frac{1}{\varepsilon} \int_{|\xi| < \frac{1}{\varepsilon} + |z|} |f(\xi)|\, v(\xi)^{\frac{1}{p}}\, v(\xi)^{-\frac{1}{p}}\, d\mu(\xi) \\ &\leq \frac{1}{\varepsilon} \|f\|_{L^p(v)} \Big(\int_{|\xi| < \frac{1}{\varepsilon} + |z|} v(\xi)^{1-p'}\, d\mu(\xi) \Big)^{\frac{1}{p'}} < \infty. \end{aligned} \tag{10}$$

Fix $f \in L^p(v)$ and $z_0 \in \operatorname{supp} \mu$. For $k = 1, 2, \ldots$ set $B_k = B(z_0, 2^{-k})$ and $S_k = B_k \setminus B_{k+1}$. Then, there exists $k_0 \geq 1$ such that $\mu(S_{k_0}) > 0$. The function $f\, \chi_{B_{k_0}} \in L^p(v)$. By hypothesis we have a zero measure set E, such that, for $z \in \mathbb{C} \setminus E$, we can find ε_z, with

$$\int_{\mathbb{C}} |f(\xi)|\, \chi_{F_z^{\varepsilon}}(\xi) < \infty, \quad \text{where } F_z^{\varepsilon} = \{\xi \in B_{k_0} : \varepsilon < |z-\xi| < \varepsilon^{-1}\},$$

for $0 < \varepsilon < \varepsilon_z$. Since $\mu(S_{k_0}) > 0$, $\mu(B_{k_0+3}) > 0$ and $\mu(E) = 0$, we can find $z_1 \in B_{k_0+3} \setminus E$ and $z_2 \in S_{k_0} \setminus E$. Take $0 < \varepsilon_0 < \min\{\varepsilon_{z_1}, \varepsilon_{z_2}, 2^{-k_0-3}\}$ and set $F_1 = F_{z_1}^{\varepsilon_0}$, $F_2 = F_{z_2}^{\varepsilon_0}$. Then we know that $f\,\chi_{F_1}, f\,\chi_{F_2} \in L^1(\mu)$ and hence $f\,\chi_{F_1 \bigcup F_2}$ does. For $\xi \in B_{k_0}$, since $z_1 \in B_{k_0+3} \subset B_{k_0}$, we observe that $|z_1 - \xi| \leq 1 < \varepsilon_0^{-1}$, and therefore

$$F_1 = \left\{\xi \in B_{k_0} : \varepsilon_0 < |z_1 - \xi| < \frac{1}{\varepsilon_0}\right\} = \{\xi \in B_{k_0} : |z_1 - \xi| > \varepsilon_0\}.$$

Since $z_2 \in B_{k_0}$, we can also obtain that $F_2 = \{\xi \in B_{k_0} : |z_2 - \xi| > \varepsilon_0\}$. Then $F_1 \bigcup F_2 = B_{k_0}$ and $f\,\chi_{F_1 \bigcup F_2} = f\,\chi_{B_{k_0}} \in L^1(\mu)$. In short, for any $z_0 \in \operatorname{supp}\mu$ there exists $r_{z_0} > 0$ such that $f\,\chi_{B(z_0, r_{z_0})} \in L^1(\mu)$. Then a compactness argument shows that for any compact K, $f\,\chi_K \in L^1(\mu)$, and it follows that $L^p(v) \subset L^1_{loc}(\mu)$. Thus, the linear operator

$$\begin{array}{rrcl} P_K : & L^p(v) & \longrightarrow & L^1(\mu) \\ & f & \longmapsto & f\,\chi_K \end{array}$$

is well defined. By using closed graph theorem and the fact that $0 < v < \infty$ μ-a.e., it is easy to see that P_K is also bounded. On the other hand

$$\begin{aligned} \|\chi_K\|_{L^{p'}(v^{1-p'})} &= \|\chi_K\, v^{-\frac{1}{p}}\|_{L^{p'}(\mu)} = \sup_{\|f\|_{L^p(\mu)}=1} \left| \int_{\mathbb{C}} \chi_K\, v^{-\frac{1}{p}}\, f\, d\mu \right| \\ &\leq \sup_{\|f\|_{L^p(\mu)}=1} \|P_K(v^{-\frac{1}{p}}\, f)\|_{L^1(\mu)} \leq \|P_K\|_{L^p(v) \longrightarrow L^1(\mu)} < \infty. \end{aligned}$$

Therefore $\chi_K \in L^{p'}(v^{1-p'})$ and, since this argument is valid for every compact, we conclude that $v^{1-p'} \in L^1_{loc}(\mu)$. □

Next, we can prove the result we have announced in Section 1.

Proof of Theorem 1.2 Implications $(b) \Longrightarrow (c)$, $(d) \Longrightarrow (e)$ are trivial. On the other hand, $(a) \Longleftrightarrow (b)$ is one of the results we proved in [GM]. In order to prove $(a) \Longrightarrow (d)$, it is enough to use Corollary 4.2 and (9). The other implications will be obtained as follows:

$$(e) \Longrightarrow (f), \quad (f) \Longrightarrow (g), \quad (g) \Longrightarrow (a) \quad \text{and} \quad (c) \Longrightarrow (a).$$

$\boxed{(e) \Longrightarrow (f)}$ The inequality of (e) says that the operator $\mathcal{C}^\star : L^p(v) \longrightarrow L^{p,\infty}(u)$ is bounded. Moreover, by (8), we know that for a continuous function f with compact support, the principal value $\lim_{\varepsilon \to 0} \mathcal{C}^\varepsilon f$ exists μ-a.e.. Since $v \in L^1_{loc}(\mu)$, the space of continuous functions with compact support is densely contained in $L^p(v)$. These things allow us to conclude that for any $f \in L^p(v)$, this principal value exists μ-almost everywhere.

$\boxed{(f) \Longrightarrow (g)}$ Fix $f \in L^p(v)$. We are under the hypotheses of Lemma 4.3 (ii), then $v^{1-p'} \in L^1_{loc}(\mu)$. This allows us to use (10). Take some $z \in \mathbb{C}$ in such a way that the limit exists, then we have some ε_0, $0 < \varepsilon_0 < 1$, such that $\sup_{0<\varepsilon\leq\varepsilon_0} |\mathcal{C}^\varepsilon f(z)| < \infty$. Furthermore, if $\varepsilon_0 \leq \varepsilon < 1$, by (10) we observe

$$|\mathcal{C}^\varepsilon f(z)| \leq \frac{1}{\varepsilon_0} \|f\|_{L^p(v)} \Big(\int_{|\xi|<\frac{1}{\varepsilon_0}+|z|} v(\xi)^{1-p'}\, d\mu(\xi) \Big)^{\frac{1}{p'}} = C(f, z, \varepsilon_0) < \infty,$$

and therefore $\mathcal{C}^\star f(z) < \infty$ for μ-a.e. $z \in \mathbb{C}$.

$\boxed{(g) \Longrightarrow (a)}$ First, since the assumptions in (ii) of Lemma 4.3 are fulfilled with $\varepsilon_z = 1$, we obtain that $v^{1-p'} \in L^1_{loc}(\mu)$ and we can use (10). We would like to apply Nikishin theorem (see [GR] p. 536) and we have to check that $\mathcal{C}^\star$ is continuous in measure. Decompose $\mathbb{C}$ as $\mathbb{C} = \bigcup_{j=1}^\infty B(0,j)$. Fix $0 < \varepsilon < 1$. For $f \in L^p(v)$ we know that $\mathcal{C}^\varepsilon f \in L^0(\mu)$, the space of all μ-measurable functions. We want to show that

$$\mathcal{C}^\varepsilon : L^p(v) \longrightarrow L^0(\mu) \quad \text{is continuous in measure,}$$

and, in order to do that, it is enough to see that for each $j \geq 1$, $\Phi^j(\lambda) \longrightarrow 0$, as $\lambda \to \infty$, where

$$\Phi^j(\lambda) = \sup_{\|f\|_{L^p(v)}=1} \mu\{z \in B(0,j) : |\mathcal{C}^\varepsilon f(z)| > \lambda\}.$$

For $\|f\|_{L^p(v)} = 1$, by using (10) we get

$$|\mathcal{C}^\varepsilon f(z)| \leq \frac{1}{\varepsilon} \Big(\int_{|\xi|<\frac{1}{\varepsilon}+j} v(\xi)^{1-p'}\, d\mu(\xi) \Big)^{\frac{1}{p'}} = C(\varepsilon, j) < \infty$$

for μ-a.e. $z \in B(0,j)$. Then $\Phi^j(\lambda) = 0$ for $\lambda > C(\varepsilon, j)$, and thus $\mathcal{C}^\varepsilon$ is continuous in measure. By means of (10), we observe that the mapping $\varepsilon \longmapsto \mathcal{C}^\varepsilon f(z)$ is continuous from the right for each $f \in L^p(v)$. Consequently, $\mathcal{C}^\star$ is the supremum of certain operators, which are continuous in measure, over a countable set, $\mathbb{Q}^+$, and it is μ-almost everywhere finite for $f \in L^p(v)$. Then by Banach continuity principle, (see [GR] p. 529),

$$\mathcal{C}^\star : L^p(v) \longrightarrow L^0(\mu) \quad \text{is continuous in measure.}$$

Now, Nikishin theorem (see [GR] p. 536) guarantees the existence of a measurable μ-a.e. positive function u, such that,

$$\int_{\{z\in\mathbb{C}:\mathcal{C}^\star f(z)>\lambda\}} u(z)\,d\mu(z) \le \left(\frac{\|f\|_{L^p(v)}}{\lambda}\right)^q, \quad f\in L^p(v),\ \lambda>0$$

where $q=\min\{p,2\}$. But, if f is continuous with compact support, $\mathcal{C}^\varepsilon f \longrightarrow \mathcal{C}f$ μ-a.e. as $\varepsilon\to 0$, and, in particular, $|\mathcal{C}f(z)| = |\lim_{\varepsilon\to 0}\mathcal{C}^\varepsilon f(z)| \le \mathcal{C}^\star f(z)$ for μ-a.e. $z\in\mathbb{C}$. This fact, together with the inequality supplied by Nikishin theorem and a density argument, allows us to extend $\mathcal{C}$ to the whole $L^p(v)$ verifying

$$\int_{\{z\in\mathbb{C}:|\mathcal{C}f(z)|>\lambda\}} u(z)\,d\mu(z) \le \left(\frac{\|f\|_{L^p(v)}}{\lambda}\right)^q, \tag{11}$$

for $f\in L^p(v), \lambda>0$. We want to obtain conditions on u, v from this weak type inequality. We will do it by using the ideas of [GM]. First, we shall see that for any $z_0\in\operatorname{supp}\mu$, a radius $r_{z_0}>0$ can be found in order to ensure

$$\int_{B(z_0,r_{z_0})} u(z)\,d\mu(z) < \infty. \tag{12}$$

We shall use the notation B_k and S_k introduced in the proof of Lemma 4.3. For $z=z_1+i\,z_2$, we write $|z|_\infty=\max\{|z_1|,|z_2|\}$. Define

$$F_1=\{z\in\mathbb{C}:|z-z_0|_\infty=z_1-z_1^0\},\qquad F_2=\{z\in\mathbb{C}:|z-z_0|_\infty=z_2-z_2^0\},$$
$$F_3=\{z\in\mathbb{C}:|z-z_0|_\infty=z_1^0-z_1\},\qquad F_4=\{z\in\mathbb{C}:|z-z_0|_\infty=z_2^0-z_2\}.$$

There is some $k_0\ge 0$ such that S_{k_0} has positive measure. Assume for instance that $\mu(S_{k_0}\cap F_1)>0$. We have some measurable set $A\subset S_{k_0}\bigcap F_1$ such that $\mu(A)>0$ and $v(A)<\infty$. Hence, just as in [GM], if $z\in B_{k_0+2}$, $|\mathcal{C}(\chi_A)(z)|\ge|\mathrm{Re}\,(\mathcal{C}(\chi_A)(z))|\ge C_A>0$. For $0<\lambda_0<C_A$, we use (11) to obtain

$$\int_{B_{k_0+2}} u(z)\,d\mu(z) \le \int_{\{z\in\mathbb{C}:|\mathcal{C}(\chi_A)(z)|>\lambda_0\}} u(z)\,d\mu(z) \le \frac{1}{\lambda_0^q}\,v(A)^{\frac{q}{p}}<\infty,$$

and it may be sufficient to take $r_{z_0}=2^{-k_0-2}$. By a compactness argument, (12) yields that $u\in L^1_{loc}(\mu)$, and in particular, u is a finite μ-a.e. function. (In addition, we can also prove that $u\in Z_{q+\gamma}$ for any $\gamma>0$, but this will not be needed in what follows).

In order to obtain conditions on v, we shall "dualize" (11). By Kolmogorov inequality (see [GR] p. 485), for $1<r<q$, we have

$$\sup_E \frac{1}{u(E)^{\frac{1}{r}-\frac{1}{q}}}\left(\int_E |\mathcal{C}f|^r\,u\,d\mu\right)^{\frac{1}{r}} \le C_{r,q}\,\|\mathcal{C}f\|_{L^{q,\infty}(u\,d\mu)} \le C\,\|f\|_{L^p(v)}, \tag{13}$$

where the supremum is taken over all measurable sets with $0 < u(E) < \infty$. Let A be a measurable set which verifies

$$\mu(A) < \infty, \quad 0 < u(A) < \infty \quad \text{and} \quad \int_A u^{1-r'}\, d\mu < \infty. \tag{14}$$

Then,

$$\|\mathcal{C}(\chi_A)\|_{L^{p'}(v^{1-p'})} = \|\mathcal{C}(\chi_A)\, v^{-\frac{1}{p}}\|_{L^{p'}(\mu)} = \sup_g \Big| \int_{\mathbb{C}} \mathcal{C}(\chi_A)\, v^{-\frac{1}{p}}\, g\, d\mu \Big|, \tag{15}$$

where the supremum is taken over all continuous functions with compact support and $\|g\|_{L^p(\mu)} = 1$. Fix such a g. As $\mu(A) < \infty$, we have $\chi_A \in L^p(\mu)$ and thus $\mathcal{C}(\chi_A)$ belongs to this space. Besides, $v^{-\frac{1}{p}}\, g \in L^{p'}(\mu)$ since $v^{1-p'} \in L^1_{loc}(\mu)$. We can use the adjoint operator $\mathcal{C}^* = -\mathcal{C}$ to obtain

$$\begin{aligned} \Big| \int_{\mathbb{C}} \mathcal{C}(\chi_A)\, v^{-\frac{1}{p}}\, g\, d\mu \Big| &= \Big| \int_{\mathbb{C}} \chi_A\, \mathcal{C}^*(v^{-\frac{1}{p}}\, g)\, d\mu \Big| \le \int_{\mathbb{C}} \chi_A\, |\mathcal{C}(v^{-\frac{1}{p}}\, g)|\, d\mu \\ &\le \Big(\int_A |\mathcal{C}(v^{-\frac{1}{p}}\, g)|^r\, u\, d\mu \Big)^{\frac{1}{r}} \Big(\int_A u^{-\frac{r'}{r}}\, d\mu \Big)^{\frac{1}{r'}} \\ &\le C\, u(A)^{\frac{1}{r}-\frac{1}{q}} \Big(\int_A u^{1-r'}\, d\mu \Big)^{\frac{1}{r'}}, \end{aligned}$$

where (13) has been used. Therefore, we have

$$\|\mathcal{C}(\chi_A)\|_{L^{p'}(v^{1-p'})} \le C\, u(A)^{\frac{1}{r}-\frac{1}{q}} \Big(\int_A u^{1-r'}\, d\mu \Big)^{\frac{1}{r'}}. \tag{16}$$

By means of this inequality, we are going to see that $v \in D_p$. For $i = 1, \dots, 4$ set E_i by putting $z_0 = 0$ in the definition of F_i. Since we know that $v^{1-p'}$ is locally integrable, we only have to find $R_i > 0$ such that

$$\int_{E_i \setminus B(0,R_i)} \frac{v(z)^{1-p'}}{(1+|z|)^{p'}}\, d\mu(z) < \infty. \tag{17}$$

If $\operatorname{supp} \mu \bigcap E_i$ is a bounded set, it might be enough to enlarge sufficiently R_i. Otherwise, we shall do the case $i = 1$ and in the other regions the proof goes analogously. We can find $R_1 > 0$ such that $\mu(B(0, R_1/2) \bigcap E_1) > 0$. Then, since $0 < u < \infty$ μ-a.e., there exists a measurable set $A \subset B(0, R_1/2) \bigcap E_1$ so that (14) holds and thus the right hand side of (16) is finite. Moreover, if $z \in E_1 \setminus B(0, R_1)$, like in [GM], it can be proved $|\mathcal{C}(\chi_A)(z)| \ge |\mathrm{Re}\,(\mathcal{C}(\chi_A)(z))| \ge \frac{C}{1+|z|} > 0$. Therefore, we get

$$\int_{\mathbb{C}} |\mathcal{C}(\chi_A)(z)|^{p'}\, v(z)^{1-p'}\, d\mu(z) \ge C^{p'} \int_{E_1 \setminus B(0,R_1)} \frac{v(z)^{1-p'}}{(1+|z|)^{p'}}\, d\mu(z),$$

and consequently (17) holds for $i = 1$. As we have observed before, this fact allows us to obtain that $v \in D_p$.

$\boxed{(c) \Longrightarrow (a)}$ Now, this implication is a consequence of the previous one. Observe that the inequality of (c) is (11) with $q = p$. From this inequality we can obtain as before that u is a finite μ-a.e. function. However, in the previous reasoning, we knew that $v^{1-p'} \in L^1_{loc}(\mu)$ and this allowed us to "dualize" (11). Since, now this condition about v is not assumed, we have to modify this argument. By decomposing the space in the dyadic level sets where $2^j \leq v(z) < 2^{j+1}$ and since $0 < v < \infty$ μ-a.e., we can prove that boundedly supported functions in $L^p(\mu) \bigcap L^p(v^{-1}) \bigcap L^\infty(\mu)$ are dense in $L^p(\mu)$. From this, the supremum in (15) can be taken over all these functions. We can use the adjoint operator to get (16) for any measurable set verifying (14) with $q = p$ and $1 < r < p$. Then, (17) holds and we only have to prove that $v^{1-p'}$ is locally integrable. For any $z_0 \in \operatorname{supp} \mu$, we have some $k_0 \geq 0$ such that S_{k_0} has positive measure. Suppose for example that $\mu(S_{k_0} \bigcap F_1) > 0$. Since $0 < u < \infty$ μ-a.e., there is a measurable set $A \subset S_{k_0} \bigcap F_1$ such that (14) holds. Then the right hand side of (16) is finite. Moreover like in the reasoning we did for u, we obtain that

$$C_A^{p'} \int_{B(z_0, r_{z_0})} v(z)^{1-p'} \, d\mu(z) \leq \int_{\mathbb{C}} |\mathcal{C}(\chi_A)(z)| \, v(z)^{1-p'} \, d\mu(z) < \infty.$$

where $r_{z_0} = 2^{-k_0-2}$. A compactness argument yields that $v^{1-p'} \in L^1_{loc}(\mu)$ and therefore $v \in D_p$. $\square$

Remark 4.4 *We should mention that (f) and (g) can also be obtained from (a) by the results of* [To2]. *Namely, if $R > 0$, $f \in L^p(v)$ and $v \in D_p$, by Hölder's inequality one obtains that*

$$\int_{|\xi| > 2R} \frac{|f(\xi)|}{|z - \xi|} \, d\mu(\xi) < \infty, \quad \textit{for every } z \in B(0, R). \tag{18}$$

On the other hand, $L^p(v) \subset L^1_{\mathrm{loc}}(\mu)$ and so $f \chi_{B(0,2R)} \in L^1(\mu)$. Thus, [To2] *yields the existence of the principal value*

$$\lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon(f \chi_{B(0,2R)})(z), \quad \textit{for } \mu\textit{-almost every } z \in B(0, R). \tag{19}$$

Let us put $\mathcal{C}^\varepsilon f(z) = \mathcal{C}^\varepsilon(f \chi_{\mathbb{C} \setminus B(0,2R)})(z) + \mathcal{C}^\varepsilon(f \chi_{B(0,2R)})(z)$. For the first term, by (18), we can prove that there exists the limit as ε goes to 0 and the supremum for $0 < \varepsilon < 1$ is finite, both facts for every $z \in B(0, R)$. The *second term is handled by realizing that, for ε small enough, it is actually $\mathcal{C}_\varepsilon(f\,\chi_{B(0,2R)})(z)$. By (19) we obtain de existence of the limit and the finiteness of the supremum for μ-almost everywhere $z \in B(0,R)$. Thus, it is clear that (f) and (g) hold.*

Next, we want to see that we can replace $\mathcal{C}^\star$ by $\mathcal{C}_\star$. For $0 < \varepsilon < 1$, the fact that $\mathcal{C}^\varepsilon f(z)$ exists only can mean that the integrand is a function in $L^1(\mu)$. However, for $\mathcal{C}_\varepsilon f(z)$, $\varepsilon > 0$, since we have only truncated the integral near z and not at infinity, two interpretations are possibly: either the integrand belongs to $L^1(\mu)$, or the integral at infinity is in fact a principal value. Since the second one is weaker, we assume that the existence of $\mathcal{C}_\varepsilon f(z)$, $\varepsilon > 0$, means

$$\exists \lim_{r\to\infty} \int_{\varepsilon<|z-\xi|<r} \frac{f(\xi)}{z-\xi}\, d\mu(\xi),$$

and in this case, we write $\mathcal{C}_\varepsilon f(z)$ for this limit. Observe that the existence of each integral only admits one interpretation:

$$\int_{\varepsilon<|z-\xi|<r} \left|\frac{f(\xi)}{z-\xi}\right| d\mu(\xi) < \infty.$$

Corollary 4.5 *The statements of Theorem* 1.2 *are also equivalent to the following:*

(d)′ There exists a μ-a.e. positive measurable function u, such that,

$$\int_{\mathbb{C}} (\mathcal{C}_\star f(z))^p\, u(z)\, d\mu(z) \le C \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z), \quad \textit{for any } f \in L^p(v).$$

(e)′ There exists a μ-a.e. positive measurable function u, such that,

$$\int_{\{z\in\mathbb{C}:\mathcal{C}_\star f(z)>\lambda\}} u(z)\, d\mu(z) \le \frac{C}{\lambda^p} \int_{\mathbb{C}} |f(z)|^p\, v(z)\, d\mu(z),$$

for any $f \in L^p(v), \lambda > 0$.

(f)′ If $f \in L^p(v)$, the principal value

$$\lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon f(z) = \lim_{\varepsilon\to 0} \int_{|z-\xi|>\varepsilon} \frac{f(\xi)}{z-\xi}\, d\mu(\xi)$$

exists for μ-a.e. $z \in \mathbb{C}$.

(g)′ If $f \in L^p(v)$, then $\mathcal{C}_\star f < \infty$ μ-a.e..

Proof. By Corollary 4.2, we know that $(a) \Longrightarrow (d)'$. It is clear that $(d)' \Longrightarrow (e)'$. We can prove that $(e)' \Longrightarrow (f)'$ just as the corresponding implication of Theorem 1.2. On the other hand, $(d)' \Longrightarrow (g)'$ since $u > 0$ μ-a.e.. Moreover, $(g)' \Longrightarrow (g)$ follows from (9). Therefore, we shall finish if we see that $(f)' \Longrightarrow (f)$.

Fix $f \in L^p(v)$, and take z_0 such that $\mathcal{C}_\varepsilon f(z_0)$ converges as ε goes to 0. This implies in particular the existence of ε_0, $0 < \varepsilon_0 < 1$, for which $\mathcal{C}_\varepsilon f(z_0)$ makes sense if $0 < \varepsilon \leq \varepsilon_0$. Given $\delta > 0$, there exists η_0, $0 < \eta_0 < \varepsilon_0$, such that

$$|\mathcal{C}_{\varepsilon_1} f(z_0) - \mathcal{C}_{\varepsilon_2} f(z_0)| = \Big| \int_{\varepsilon_1 < |z_0 - \xi| \leq \varepsilon_2} \frac{f(\xi)}{z_0 - \xi}\, d\mu(\xi) \Big| < \frac{\delta}{2}, \tag{20}$$

for $0 < \varepsilon_1 < \varepsilon_2 \leq \eta_0$. On the other hand, since $\mathcal{C}_{\varepsilon_0, r} f(z_0)$ converges as r goes to infinity, we can find R_0 such that, if $0 < R_0 < r_2 < r_1$, we have

$$\Big| \int_{r_2 \leq |z_0 - \xi| < r_1} \frac{f(\xi)}{z_0 - \xi}\, d\mu(\xi) \Big| = |\mathcal{C}_{\varepsilon_0, r_1} f(z_0) - \mathcal{C}_{\varepsilon_0, r_2} f(z_0)| < \frac{\delta}{2}. \tag{21}$$

Set $\eta = \min\{\eta_0, R_0^{-1}\}$. If $0 < \varepsilon_1 < \varepsilon_2 < \eta$ we can use (20), and (21) holds with $r_1 = \varepsilon_1^{-1}$, $r_2 = \varepsilon_2^{-1}$. Thus,

$$\begin{aligned} &|\mathcal{C}^{\varepsilon_1} f(z_0) - \mathcal{C}^{\varepsilon_2} f(z_0)| \\ \leq\ & \Big| \int_{\varepsilon_1 < |z_0 - \xi| \leq \varepsilon_2} \frac{f(\xi)}{z_0 - \xi}\, d\mu(\xi) \Big| + \Big| \int_{\frac{1}{\varepsilon_2} \leq |z_0 - \xi| < \frac{1}{\varepsilon_1}} \frac{f(\xi)}{z_0 - \xi}\, d\mu(\xi) \Big| \\ <\ & \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, it follows that $\mathcal{C}^\varepsilon f(z_0)$ converges as $\varepsilon \to 0$, and we have (f). □

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