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Functional Calculus for the Ornstein–Uhlenbeck Operator

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Let γ be the Gauss measure on $\mathbb{R}^d$ and $\mathscr{L}$ the Ornstein–Uhlenbeck operator, which is self adjoint in $L^2(\gamma)$. For every p in $(1, \infty)$, $p \neq 2$, set $\phi_p^* = \arcsin\ |2/p-1|$ and consider the sector $\mathbf{S}_{\phi_p^*} = \{z \in \mathbb{C} : |\arg z| < \phi_p^*\}$. The main result of this paper is that if M is a bounded holomorphic function on $\mathbf{S}_{\phi_p^*}$ whose boundary values on $\partial \mathbf{S}_{\phi_p^*}$ satisfy suitable Hörmander type conditions, then the spectral operator $M(\mathscr{L})$ extends to a bounded operator on $L^p(\gamma)$ and hence on $L^q(\gamma)$ for all q such that $|1/q-1/2| \leqslant |1/p-1/2|$. The result is sharp, in the sense that $\mathscr{L}$ does not admit a bounded holomorphic functional calculus in a sector smaller than $\mathbf{S}_{\phi_p^*}$.

Key Words: Ornstein–Uhlenbeck operator; functional calculus; spectral multiplier; Hörmander–Mihlin condition.

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We consider the Gauss measure on $\mathbb{R}^d$, i.e., the probability measure γ with density

$$\gamma_0(x) = \pi^{-d/2} \exp(-|x|^2)$$

with respect to Lebesgue measure. The Ornstein–Uhlenbeck operator

$$-\tfrac{1}{2}\Delta + x \cdot \nabla$$

is essentially self-adjoint in $L^2(\gamma)$; we denote by $\mathscr{L}$ its self-adjoint extension. The spectrum of $\mathscr{L}$ is $\mathbb{N} = \{0, 1, ...\}$. Let $\{\mathscr{P}_n\}_{n \in \mathbb{N}}$ be the spectral resolution of the identity for which

$$\mathscr{L}f = \sum_{n=0}^{\infty} n\mathscr{P}_n f \qquad \forall f \in \mathrm{Dom}(\mathscr{L}).$$

It is well known [B] that if p is in $(1, \infty)$ and n is in $\mathbb{N}$, then $\mathscr{P}_n$ extends to a bounded operator on $L^p(\gamma)$. Furthermore, if p is in $[1, \infty)$, the projection $\mathscr{P}_0$ extends to a nontrivial contraction operator on $L^p(\gamma)$.

For each $t > 0$, the *Ornstein–Uhlenbeck semigroup* $\mathscr{H}_t$ is defined by

$$\mathscr{H}_t f = \sum_{n=0}^{\infty} e^{-tn} \mathscr{P}_n f \qquad \forall f \in L^2(\gamma).$$

It is known that $\{\mathscr{H}_t\}_{t \geqslant 0}$ extends to a markovian semigroup, which has been the object of many studies, both in the finite and in the infinite-dimensional case. A good reference about the Ornstein–Uhlenbeck semigroup is [B] (see also [Me]), where additional references can be found. In this paper we shall consider only the finite-dimensional case. Some results involving maximal operators and Riesz transforms associated to this semigroup are described in the survey [Sj].

Suppose that M: $\mathbb{N} \to \mathbb{C}$ is a bounded sequence. By the spectral theorem, we may form the operator $M(\mathscr{L})$, defined by

$$M(\mathscr{L}) f = \sum_{n=0}^{\infty} M(n)\, \mathscr{P}_n f \qquad \forall f \in L^2(\gamma);$$

clearly $M(\mathscr{L})$ is bounded on $L^2(\gamma)$. We call $M(\mathscr{L})$ the *spectral operator* associated to the *spectral multiplier M*.

The purpose of this paper is to develop a functional calculus for $\mathscr{L}$, i.e., to find sufficient conditions on the spectral multiplier M for the spectral operator $M(\mathscr{L})$, initially defined in $L^2(\gamma) \cap L^p(\gamma)$, to extend to a bounded operator on $L^p(\gamma)$, for some p in $(1, \infty)$.

On the one hand, we show that if $p \neq 2$, then there is no reasonable non-holomorphic functional calculus in $L^p(\gamma)$ for $\mathscr{L}$. In particular, we prove that there is no analogue of the classical Hörmander multiplier theorem in this context. In fact, for each $p \neq 2$ there exists a spectral multiplier M_p, such that $M_p(\mathscr{L})$ does not extend to a bounded operator on $L^p(\gamma)$, and which is the restriction of a function, also denoted by M_p, analytic in a neighbourhood of $\mathbb{R}^+$, that satisfies the conditions

$$\sup_{\lambda > 0} |\lambda^j D^j M_p(\lambda)| < \infty \qquad \forall j \in \mathbb{N}.$$

On the other hand, it follows from an abstract result of Stein [S, Chap. 4] that if $M: \mathbb{N} \to \mathbb{C}$ is a bounded sequence and there exists a holomorphic function $\tilde{M}$ of Laplace transform type, such that

$$\tilde{M}(k) = M(k), \qquad k = 1, 2, 3, ...,$$

then $M(\mathscr{L})$ extends to an operator bounded on $L^p(\gamma)$ for every p in $(1, \infty)$. Notice that we do not impose any restriction on $M(0)$. Since $\mathscr{P}_0$ is bounded on $L^p(\gamma)$, the operator $M(\mathscr{L})$ is bounded on $L^p(\gamma)$ if and only if $M(\mathscr{L}) - M(0)\,\mathscr{P}_0$ is. This has recently been improved by García-Cuerva *et al.* [GMST], who showed that $M(\mathscr{L})$ is also of weak type $(1, 1)$ under the same assumptions.

Furthermore, if we fix p in $(1, \infty)$, it is interesting to determine the "minimal regularity conditions" on M which imply that $M(\mathscr{L})$ is bounded on $L^p(\gamma)$. These conditions are sometimes best expressed in terms of Banach spaces of holomorphic functions. If $\psi \in (0, \pi)$, we denote by $\mathbf{S}_\psi$ the open sector

$$\{z \in \mathbb{C} : |\arg z| < \psi\},$$

and by $H^\infty(\mathbf{S}_\psi)$ the space of bounded holomorphic functions on $\mathbf{S}_\psi$. A consequence of an abstract result of Cowling [C, Theorem 2] is that if $\psi > \pi\,|1/q - 1/2|$, $M: \mathbb{N} \to \mathbb{C}$ is a bounded sequence and there exists $\tilde{M}$ in $H^\infty(\mathbf{S}_\psi)$ such that

$$\tilde{M}(k) = M(k), \qquad k = 1, 2, 3, ...,$$

then $M(\mathscr{L})$ extends to a bounded operator on $L^q(\gamma)$.

In this paper we improve this result for the (finite-dimensional) Ornstein–Uhlenbeck operator, by showing that analyticity in a smaller sector suffices to give bounded operators on $L^p(\gamma)$.

The problem of the existence of a nonholomorphic functional calculus for generators of Markov semigroups has attracted considerable attention in recent years. So far, only a few examples have been understood, see, for instance Christ and Müller [ChM] and Hebisch [H].

For the statement of our main result, we need the following notation. Suppose that J is a nonnegative integer and that $\psi \in (0, \pi/2)$. We denote by $H^\infty(\mathbf{S}_\psi; J)$ the Banach space of all M in $H^\infty(\mathbf{S}_\psi)$ for which there exists a constant C such that

$$\sup_{R>0} \int_R^{2R} |\lambda^j D^j M(e^{\pm i\psi}\lambda)|^2 \frac{d\lambda}{\lambda} \leqslant C^2 \qquad \forall j \in \{0, 1, ..., J\}, \tag{1}$$

endowed with the norm

$$\|M\|_{\psi; J} = \inf\{C\colon (1) \text{ holds}\}.$$

Condition (1) is called a *Hörmander condition of order* J [Hö]. Note that (1) implies that $\sup_{z \in \mathbf{S}_\psi} |M(z)| \leqslant 2C$, if $J > 0$.

Our main result is the following

THEOREM 1. *Suppose that* $1 < p < \infty$ *and* $p \neq 2$, *and set* $\phi_p^* = \text{arc sin} |2/p - 1|$. *Let* $M\colon \mathbb{N} \to \mathbb{C}$ *be a bounded sequence and assume that there exists a bounded holomorphic function* $\tilde{M}$ *in* $\mathbf{S}_{\phi_p^*}$ *such that*

$$\tilde{M}(k) = M(k), \qquad k = 1, 2, 3, ...$$

Then the following hold:

(i) *if* $\tilde{M} \in H^\infty(\mathbf{S}_{\phi_p^*}; 4)$, *then* $M(\mathscr{L})$ *extends to a bounded operator on* $L^p(\gamma)$ *and hence on* $L^q(\gamma)$ *for all* q *such that* $|1/q - 1/2| \leqslant |1/p - 1/2|$;

(ii) *if* $\tilde{M} \in H^\infty(\mathbf{S}_{\phi_p^*})$ *and* $|1/q - 1/2| < |1/p - 1/2|$, *then* $M(\mathscr{L})$ *extends to a bounded operator on* $L^q(\gamma)$.

The next result shows that in Theorem 1, the size of the region of holomorphy, measured by the aperture of the cone, cannot be reduced.

THEOREM 2. *Let* p *and* ϕ_p^* *be as in Theorem* 1. *If* $\psi < \phi_p^*$, *there exists a function* M *which decays exponentially at infinity and belongs to* $H^\infty(\mathbf{S}_\psi; J)$ *for every positive integer* J, *such that* $M(\mathscr{L})$ *does not extend to a bounded operator on* $L^p(\gamma)$.

We remark that Theorem 1 may be sharpened by means of spaces $H^\infty(\mathbf{S}_{\phi_p^*}; J)$ with nonintegral J.

A significant feature of Theorem 1 is that the number of derivatives on M required in (i) is independent of the dimension d. However, our estimates depend strongly on d, so that our methods fail to give a multiplier result for the infinite dimensional Ornstein–Uhlenbeck operator. Note that Cowling's result holds in the infinite-dimensional case too. We recall that other important operators related to the Ornstein–Uhlenbeck semigroup, such as the Riesz transforms, have $L^p(\gamma)$ bounds independent of the dimension. The reader is referred to the elegant analytic proof of Pisier [Pi].

Theorems 1 and 2 are proved in Section 3. The main ingredient of the proof of Theorem 1 will be an estimate

$$(2)\qquad |||(\mathscr{L}+\varepsilon\mathscr{I})^{iu}|||_p \leqslant C(1+|u|)^{5/2}\, e^{\phi_p^*\,|u|} \qquad \forall \varepsilon\in(0,1] \qquad \forall u\in\mathbb{R},$$

where $|||\cdot|||_p$ denotes the operator norm on $L^p(\gamma)$ and $C>0$ is a constant. This will be combined with an abstract multiplier result for generators of holomorphic semigroups, which is a variant of an earlier result of Meda [M, Theorem 4] (see also [CM, Theorem 2.1]). The abstract multiplier result is proved in Section 2. The estimate (2) will be obtained as an easy consequence of Propositions 3.1 and 3.2, which contain norm estimates concerning two auxiliary operators, $J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})$ and $K^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})$, introduced at the beginning of Section 3. The norm estimates for $J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})$, in turn, hinge on pointwise estimates off the diagonal for the distributional kernels of the complex powers of the resolvent operator $(\mathscr{L}+\varepsilon\mathscr{I})^{-1}$. This analysis is rather technical and occupies Sections 4 and 5.

One of the main ingredients of our approach is a careful analysis of the complex time Ornstein–Uhlenbeck semigroup. The notation and some preliminary results concerning the Ornstein–Uhlenbeck semigroup are contained in Section 1.

Maximal estimates for the complex Ornstein–Uhlenbeck semigroup will appear in a forthcoming paper.

1. NOTATION AND PRELIMINARY RESULTS

We shall consider L^p spaces both with respect to Lebesgue measure and Gauss measure, which we denote by $L^p(\mathbb{R}^d)$ and $L^p(\gamma)$, respectively.

Suppose that $\mathscr{M}$ is a continuous linear operator from $C_c^\infty(\mathbb{R}^d)$ into distributions. By the Schwartz kernel theorem there is a unique distribution $m_S\in\mathscr{D}'(\mathbb{R}^d\times\mathbb{R}^d)$ such that

$$\langle \mathscr{M}\phi, \psi\rangle_{\mathbb{R}^d} = \langle m_S, \psi\otimes\phi\rangle_{\mathbb{R}^{2d}} \qquad \forall \phi, \psi\in C_c^\infty(\mathbb{R}^d),$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2d}}$ denote the pairings between test functions and distributions in $\mathbb{R}^d$ and in $\mathbb{R}^{2d}$, respectively. We call the distribution $(1\otimes\gamma_0^{-1})\, m_S$ the *kernel of* $\mathscr{M}$ and denote it by m. The justification for this notation is that if m_S is locally integrable, then $\mathcal{M}$ may be represented as an integral operator with kernel m with respect to the Gauss measure. Indeed,

$$\mathcal{M}\phi(x) = \int_{\mathbb{R}^d} m_S(x, y)\, \phi(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} m(x, y)\, \phi(y)\, \mathrm{d}\gamma(y) \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d).$$

If $\mathcal{M}$ is a bounded operator on $L^2(\gamma)$, then it maps $C_c^\infty(\mathbb{R}^d)$ functions continuously into distributions, so that we may consider its kernel. In particular, if $\mathcal{R}z \geqslant 0$, we denote by h_z the kernel of the operator $\mathcal{H}_z$ spectrally defined by

$$\mathcal{H}_z f = \sum_{n=0}^{\infty} e^{-zn} \mathcal{P}_n f \qquad \forall f \in L^2(\gamma);$$

h_z is called the *Mehler kernel* and is given by the smooth function

$$(3) h_z(x, y) = (1 - e^{-2z})^{-d/2}$$
$$\times \exp\left[\frac{1}{2}\,\frac{1}{e^z + 1}\,|x + y|^2 - \frac{1}{2}\,\frac{1}{e^z - 1}\,|x - y|^2\right] \qquad \forall x, y \in \mathbb{R}^d,$$

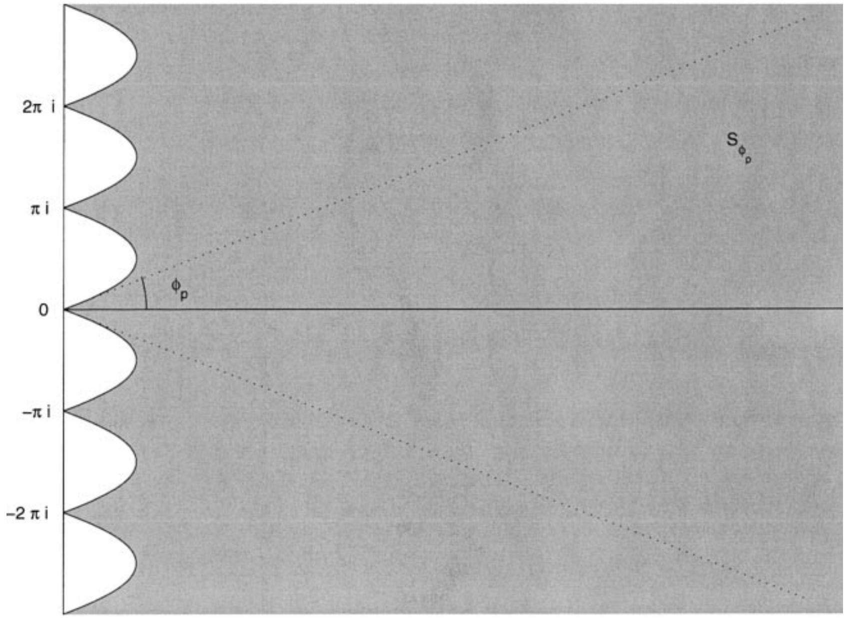
if $z \notin i\pi\mathbb{Z}$. For k in $\mathbb{Z}$, the distribution $h_{ik\pi}$ is defined by

$$\langle h_{ik\pi}, \phi \rangle = \int_{\mathbb{R}^d} \phi(x, (-1)^k x) \exp(|x|^2)\, \mathrm{d}x \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d).$$

It is easy to check that $\{h_z\}_{\mathcal{R}z \geqslant 0}$ is an analytic family of distributions, and that $\{\mathcal{H}_z\}_{\mathcal{R}z \geqslant 0}$ is an analytic family of continuous operators from $C_c^\infty(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$. In particular, if $\phi \in C_c^\infty(\mathbb{R}^d)$ and $k \in \mathbb{Z}$ we have that $\mathcal{H}_{ik\pi}\phi(x) = \phi((-1)^k x)$. Further properties of $\{\mathcal{H}_z\}_{\mathcal{R}z \geqslant 0}$ are contained in Proposition 1.1 below. For every p in $(1, \infty)$, $p \neq 2$, set $\phi_p = \arccos |2/p - 1|$, and denote by $\mathbf{E}_p$ the set

$$\{x + iy \in \mathbb{C} : |\sin y| \leqslant (\tan \phi_p) \sinh x\};$$

see Fig 1. If $p = 2$, define ϕ_p to be $\pi/2$ and $\mathbf{E}_p$ to be $\overline{\mathbf{S}}_{\pi/2}$. The set $\mathbf{E}_p$ is a closed πi-periodic subset of the right half-plane. The rays $[0, e^{\pm i\phi_p}\infty)$ are contained in $\mathbf{E}_p$ and are tangent to the boundary of $\mathbf{E}_p$ at the origin. Note that if $1/p + 1/p' = 1$, then $\mathbf{E}_p = \mathbf{E}_{p'}$, and that $\mathbf{E}_p \subset \mathbf{E}_q$ if $1 < p < q < 2$.



Region E_p

FIGURE 1

PROPOSITION 1.1. *Suppose that* $1 \leqslant p \leqslant \infty$. *The following hold*:

(i) *the semigroup* $\{\mathscr{H}_t\}_{t \geqslant 0}$ *is markovian*;

(ii) *if* $t > 0$ *and* $1 < p < 2$, *then* $\mathscr{H}_t$ *is bounded from* $L^p(\gamma)$ *to* $L^2(\gamma)$ *if and only if* $t \geqslant -\log\sqrt{p-1}$, *in which case it is a contraction*;

(iii) *the operator* $\mathscr{H}_z$ *extends to a bounded operator on* $L^p(\gamma)$ *if and only if* $z \in \mathbf{E}_p$, *in which case it is a contraction. Furthermore, the map* $z \mapsto \mathscr{H}_z$ *from* $\mathbf{E}_p$ *to the Banach algebra of bounded operators on* $L^p(\gamma)$ *is continuous in the strong operator topology, and its restriction to the interior of* $\mathbf{E}_p$ *is analytic.*

This result is well known. In particular, (ii) is due to Nelson [N], and (iii) to Epperson [E]. The reader is referred to [B] for (i) and more on the Ornstein–Uhlenbeck semigroup.

Positive constants are denoted either by c or by C; these may differ from one occurrence to another. The expression

$$A(t) \sim B(t) \qquad \forall t \in \mathbf{D},$$

where **D** is some subset of the domains of A and of B, means that there exist constants C and C' such that

$$C\,|A(t)| \leqslant |B(t)| \leqslant C'\,|A(t)| \qquad \forall t \in \mathbf{D}.$$

2. AN ABSTRACT HÖRMANDER TYPE MULTIPLIER THEOREM

In this section we prove a result concerning the existence of a bounded holomorphic functional calculus for infinitesimal generators of symmetric contraction semigroups. We shall use this result in Section 3 in our study of the Ornstein–Uhlenbeck operator.

Let X be a σ-finite measure space and $\mathscr{G}$ a positive linear operator on $L^2(X)$, possibly unbounded, but with dense domain. Let $\{\mathscr{E}_\lambda\}$ be the spectral resolution of the identity for which

$$\mathscr{G}f = \int_0^\infty \lambda \,\mathrm{d}\mathscr{E}_\lambda f \qquad \forall f \in \mathrm{Dom}(\mathscr{G}).$$

For every positive real number t, we define the operator $\mathscr{T}_t$ by

$$\mathscr{T}_t f = \int_0^\infty e^{-t\lambda}\,\mathrm{d}\mathscr{E}_\lambda f \qquad \forall f \in L^2(X).$$

We assume that each $\mathscr{T}_t$ has the contraction property

$$\|\mathscr{T}_t f\|_p \leqslant \|f\|_p \qquad \forall f \in L^2(X) \cap L^p(X)$$

whenever $1 \leqslant p \leqslant \infty$. A semigroup $\{\mathscr{T}_t\}_{t \geqslant 0}$ with the above properties is called a *symmetric contraction semigroup*, and $\mathscr{G}$ will be called the *infinitesimal generator* of $\{\mathscr{T}_t\}_{t \geqslant 0}$. Note that in many texts on semigroups, the generator of the semigroup is $-\mathscr{G}$ instead of $\mathscr{G}$.

Let M be a complex-valued, Borel measurable function on $\mathbb{R}^+$. The multiplier operator $M(\mathscr{G})$ is then defined on a suitable subspace of $L^2(X)$ by

$$M(\mathscr{G})f = \int_0^\infty M(\lambda)\,\mathrm{d}\mathscr{E}_\lambda f.$$

By spectral theory, if M is bounded, then $M(\mathscr{G})$ is bounded on $L^2(X)$. An important problem is to find conditions on M (and on the semigroup), so that the operator $M(\mathscr{G})$ extends to a bounded operator on $L^p(X)$ for some $p \in (1, \infty)$.

Recall that the Mellin transform $\mathscr{M}f$ of a function $f \in L^1(\mathbb{R}^+, \mathrm{d}\lambda/\lambda)$ is defined by

$$\mathscr{M}f(u) = \int_0^\infty f(\lambda)\, \lambda^{-iu} \frac{\mathrm{d}\lambda}{\lambda} \qquad \forall u \in \mathbb{R}.$$

Let M be a complex-valued, Borel measurable function on $\mathbb{R}^+$. Given a positive integer N, we denote by M_N: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ the function defined by

$$M_N(t, \lambda) = (t\lambda)^N \exp(-t\lambda)\, M(\lambda),$$

and by $\mathscr{M}M_N(t, \cdot)$ the Mellin transform of $M_N(t, \cdot)$.

If $\mathscr{T}$ is a bounded linear operator on $L^p(X)$, we denote by $|||\mathscr{T}|||_p$ its operator norm.

THEOREM 2.1. *Let $\mathscr{G}$ be the infinitesimal generator of a symmetric contraction semigroup and assume that the spectral projection $\mathscr{E}_0$ is trivial. Suppose that $1 < p < \infty$ and that M is a Borel measurable function on $\mathbb{R}^+$. If for some positive integer N*

$$\int_{-\infty}^\infty \sup_{t>0} |\mathscr{M}M_N(t, u)|\; |||\mathscr{G}^{iu}|||_p\, \mathrm{d}u < \infty,$$

then $M(\mathscr{G})$ extends to a bounded operator on $L^p(X)$.

This result was proved by Meda [M, Theorem 1]. A more elegant proof of the same result, due to Cowling and Meda, is in [CM, Theorem 2.1].

Suppose that $M \in H^\infty(\mathbf{S}_\psi)$. It is well known that M admits a bounded extension, also denoted by M, to $\overline{\mathbf{S}}_\psi$. For $|\theta| \leqslant \psi$, let M_θ: $\mathbb{R}^+ \to \mathbb{C}$ denote the function defined by

$$M_\theta(\lambda) = M(e^{i\theta}\lambda).$$

Suppose that J is a positive integer. We say that M_θ satisfies a Hörmander condition of order J if there exists a constant C such that

$$\sup_{R>0} \int_R^{2R} |\lambda^j D^j M_\theta(\lambda)|^2 \frac{\mathrm{d}\lambda}{\lambda} \leqslant C^2 \qquad \forall j \in \{0, 1, ..., J\}.$$

The smallest constant C for which this inequality holds is called the Hörmander J-constant of M_θ, and is denoted by $\|M_\theta\|_{\text{Hörm } J}$. Clearly, if M is in $H^\infty(\mathbf{S}_\psi; J)$, then

$$\|M\|_{\psi; J} = \max(\|M_\psi\|_{\text{Hörm } J}, \|M_{-\psi}\|_{\text{Hörm } J}).$$

We now state the main result of this section. Its proof is a slight modification of the proof of [M, Theorem 4]. Our result is related to a previous result of Cowling *et al.* [CDMY, Theorem 5.4] on the H^∞ functional calculus for a certain class of operators acting on Banach spaces.

THEOREM 2.2. *Let $\mathscr{G}$ be the infinitesimal generator of a symmetric contraction semigroup and assume that $\mathscr{E}_0 = 0$. Suppose that $1 < p < \infty$ and that there exist positive constants C and σ, and a constant $\theta \in (0, \pi/2)$ such that*

$$|||\mathscr{G}^{iu}|||_p \leqslant C(1+|u|)^\sigma \exp(\theta\,|u|) \qquad \forall u \in \mathbb{R}.$$

If $J > \sigma + 1$ and $M \in H^\infty(\mathbf{S}_\theta; J)$, then $M(\mathscr{G})$ extends to a bounded operator on $L^p(X)$, and

$$|||M(\mathscr{G})|||_p \leqslant C\,\|M\|_{\theta; J}.$$

Proof. We show that M satisfies the hypotheses of Theorem 2.1.

Let ψ be a $C_c^\infty(\mathbb{R})$ function supported in [1/2, 2] and such that

$$\sum_{k=-\infty}^{\infty} \psi(2^k \lambda) = 1 \qquad \forall \lambda \in \mathbb{R}^+.$$

Observe that

$$\begin{aligned} \mathscr{M}M_N(t,u) &= \int_0^\infty (t\lambda)^N e^{-t\lambda} M(\lambda)\,\lambda^{-iu}\,\frac{\mathrm{d}\lambda}{\lambda} \\ &= e^{(iN+u)\,\theta} \int_0^\infty (t\lambda)^N \exp(-e^{i\theta}t\lambda)\,M_\theta(\lambda)\,\lambda^{-iu}\,\frac{\mathrm{d}\lambda}{\lambda} \end{aligned}$$

by Cauchy's integral theorem. A change of variables $(t\lambda = v)$ shows that

$$\begin{aligned} e^{-\theta u}\,\mathscr{M}M_N(t,u) &= e^{iN\theta}\,t^{iu} \int_0^\infty v^{N-iu} \exp(-e^{i\theta}v)\,M_\theta(v/t)\,\frac{\mathrm{d}v}{v} \\ &= e^{iN\theta}\,t^{iu} \sum_{k=-\infty}^{\infty} \int_0^\infty v^{N-iu} \exp(-e^{i\theta}v)\,M_\theta(v/t)\,\psi(2^k v)\,\frac{\mathrm{d}v}{v}. \end{aligned}$$

The rest of the proof is a trivial modification of the proof of [M, Theorem 4]. We omit the details. ∎

In view of the application to the Ornstein–Uhlenbeck semigroup, we need a version of Theorem 2.2 for generators of symmetric contraction semigroups whose spectral projection $\mathscr{E}_0$ need not be trivial. This is the content of the next corollary.

COROLLARY 2.3. *Let $\mathcal{G}$ be the generator of the symmetric contraction semigroup $\{\mathcal{T}_t\}$. Suppose that $1<p<\infty$ and that there exist positive constants C and σ, and a constant $\theta\in(0,\pi/2)$ such that*

$$|||(\mathcal{G}+\varepsilon\mathcal{I})^{iu}|||_p \leqslant C(1+|u|)^\sigma \exp(\theta\,|u|) \qquad \forall\varepsilon\in(0,1] \qquad \forall u\in\mathbb{R}.$$

Let M: $[0,\infty)\to\mathbb{C}$ be a bounded Borel measurable function and suppose that there exists $\tilde{M}\in H^\infty(\mathbf{S}_\theta;J)$ for some $J>\sigma+1$ such that

$$\tilde{M}(\lambda)=M(\lambda) \qquad \forall\lambda\in\mathbb{R}^+.$$

Then $M(\mathcal{G})$ extends to a bounded operator on $L^p(X)$, and

$$|||M(\mathcal{G})|||_p \leqslant C(|M(0)|+\|M\|_{\theta;J}).$$

Proof. It is immediate to check that $\mathcal{G}+\varepsilon\mathcal{I}$ is the infinitesimal generator of the symmetric contraction semigroup $\{e^{-\varepsilon t}\mathcal{T}_t\}_{t\geqslant 0}$ and that its spectrum is contained in $[\varepsilon,\infty)$. Therefore, we may apply Theorem 2.2 and deduce that there exists a constant C such that

$$|||\tilde{M}(\mathcal{G}+\varepsilon\mathcal{I})|||_p \leqslant C\,\|\tilde{M}\|_{\theta;J}.$$

By spectral theory

$$\mathcal{E}_0 f=\lim_{t\to\infty}\mathcal{T}_t f \qquad \forall f\in L^2(X). \tag{4}$$

Since $\mathcal{T}_t$ is a contraction on $L^p(X)$, it follows that $\mathcal{E}_0$ is contractive on $L^p(X)$ for every p in $[1,\infty)$. Consequently, $\mathcal{I}-\mathcal{E}_0$ is bounded on $L^p(X)$, so that

$$|||(\mathcal{I}-\mathcal{E}_0)\,\tilde{M}(\mathcal{G}+\varepsilon\mathcal{I})|||_p \leqslant C\,\|\tilde{M}\|_{\theta;J}. \tag{5}$$

Observe that for every $\varepsilon>0$

$$\tilde{M}(\mathcal{G}+\varepsilon\mathcal{I})f=\tilde{M}(\varepsilon)\,\mathcal{E}_0 f+\int_{0^+}^\infty \tilde{M}(\lambda+\varepsilon)\,\mathrm{d}\mathcal{E}_\lambda f \qquad \forall f\in L^2(X).$$

Thus, if $\varepsilon_k\to 0^+$

$$\begin{aligned}(\mathcal{I}-\mathcal{E}_0)\,\tilde{M}(\mathcal{G}+\varepsilon_k\mathcal{I})f&=\int_{0^+}^\infty \tilde{M}(\lambda+\varepsilon_k)\,\mathrm{d}\mathcal{E}_\lambda f\\ &\to\int_{0^+}^\infty \tilde{M}(\lambda)\,\mathrm{d}\mathcal{E}_\lambda f\\ &=M(\mathcal{G})f-M(0)\,\mathcal{E}_0 f \qquad \forall f\in L^2(X),\end{aligned}$$

whence $(\mathscr{I}-\mathscr{E}_0)\,\tilde{M}(\mathscr{G}+\varepsilon_k\mathscr{I})$ converges to $M(\mathscr{G})-M(0)\,\mathscr{E}_0$ in the strong operator topology of $L^p(X)$ by (5). Therefore (5) implies that

$$|||M(\mathscr{G})-M(0)\,\mathscr{E}_0|||_p\leqslant C\,\|\tilde{M}\|_{\theta;\,J},$$

and finally that

$$|||M(\mathscr{G})|||_p\leqslant |M(0)|+C\,\|\tilde{M}\|_{\theta;\,J},$$

as required. ∎

Remark 2.4. Suppose that the symmetric contraction semigroup $\{\mathscr{T}_t\}$ preserves the class of real functions (in particular, this holds if $\{\mathscr{T}_t\}$ is a submarkovian semigroup). Assume that for some p in $(1,2)$ there exist positive constants C and σ, and a constant $\theta\in(0,\pi/2)$ such that

$$|||(\mathscr{G}+\varepsilon\mathscr{I})^{-iu}|||_p\leqslant C\,(1+u)^\sigma\exp(\theta u)\qquad\forall\varepsilon\in(0,1]\qquad\forall u\in\mathbb{R}^+. \tag{6}$$

We claim that

$$|||(\mathscr{G}+\varepsilon\mathscr{I})^{-iu}|||_p\leqslant C\,(1+|u|)^\sigma\exp(\theta\,|u|)\qquad\forall\varepsilon\in(0,1]\qquad\forall u\in\mathbb{R} \tag{6'}$$

and that an estimate similar to this holds, with p replaced by its conjugate index p'.

Indeed, since $\{\mathscr{T}_t\}$ preserves the class of real functions, the same holds for its infinitesimal generator $\mathscr{G}$ and for the spectral projections $\{\mathscr{E}_\lambda\}$. Therefore $\overline{\mathscr{E}_\lambda f}=\mathscr{E}_\lambda\bar{f}$, whence

$$\begin{aligned}\overline{(\mathscr{G}+\varepsilon\mathscr{I})^{iu}f}&=\overline{\varepsilon^{iu}\mathscr{E}_0 f}+\overline{\int_{0^+}^{\infty}(\lambda+\varepsilon)^{iu}\,\mathrm{d}\mathscr{E}_\lambda f}\\&=\varepsilon^{-iu}\mathscr{E}_0\bar{f}+\int_{0^+}^{\infty}(\lambda+\varepsilon)^{-iu}\,\mathrm{d}\mathscr{E}_\lambda\bar{f}\\&=(\mathscr{G}+\varepsilon\mathscr{I})^{-iu}\,\bar{f}\qquad\forall f\in L^2(X).\end{aligned}$$

If $f\in L^2(X)\cap L^p(X)$, then (6) implies that for all $v\in\mathbb{R}^+$

$$\begin{aligned}\|(\mathscr{G}+\varepsilon\mathscr{I})^{iv}\,f\|_p&=\|\overline{(\mathscr{G}+\varepsilon\mathscr{I})^{iv}\,f}\|_p\\&=\|(\mathscr{G}+\varepsilon\mathscr{I})^{-iv}\,\bar{f}\|_p\\&\leqslant C(1+v)^\sigma\exp(\theta v)\,\|f\|_p\qquad\forall\varepsilon\in(0,1];\end{aligned}$$

a density argument then shows that (6′) holds for all $u\in\mathbb{R}$.

Furthermore, for every $f \in L^2(X) \cap L^p(X)$ and every $g \in L^2(X) \cap L^{p'}(X)$

$$\begin{aligned}((\mathcal{G}+\varepsilon\mathcal{I})^{iu} f, g) &= (f, ((\mathcal{G}+\varepsilon\mathcal{I})^{iu})^\star g)\\ &= (f, (\mathcal{G}+\varepsilon\mathcal{I})^{-iu} g),\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\gamma)$ and * the Hilbert space adjoint. We have proved that for every $u \in \mathbb{R}$ the operator $(\mathcal{G}+\varepsilon\mathcal{I})^{iu}$ extends to a bounded operator on $L^p(X)$. It follows that for every $u \in \mathbb{R}$ the operator $(\mathcal{G}+\varepsilon\mathcal{I})^{-iu}$ extends to a bounded operator on $L^{p'}(X)$, as required to finish the proof of the claim.

3. THE MAIN RESULT

In this section we prove our main result, Theorem 1, modulo two propositions. Theorem 2 is also proved. The strategy for part (i) of Theorem 1 is to show that if $1<p<2$

$$|||(\mathcal{L}+\varepsilon\mathcal{I})^{-iu}|||_p \leqslant C(1+u)^{5/2}\exp(\phi_p^* u) \qquad \forall \varepsilon \in (0,1] \qquad \forall u \in \mathbb{R}^+,$$

and then to apply Remark 2.4 and Corollary 2.3.

First we need a little more notation. We denote by $\tau\colon (\mathbb{C}\setminus\mathbb{R}) \cup (-1,1) \to \mathbb{C}$ the transformation

$$\tau(\zeta) = \log\frac{1+\zeta}{1-\zeta},$$

where $\log w$ is real when $w>0$. It is straightforward to check that τ is a biholomorphic transformation of $(\mathbb{C}\setminus\mathbb{R}) \cup (-1,1)$ onto the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$. In particular, if $1<p<2$, then τ maps $\mathbf{S}_{\phi_p}\setminus[1,\infty)$ onto the interior of $\mathbf{E}_p \cap \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ and the ray $[0, e^{i\phi_p}\infty)$ onto $\partial\mathbf{E}_p \cap \{z \in \mathbb{C} : 0 \leqslant \mathrm{Im}\, z < \pi\}$.

Observe that if $z = \tau(\zeta)$, then

$$1-e^{-2z} = \frac{4\zeta}{(1+\zeta)^2}, \qquad \frac{1}{2}\,\frac{1}{e^z+1} = \frac{1}{4} - \frac{\zeta}{4}, \qquad \text{and} \qquad -\frac{1}{2}\,\frac{1}{e^z-1} = \frac{1}{4} - \frac{1}{4\zeta}.$$

From Mehler's formula (3) for the heat kernel, we deduce immediately that

$$h_{\tau(\zeta)}(x,y) = \frac{(1+\zeta)^d}{(4\zeta)^{d/2}}\exp\left[\frac{|x|^2+|y|^2}{2} - \frac{1}{4}(\zeta\,|x+y|^2 + \zeta^{-1}\,|x-y|^2)\right]. \tag{7}$$

If z and w are complex numbers, we denote by $[z,w]$ the closed segment in the complex plane joining z and w. We denote by z_p the point $\tau(e^{i\phi_p}/2)$, which is in $\partial \mathbf{E}_p$, by α_p^* the set $\tau([0, e^{i\phi_p}/2])$, and by α_p the regular curve $t \mapsto e^{i\phi_p}t$, $0 \leqslant t \leqslant 1/2$. Further, β_p^* will denote the union of the segment $[z_p, e^{i\phi_p}]$ and the ray $[e^{i\phi_p}, e^{i\phi_p}\infty)$, and β_p a piecewise regular curve with range β_p^*.

For every complex number w such that $\mathscr{R}w > 0$, we define the functions $J^{p,w}\colon \mathbb{R}^+ \to \mathbb{C}$ and $K^{p,w}\colon \mathbb{R}^+ \to \mathbb{C}$ by

$$J^{p,w}(\lambda) = \frac{1}{\Gamma(w)} \int_{\alpha_p} z^w \, e^{-\lambda z} \frac{dz}{z} \qquad \text{and}$$

$$K^{p,w}(\lambda) = \frac{1}{\Gamma(w)} \int_{\beta_p} z^w \, e^{-\lambda z} \frac{dz}{z}.$$

Observe that the function $w \mapsto K^{p,w}(\lambda)$ is entire. The function $w \mapsto J^{p,w}(\lambda)$ is analytic in the half plane $\mathscr{R}w > 0$. A complex integration by parts shows that if $\mathscr{R}w > 0$

$$J^{p,w}(\lambda) = \frac{\lambda}{\Gamma(w+1)} \int_{\alpha_p} z^w \, e^{-\lambda z} \, dz + \frac{z_p^w \exp(-z_p \lambda)}{\Gamma(w+1)}. \tag{8}$$

The right hand side is analytic in the half plane $\operatorname{Re} w > -1$. We shall use (8) to define $J^{p,w}(\lambda)$ for $-1 < \operatorname{Re} w \leqslant 0$. In particular, $J^{p,w}(\lambda)$ is defined for $w \in i\mathbb{R}$.

For every $\varepsilon > 0$ we define the operators $J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ and $K^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ by the formulae

$$J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I}) f = \sum_{n=0}^{\infty} J^{p,w}(n + \varepsilon) \, \mathscr{P}_n f \qquad \text{and}$$

$$K^{p,w}(\mathscr{L} + \varepsilon \mathscr{I}) f = \sum_{n=0}^{\infty} K^{p,w}(n + \varepsilon) \, \mathscr{P}_n f,$$

on their natural domains. It is easy to show (see the proof of Theorem 1(i) below) that if $u \in \mathbb{R}$

$$(\mathscr{L} + \varepsilon \mathscr{I})^{-iu} f = J^{p,iu}(\mathscr{L} + \varepsilon \mathscr{I}) f + K^{p,iu}(\mathscr{L} + \varepsilon \mathscr{I}) f \qquad \forall f \in L^2(X).$$

Thus, we are led to the problem of finding $L^p(\gamma)$ estimates for $J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ and $K^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$. Our main results concerning $J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ and $K^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ are Proposition 3.1 and Proposition 3.2 below. The proof of Proposition 3.1, which is quite technical and requires a detailed analysis of the kernel of $J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})$, will be given in Section 5.

PROPOSITION 3.1. *Suppose that* $1<p<2$. *Then there exists* C *such that*

$$|||J^{p,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p \leqslant C(1+u)^{5/2}\, e^{\phi_p^* u} \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in(0,1].$$

PROPOSITION 3.2. *Suppose that* $1<p<2$. *Then there exists* C *such that*

$$|||(\mathscr{I}-\mathscr{P}_0)\,K^{p,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p \leqslant C(1+u)^{1/2}\, e^{\phi_p^* u} \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in\mathbb{R}^+.$$

Proof. Define $t_p=-\log\sqrt{p-1}$. We claim that there exists C such that

$$|||(\mathscr{I}-\mathscr{P}_0)\,e^{-\varepsilon z}\,\mathscr{H}_z|||\;p \leqslant C\min(1, e^{-(\operatorname{Re} z - t_p)}) \qquad \forall z\in\mathbf{E}_p.$$

Indeed, on the one hand Proposition 1.1(iii) and the boundedness of $\mathscr{I}-\mathscr{P}_0$ in $L^p(\gamma)$ imply that

$$|||(\mathscr{I}-\mathscr{P}_0)\,e^{-\varepsilon z}\mathscr{H}_z|||_p \leqslant C\,|||e^{-\varepsilon z}\mathscr{H}_z|||_p \leqslant C \qquad \forall z\in\mathbf{E}_p.$$

On the other hand, if $\operatorname{Re} z\geqslant t_p$, then

$$\begin{aligned}
\|(\mathscr{I}-\mathscr{P}_0)\,e^{-\varepsilon z}\mathscr{H}_z f\|_p &\leqslant \|\mathscr{H}_z(\mathscr{I}-\mathscr{P}_0) f\|_2\\
&= \|\mathscr{H}_{\operatorname{Re} z}(\mathscr{I}-\mathscr{P}_0) f\|_2\\
&= \left(\sum_{n=1}^{\infty} e^{-2(\operatorname{Re} z - t_p + t_p)\,n}\,\|\mathscr{P}_n f\|_2^2\right)^{1/2}\\
&\leqslant e^{-(\operatorname{Re} z - t_p)}\,\|\mathscr{H}_{t_p}(\mathscr{I}-\mathscr{P}_0) f\|_2\\
&\leqslant e^{-(\operatorname{Re} z - t_p)}\,\|(\mathscr{I}-\mathscr{P}_0) f\|_p\\
&\leqslant C\,e^{-(\operatorname{Re} z - t_p)}\,\|f\|_p \qquad \forall f\in L^p(\gamma)\cap L^2(\gamma).
\end{aligned}$$

The first inequality follows from Hölder's inequality and the fact that $\gamma(\mathbb{R}^d)=1$, the second is a consequence of spectral theory, the third follows from the hypercontractivity of $\mathscr{H}_t$ (Proposition 1.1(ii)) and the fourth from the boundedness of $\mathscr{I}-\mathscr{P}_0$ on $L^p(\gamma)$. A density argument then shows that $|||(\mathscr{I}-\mathscr{P}_0)\,e^{-\varepsilon z}\mathscr{H}_z|||_p \leqslant C\,e^{-(\mathscr{R}z - t_p)}$, as required to finish the proof of the claim.

Recall that for every $s\in\mathbb{R}$, $|\Gamma(s+iu)|\sim|u|^{s-1/2}\,e^{-\pi|u|/2}$ as $|u|$ tends to ∞. Observe that

$$|z^{iu}|\leqslant e^{-\phi_p u} \qquad \forall z\in\beta_p^* \qquad \forall u\in\mathbb{R}^+,$$

because $\arg z \geqslant \phi_p$ for every z in β_p^*. Thus,

$$\left|\!\left|\!\left| \frac{1}{\Gamma(-iu)} \int_{\beta_p} z^{iu}\, e^{-\varepsilon z} (\mathscr{I} - \mathscr{P}_0)\, \mathscr{H}_z \frac{\mathrm{d}z}{z} \right|\!\right|\!\right|_p$$

$$\leqslant C \left| \frac{1}{\Gamma(-iu)} \right| \int_{\beta_p} |z^{iu}| \min(1, e^{-(\mathrm{Re}\, z - t_p)}) \frac{|\mathrm{d}z|}{|z|}$$

$$\leqslant C\, (1+u)^{1/2}\, e^{\phi_p^* u},$$

as required. ∎

Now we prove our main result, Theorem 1, which we restate for the reader's convenience.

THEOREM 1. *Suppose that* $1 < p < \infty$ *and* $p \neq 2$, *and set* $\phi_p^* = \arcsin |2/p - 1|$. *Let* $M\colon \mathbb{N} \to \mathbb{C}$ *be a bounded sequence and assume that there exists a bounded holomorphic function* $\tilde{M}$ *in* $\mathbf{S}_{\phi_p^*}$ *such that*

$$\tilde{M}(k) = M(k), \qquad k = 1, 2, 3, \ldots.$$

Then the following hold:

(i) *if* $\tilde{M} \in H^\infty(\mathbf{S}_{\phi_p^*}; 4)$, *then* $M(\mathscr{L})$ *extends to a bounded operator on* $L^p(\gamma)$ *and hence on* $L^q(\gamma)$ *for all* q *such that* $|1/q - 1/2| \leqslant |1/p - 1/2|$;

(ii) *if* $\tilde{M} \in H^\infty(\mathbf{S}_{\phi_p^*})$ *and* $|1/q - 1/2| < |1/p - 1/2|$, *then* $M(\mathscr{L})$ *extends to a bounded operator on* $L^q(\gamma)$.

Proof. We first prove (i). By duality we may assume that $1 < p < 2$. Suppose that $\mathscr{R}w > 0$. Recall the following classical formula

$$\lambda^{-w} = \frac{1}{\Gamma(w)} \int_0^\infty t^w\, e^{-t\lambda} \frac{\mathrm{d}t}{t} \qquad \forall \lambda > 0.$$

By Cauchy's integral theorem applied to the analytic function $z \mapsto z^{w-1}\, e^{-\lambda z}$,

$$\lambda^{-w} = \frac{1}{\Gamma(w)} \int_{\alpha_p + \beta_p} z^w\, e^{-\lambda z} \frac{\mathrm{d}z}{z}$$

$$= J^{p,w}(\lambda) + K^{p,w}(\lambda).$$

If $\operatorname{Re} w \leqslant 0$ we interpret this formula by analytic continuation. Then, by spectral theory,

$$\begin{aligned}(\mathscr{L}+\varepsilon\mathscr{I})^{-iu} &= \mathscr{P}_0(\mathscr{L}+\varepsilon\mathscr{I})^{-iu}+(\mathscr{I}-\mathscr{P}_0)(\mathscr{L}+\varepsilon\mathscr{I})^{-iu}\\ &= \mathscr{P}_0(\mathscr{L}+\varepsilon\mathscr{I})^{-iu}+(\mathscr{I}-\mathscr{P}_0)\, J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})\\ &\quad +(\mathscr{I}-\mathscr{P}_0)\, K^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I}) \qquad \forall u\in\mathbb{R}^+.\end{aligned}$$

If f is in $L^2(\gamma)$ and hence in $L^p(\gamma)$, then by spectral theory and the fact that $|||\mathscr{P}_0|||_p=1$

$$\|\mathscr{P}_0(\mathscr{L}+\varepsilon\mathscr{I})^{-iu} f\|_p=\|\varepsilon^{-iu}\,\mathscr{P}_0 f\|_p=\|\mathscr{P}_0 f\|_p\leqslant\|f\|_p \qquad \forall u\in\mathbb{R}.$$

A density argument then shows that

$$|||\mathscr{P}_0(\mathscr{L}+\varepsilon\mathscr{I})^{-iu}|||_p\leqslant 1 \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in(0,1].$$

Since $\mathscr{I}-\mathscr{P}_0$ is bounded on $L^p(\gamma)$, by Proposition 3.1 we have that

$$|||(\mathscr{I}-\mathscr{P}_0)\, J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p\leqslant C\,(1+u)^{5/2}\, e^{\phi_p^* u} \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in(0,1].$$

Finally, Proposition 3.2 implies

$$|||(\mathscr{I}-\mathscr{P}_0)\, K^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p\leqslant C\,(1+u)^{1/2}\, e^{\phi_p^* u} \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in(0,1].$$

Therefore, we may conclude that

$$|||(\mathscr{L}+\varepsilon\mathscr{I})^{-iu}|||\, p\leqslant C\,(1+u)^{5/2}\, e^{\phi_p^* u} \qquad \forall u\in\mathbb{R}^+ \qquad \forall\varepsilon\in(0,1].$$

Then (i) follows from Corollary 2.3 and Remark 2.4.

We now prove (ii). Since $\widetilde{M}\in H^\infty(\mathbf{S}_{\phi_p^*})$, then by Cauchy's integral theorem it is in $H^\infty(\mathbf{S}_{\psi_q}; J)$ for any nonnegative integer J and for any q such that $|1/q-1/2|<|1/p-1/2|$. Then by (i) (with q instead of p) $M(\mathscr{L})$ extends to a bounded operator on $L^q(\gamma)$, as required.

The proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Suppose that $\psi<v<\phi_p^*$ and that $\delta>0$. We define $M_{v,\,\delta}$ by

$$M_{v,\,\delta}(z)=\exp[-\delta e^{i(\pi/2-v)}z].$$

Clearly, $M_{v,\,\delta}$ is in $H^\infty(\mathbf{S}_v)$, hence in $H^\infty(\mathbf{S}_\psi; J)$ for every nonnegative integer J by the Cauchy integral theorem. The corresponding spectral operator is the operator $\mathscr{H}_{\delta e^{i(\pi/2-\nu)}}$. If δ is sufficiently small, the point $\delta e^{i(\pi/2-\nu)}$ is not in $\mathbf{E}_p$, so that $\mathscr{H}_{\delta e^{i(\pi/2-\nu)}}$ is unbounded on $L^p(\gamma)$, by Proposition 1.1(iii).

Theorem 2 is proved. ∎

4. ESTIMATES FOR SOME KERNELS

Suppose that $\varepsilon > 0$, $1 < p < 2$ and w is a complex number. Let $r_\varepsilon^{p,w}$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be defined by

$$r_\varepsilon^{p,w}(x, y) = \frac{1}{\Gamma(w)} \int_{\alpha_p} z^w e^{-\varepsilon z} h_z(x, y) \frac{dz}{z}, \qquad x \neq y,$$

and $r_\varepsilon^{p,w}(x, x) = 0$. It is not hard to prove that this integral is absolutely convergent. We omit this verification, as it is implicit in Proposition 4.1 below. Note that the change of variables $z = \tau(\zeta)$ and formula (7) for the Mehler kernel show that

$$r_\varepsilon^{p,w}(x, y) = \frac{e^{(|x|^2 + |y|^2)/2}}{2^d \Gamma(w)} \int_{\tau^{-1} \circ \alpha_p} \tau(\zeta)^{w-1} \frac{(1+\zeta)^d}{\zeta^{d/2} e^{\varepsilon \tau(\zeta)}} \times e^{-(\zeta |x+y|^2 + \zeta^{-1} |x-y|^2)/4} \tau'(\zeta) \, d\zeta. \tag{9}$$

The function $r_\varepsilon^{p,w}$ agrees with the kernel of the operator $J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ off the diagonal (see Proposition 5.2 below). In this section we prove pointwise estimates for $r_\varepsilon^{p,w}$, which will be crucial for the study of the operator $J^{p,w}(\mathscr{L} + \varepsilon \mathscr{I})$ we shall perform in Section 5. In Proposition 4.1 we show that $r_\varepsilon^{p,w}$ satisfies standard estimates in a convenient neighbourhood of the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$. Define the *local region* by

$$L = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leqslant \min(1, |x+y|^{-1})\}.$$

Pointwise estimates in the complement of L are proved in Proposition 4.3.

PROPOSITION 4.1. *Suppose that $1 < p < 2$ and that $N > 0$ is an integer. Then there exists C such that for every $\varepsilon \in (0, 1]$, and every complex number w with $-N \leqslant \mathscr{R}w \leqslant d/2 - 1/N$, the following hold*:

(i) *if $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ then*

$$|r_\varepsilon^{p,w}(x, y)| \leqslant C \frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(w)|} e^{(|x|^2 + |y|^2)/2} |x - y|^{2 \operatorname{Re} w - d};$$

(ii) *if* (x, y) *is in the local region* L, *and* $x \neq y$, *then*

$$
\begin{aligned}
&|\nabla_x r_\varepsilon^{p,w}(x,y)| + |\nabla_y r_\varepsilon^{p,w}(x,y)| \\
&\qquad \leqslant C \frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(w)|} e^{(|x|^2+|y|^2)/2} |x-y|^{2\operatorname{Re} w - d - 1}.
\end{aligned}
$$

Proof. We assume that $x \neq y$, because otherwise the conclusion is obvious. We shall need the integral $I(x, y; w, k)$, $k \in \mathbb{R}$, defined by

$$
\int_{\tau^{-1}\circ\alpha_p} \tau(\zeta)^{w-1} \frac{(1+\zeta)^d}{\zeta^k e^{\varepsilon\tau(\zeta)}} e^{-(\zeta|x+y|^2+\zeta^{-1}|x-y|^2)/4} \tau'(\zeta)\, \mathrm{d}\zeta.
$$

We parametrise $\tau^{-1}\circ\alpha_p$ by $\zeta = e^{i\phi_p} t$, where t is in $[0, 1/2]$. Since

$$
\left| \tau(\zeta)^{w-1} \frac{(1+\zeta)^d}{\zeta^k e^{\varepsilon\tau(\zeta)}} \tau'(\zeta) \right| \leqslant C\, e^{-\phi_p \operatorname{Im} w}\, t^{\operatorname{Re} w - k - 1}
$$

by elementary complex analysis, we obtain that

$$
\begin{aligned}
|I(x,y;w,k)| &\leqslant C\, e^{-\phi_p \operatorname{Im} w} \int_0^{1/2} t^{\operatorname{Re} w - k}\, e^{-\cos\phi_p |x-y|^2/4t} \frac{\mathrm{d}t}{t} \\
&= C \frac{4^{k-\operatorname{Re} w} e^{-\phi_p \operatorname{Im} w}}{(\cos\phi_p)^{k-\operatorname{Re} w}} |x-y|^{2\operatorname{Re} w - 2k} \\
&\qquad \times \int_{\cos\phi_p |x-y|^2/2}^{\infty} v^{k-\operatorname{Re} w}\, e^{-v} \frac{\mathrm{d}v}{v} \\
&\leqslant C \begin{cases} e^{-\phi_p \operatorname{Im} w} |x-y|^{2\operatorname{Re} w - 2k} & \text{if } \operatorname{Re} w < k \\ e^{-\phi_p \operatorname{Im} w}(1 + |\log|x-y||) & \text{if } \operatorname{Re} w = k \\ e^{-\phi_p \operatorname{Im} w} & \text{if } \operatorname{Re} w > k. \end{cases}
\end{aligned}
$$

Since $\mathscr{R}w < d/2$ by hypothesis, we deduce from (9) that

$$
\begin{aligned}
|r_\varepsilon^{p,w}(x,y)| &\leqslant C \frac{e^{(|x|^2+|y|^2)/2}}{|\Gamma(w)|} |I(x,y;w,d/2)| \\
&\leqslant C \frac{e^{(|x|^2+|y|^2)/2}}{|\Gamma(w)|} e^{-\phi_p \operatorname{Im} w} |x-y|^{2\operatorname{Re} w - d},
\end{aligned}
$$

as required to prove (i).

To prove (ii), it suffices to estimate $\nabla_x r_\varepsilon^{p,w}$, because $r_\varepsilon^{p,w}$ is symmetric. By differentiating (9) under the integral sign, it is easy to check that

$$\nabla_x r_\varepsilon^{p,w}(x,y) = x\, r_\varepsilon^{p,w}(x,y) + \frac{e^{(|x|^2+|y|^2)/2}}{2^d \Gamma(w)} \nabla_x I(x,y;w,d/2),$$

and that

$$\nabla_x I(x,y;w,d/2) = -\tfrac{1}{2}(x+y)\, I(x,y;w,d/2-1) \\ -\tfrac{1}{2}(x-y)\, I(x,y;w,d/2+1).$$

We remark that if $(x,y)\in L$, and $x\neq y$, then

$$|x| = \tfrac{1}{2}\,|x-y+x+y| \leqslant \tfrac{1}{2}(|x-y|+|x+y|) \leqslant |x-y|^{-1},$$

so that $\max(|x|,|x+y|)\leqslant |x-y|^{-1}$. Thus, from the estimates for I proved above we deduce that

$$\begin{aligned} |\nabla_x I(x,y;w,d/2)| &\leqslant \frac{1}{2\,|x-y|}\,|I(x,y;w,d/2-1)| \\ &\quad + \frac{1}{2}\,|x-y|\,|I(x,y;w,d/2+1)| \\ &\leqslant Ce^{-\phi_p \operatorname{Im} w}\,|x-y|^{2\operatorname{Re} w-d-1} \qquad \forall (x,y)\in L. \end{aligned}$$

Consequently,

$$\begin{aligned} |\nabla_x r_\varepsilon^{p,w}(x,y) &\leqslant |x-y|^{-1}\,|r_\varepsilon^{p,w}(x,y)| \\ &\quad + \frac{e^{(|x|^2+|y|^2)/2}}{2^d\,|\Gamma(w)|}\,|\nabla_x I(x,y;w,d/2)| \\ &\leqslant C\,\frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(w)|}\,e^{(|x|^2+|y|^2)/2}\,|x-y|^{2\operatorname{Re} w-d-1} \qquad \forall (x,y)\in L, \end{aligned}$$

as required. ∎

We now estimate $r_\varepsilon^{p,iu}$, $u\in\mathbb{R}$, off the local region. A similar analysis may be carried out for $r_\varepsilon^{p,w}$ for all complex w. We need a little more notation.

Suppose that a is in $\mathbb{R}^+$ and that $b\geqslant 0$. Let $F_{a,b}\colon \mathbb{R}^+\to\mathbb{C}$ be defined by

$$F_{a,b}(s) = -a(s+s^{-1}-2) + iab(s^{-1}-s).$$

Various estimates for $r_\varepsilon^{p,w}$ will involve integrals of $\exp(F_{a,b})$ for different values of the parameters a, b and w. We study such integrals in the following technical lemma, which will be used in Proposition 4.3.

LEMMA 4.2. *Suppose that δ, κ and N are in $\mathbb{R}^+$. Then there exists C (depending on δ, κ and N) such that the following hold*

(i) *for every $a \in \mathbb{R}^+$, every $b \geqslant 0$, every complex number v with $|\mathrm{Re}\, v| \leqslant N$, and every $\sigma \geqslant \delta > 1/2$ such that $a\sigma \geqslant \kappa$*

$$\left| \int_0^{1/2} s^v \, e^{F_{a,b}(s/\sigma)} \frac{\mathrm{d}s}{s} \right| \leqslant C\,(a\sigma)^{-1}\, e^{-2(1-1/2\delta)^2 a\sigma};$$

(ii) *for every $a \in [\kappa, \infty)$, every $b \in [0, N]$, every complex number v with $|\mathrm{Re}\, v| \leqslant N$ and every $\sigma \in (0, \delta)$*

$$\left| \int_0^{1/2} s^v \, e^{F_{a,b}(s/\sigma)} \frac{\mathrm{d}s}{s} \right| \leqslant \begin{cases} C(1+|\mathrm{Im}\, v|)\, \sigma^{\mathrm{Re}\, v}\, a^{-1/2} & \text{if } \; b = 0 \\ C(1+|\mathrm{Im}\, v|)\, \sigma^{\mathrm{Re}\, v}\, (ab)^{-1} & \text{if } \; b > 0. \end{cases}$$

Proof. For notational convenience we write F instead of $F_{a,b}$ and δ' instead of $1/(2\delta)$ during this proof.

We first prove (i). It is easy to check that if v is in $(0, \delta']$, then $\mathrm{Re}\, F(v) \leqslant -(1-\delta')^2 a/v$. Since s/σ is in $(0, \delta']$,

$$\begin{aligned} \left| \int_0^{1/2} s^v \, e^{F(s/\sigma)} \frac{\mathrm{d}s}{s} \right| &\leqslant \int_0^{1/2} s^{\mathrm{Re}\, v}\, e^{\mathrm{Re}\, F(s/\sigma)} \frac{\mathrm{d}s}{s} \\ &\leqslant \int_0^{1/2} s^{\mathrm{Re}\, v}\, e^{-(1-\delta')^2 a\sigma/s} \frac{\mathrm{d}s}{s} \\ &= ((1-\delta')^2 a\sigma)^{\mathrm{Re}\, v} \int_{2(1-\delta')^2 a\sigma}^{\infty} v^{-\mathrm{Re}\, v}\, e^{-v} \frac{\mathrm{d}v}{v} \\ &\sim (a\sigma)^{-1}\, e^{-2(1-\delta')^2 a\sigma}, \end{aligned}$$

as required to prove (i).

We now prove (ii). Without loss of generality, we may assume that $\delta > 1/2$. Clearly,

$$\left| \int_0^{1/2} s^v \, e^{F(s/\sigma)} \frac{\mathrm{d}s}{s} \right| \leqslant \left| \int_0^{\sigma/2\delta} s^v e^{F(s/\sigma)} \frac{\mathrm{d}s}{s} \right| + \left| \int_{\sigma/2\delta}^{1/2} s^v e^{F(s/\sigma)} \frac{\mathrm{d}s}{s} \right|.$$

Arguing much as in the proof of (i), we see that

$$\left|\int_0^{\sigma/2\delta} s^\nu e^{F(s/\sigma)} \frac{ds}{s}\right| \leqslant \int_0^{\sigma/2\delta} s^{\operatorname{Re}\nu} e^{\operatorname{Re} F(s/\sigma)} \frac{ds}{s} \tag{10}$$

$$\leqslant \int_0^{\sigma/2\delta} s^{\operatorname{Re}\nu} e^{-(1-\delta')^2 a\sigma/s} \frac{ds}{s}$$

$$= ((1-\delta')^2 a\sigma)^{\operatorname{Re}\nu} \int_{2\delta(1-\delta')^2 a}^{\infty} v^{-\operatorname{Re}\nu} e^{-v} \frac{dv}{v}$$

$$\leqslant C\sigma^{\operatorname{Re}\nu} a^{-1} e^{-2\delta(1-\delta')^2 a}.$$

By changing variables, we get

$$\left|\int_{\sigma/2\delta}^{1/2} s^\nu e^{F(s/\sigma)} \frac{ds}{s}\right| = \sigma^{\operatorname{Re}\nu} \left|\int_{\delta'}^{1/2\sigma} v^\nu e^{F(v)} \frac{dv}{v}\right|.$$

We claim that there exists C such that

$$\left|\int_{\delta'}^{1/2\sigma} v^\nu e^{F(v)} \frac{dv}{v}\right| \leqslant \begin{cases} C(1+|\nu|)\, a^{-1/2} & \text{if} \quad b=0 \\ C(1+|\nu|)(ab)^{-1} & \text{if} \quad b>0. \end{cases} \tag{11}$$

Assuming the claim, we immediately get (ii) from (10) and (11).

We now prove the claim, considering the two cases $1/(4-1/\delta) \leqslant \sigma \leqslant \delta$ and $\sigma < 1/(4-1/\delta)$ separately.

Suppose first that $1/(4-1/\delta) \leqslant \sigma \leqslant \delta$. By the mean value theorem, we may write $v^{\nu-1} = 1 + R(v; \nu)$, where

$$|R(v;\nu)| \leqslant C(1+|\nu|)\, |v-1| \qquad \forall v \in [\delta', 2-\delta'].$$

Correspondingly, we write

$$\int_{\delta'}^{1/2\sigma} v^\nu e^{F(v)} \frac{dv}{v} = \int_{\delta'}^{1/2\sigma} e^{F(v)}\, dv + \int_{\delta'}^{1/2\sigma} R(v;\nu)\, e^{F(v)}\, dv.$$

If v is in $[\delta', 2-\delta']$, then $\mathscr{R}F(v) = -a((v-1)^2/v) \leqslant -a((v-1)^2/(2-\delta'))$. Therefore,

$$\left|\int_{\delta'}^{1/2\sigma} R(v,\nu)\, e^{F(v)}\, dv\right| \leqslant C(1+|\nu|) \int_{\delta'}^{2-\delta'} |v-1|\, e^{-a(v-1)^2/(2-\delta')}\, dv$$

$$= 2C(1+|\nu|)(2-\delta')\, a^{-1} \int_0^{(1-\delta')\sqrt{a/(2-\delta')}} v e^{-v^2}\, dv$$

$$\leqslant C(1+|\operatorname{Im}\nu|)(1+a)^{-1}.$$

We now estimate $\int_{\delta'}^{1/2\sigma} e^{F(v)}\,\mathrm{d}v$. If $b=0$, then

$$\begin{aligned}\int_{\delta'}^{1/2\sigma} e^{F(v)}\,\mathrm{d}v &\leqslant \int_{\delta'}^{2-\delta'} e^{-a(v-1)^2/(2-\delta')}\,\mathrm{d}v \\ &= 2\left(\frac{2-\delta'}{a}\right)^{1/2}\int_0^{(1-\delta')\sqrt{a/(2-\delta')}} e^{-s^2}\,\mathrm{d}s \\ &\leqslant C\,(1+a)^{-1/2}.\end{aligned}$$

If $b>0$, an integration by parts shows that

$$\begin{aligned}\int_{\delta'}^{1/2\sigma} e^{F(v)}\,\mathrm{d}v &= \int_{\delta'}^{1/2\sigma} e^{\mathrm{Re}\,F(v)}\,e^{i\,\mathrm{Im}\,F(v)}\,\mathrm{d}v \\ &= \left.\frac{e^{F(v)}}{i\,\mathrm{Im}\,F'(v)}\right|_{\delta'}^{\sigma/2} + i\int_{\delta'}^{1/2\sigma}\left(\frac{\mathrm{Re}\,F'(v)}{\mathrm{Im}\,F'(v)} - \frac{\mathrm{Im}\,F''(v)}{(\mathrm{Im}\,F'(v))^2}\right) e^{F(v)}\,\mathrm{d}v.\end{aligned}$$

Since $\mathrm{Im}\,F'(v) = -ab(1+v^{-2})$, $\mathrm{Im}\,F''(v) = 2abv^{-3}$ and $\mathrm{Re}\,F'(v) = a(v^{-2}-1)$,

$$\left|\int_{\delta'}^{1/2\sigma} e^{F(v)}\,\mathrm{d}v\right| \leqslant C(ab)^{-1}\left(1 + a\int_{\delta'}^{2-\delta'} |v-1|\,e^{\mathrm{Re}\,F(v)}\,\mathrm{d}v\right).$$

We have already shown that the last integral is bounded by $C\,(1+a)^{-1}$. Thus, we may conclude that

$$\left|\int_{\delta'}^{1/2\sigma} e^{F(v)}\,\mathrm{d}v\right| \leqslant C(ab)^{-1},$$

as required to finish the proof of the claim in the case where $1/(4-1/\delta) \leqslant \sigma \leqslant \delta$.

We now consider the case where $\sigma < 1/(4-1/\delta)$. Clearly,

$$\left|\int_{\delta'}^{1/2\sigma} v^{\nu}\,e^{F(v)}\,\frac{\mathrm{d}v}{v}\right| \leqslant \left|\int_{\delta'}^{2-\delta'} v^{\nu} e^{F(v)}\,\frac{\mathrm{d}v}{v}\right| + \left|\int_{2-\delta'}^{1/2\sigma} v^{\nu} e^{F(v)}\,\frac{\mathrm{d}v}{v}\right|.$$

From (11) (with $\sigma = 1/(4-1/\delta)$) we see that

$$\left|\int_{\delta'}^{2-\delta'} v^{\nu} e^{F(v)}\,\frac{\mathrm{d}v}{v}\right| \leqslant \begin{cases} C(1+|\mathrm{Im}\,\nu|)\,a^{-1/2} & \text{if}\quad b=0 \\ C(1+|\mathrm{Im}\,\nu|)(ab)^{-1} & \text{if}\quad b>0. \end{cases}$$

Since $v \mapsto \operatorname{Re} F(v)/v$ is decreasing in $[1, \infty)$, we have that $\operatorname{Re} F(v) \leqslant -(1-\delta')^2\, av/(2-\delta')^2$ on $[2-\delta', \infty)$, and so

$$\begin{aligned}\left|\int_{2-\delta'}^{1/2\sigma} v^{\nu}\, e^{F(v)}\, \frac{dv}{v}\right| &\leqslant \int_{2-\delta'}^{1/2\sigma} v^{\operatorname{Re}\nu} e^{\operatorname{Re} F(v)}\, \frac{dv}{v} \\ &\times \int_{2-\delta'}^{\infty} v^{\operatorname{Re}\nu}\, e^{-(1-\delta')^2\, av/(2-\delta')^2}\, \frac{dv}{v} \\ &= \left(\frac{(1-\delta')^2\, a}{(2-\delta')^2}\right)^{-\operatorname{Re}\nu} \int_{(1-\delta')^2\, a/(2-\delta')}^{\infty} s^{\operatorname{Re}\nu}\, e^{-s}\, \frac{ds}{s} \\ &\sim a^{-1}\, e^{-(1-\delta')^2\, a/(2-\delta')},\end{aligned}$$

as required to finish the proof of the claim and of the lemma. ▮

We now estimate $r_{\varepsilon}^{p,\, iu}$ off the local region L. We call the complementary set of L the *global region* and denote it by G. Explicitly,

$$G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| > \min(1, |x+y|^{-1})\}.$$

For $0 < \eta < 1$, let D^{η} be defined by

$$D^{\eta} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \eta\, |x+y|\}.$$

PROPOSITION 4.3. *Suppose that $1 < p < 2$ and that $0 < \eta < 1$. Then there exists C such that for every $\varepsilon \in (0, 1]$ and every $u \in \mathbb{R}^+$ the following hold*:

(i) *for every $(x, y) \in G \cap D^{\eta}$*

$$|r_{\varepsilon}^{p,\, iu}(x, y)| \leqslant C(1+u)^2\, \frac{e^{-\phi_p u}}{|\Gamma(iu)|}\, \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}} \left(1 + \frac{|x-y|^{3/2}}{|x+y|^{1/2}}\right) \times e^{(|x|^2+|y|^2-(\cos\phi_p)\,|x-y|\,|x+y|)/2};$$

(ii) *for $1/2 < \eta < 1$ and for every $(x, y) \in G \setminus D^{\eta}$*

$$|r_{\varepsilon}^{p,\, iu}(x, y)| \leqslant C(1+u)\, \frac{e^{-\phi_p u}}{\Gamma(iu)}\, \frac{e^{-\mu\,|x-y|^2}}{|x-y|^2} \times e^{(|x|^2+|y|^2-(\cos\phi_p)\,|x-y|\,|x+y|)/2},$$

where $\mu = (1-1/2\eta)^2\,(\cos\phi_p)/2$.

Proof. We consider (9). By elementary complex analysis

$$\tau(\zeta)^{iu-1}\, \frac{(1+\zeta)^d}{\zeta^{d/2}\, e^{\varepsilon\tau(\zeta)}}\, \tau'(\zeta) = 2^{iu}\, \zeta^{iu-d/2-1} + R(\zeta; iu, \varepsilon),$$

where the remainder R satisfies the estimate

$$|R(\zeta;\, iu,\, \varepsilon)| \leqslant C(1+u)\, e^{-\phi_p u}\, |\zeta|^{-d/2} \qquad \forall \zeta \in \tau^{-1}(\alpha_p^*),$$

for some C independent of u. Then, we may write

$$r_\varepsilon^{p,\, iu}(x,\, y) = \frac{e^{(|x|^2+|y|^2)/2}}{2^{d-iu}\,\Gamma(iu)}\, A(x,\, y;\, iu) + \frac{e^{(|x|^2+|y|^2)/2}}{2^d\,\Gamma(iu)}\, B(x,\, y;\, iu), \tag{12}$$

where

$$A(x,\, y;\, iu) = \int_{\tau^{-1}\circ\alpha_p} \zeta^{iu-d/2-1}\, e^{-(\zeta\,|x+y|^2+\zeta^{-1}\,|x-y|^2)/4}\, \mathrm{d}\zeta$$

and

$$B(x,\, y;\, iu) = \int_{\tau^{-1}\circ\alpha_p} R(\zeta;\, iu,\, \varepsilon)\, e^{-(\zeta\,|x+y|^2+\zeta^{-1}\,|x-y|^2)/4}\, \mathrm{d}\zeta.$$

We parametrise $\tau^{-1}\circ\alpha_p$ by $\zeta = e^{i\phi_p}t$, where $0 \leqslant t \leqslant 1/2$. It is easy to check that

$$\begin{aligned} -\tfrac{1}{4}(\zeta\,|x+y|^2+\zeta^{-1}\,|x-y|^2 &= -a(t/\sigma+\sigma/t) + iab(\sigma/t - t/\sigma) \\ &= F_{a,\,b}(t/\sigma) - 2a, \end{aligned}$$

where $a = (\cos\phi_p)\,|x+y|\,|x-y|/4$, $b = \tan\phi_p$, $\sigma = |x-y|/|x+y|$ and $F_{a,\,b}$ is as in Lemma 4.2. A simple computation shows that

$$A(x,\, y;\, iu) = e^{i\phi_p(iu-d/2)}\, e^{-2a} \int_0^{1/2} t^{iu-d/2}\, e^{F_{a,\,b}(t/\sigma)}\, \frac{\mathrm{d}t}{t}, \tag{13}$$

and similarly that

$$\begin{aligned} |B(x,\, y;\, iu)| &\leqslant \int_{\tau^{-1}\circ\alpha_p} |R(\zeta;\, iu,\, \varepsilon)|\, e^{-\mathrm{Re}(\zeta\,|x+y|^2+\zeta^{-1}\,|x-y|^2)/4}\, \mathrm{d}\zeta \\ &\leqslant C(1+u)\, e^{-\phi_p u}\, e^{-2a} \int_0^{1/2} t^{1-d/2}\, e^{F_{a,\,0}(t/\sigma)}\, \frac{\mathrm{d}t}{t}. \end{aligned} \tag{14}$$

We now prove (i). We claim that if $(x,\, y) \in G \cap D^\eta$, then $|x-y|\,|x+y| > 1$. Indeed, if $\min(1,\, |x+y|^{-1}) = |x+y|^{-1}$, then $|x-y|\,|x+y| > 1$. If, instead, $\min(1,\, |x+y|^{-1}) = 1$, then $|x+y| \leqslant 1$, and $|x-y| > 1$ because $(x,\, y) \in G$. Thus,

$$|x+y| \leqslant 1 < |x-y| < \eta\,|x+y|,$$

which contradicts $\eta < 1$, and the claim is proved. Consequently, $a \geqslant (\cos \phi_p)/4$. We may apply Lemma 4.2(ii) (with $\kappa = (\cos \phi_p)/4$, $v = iu - d/2$ and $\delta = \eta$) to estimate the absolute value of the integral in (13) and (with the same values of κ and δ, but with $v = 1 - d/2$ and $b = 0$) to estimate the last integral in (14). We obtain that

$$|A(x, y; iu)| \leqslant C(1+u)\, e^{-\phi_p u} \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\, e^{-(\cos \phi_p)\, |x+y|\, |x-y|/2},$$

and that

$$|B(x, y; iu) \leqslant C(1+u)^2\, e^{-\phi_p u} \frac{|x+y|^{d/2-3/2}}{|x-y|^{d/2-1/2}}\, e^{-(\cos \phi_p)\, |x+y|\, |x-y|/2}.$$

By combining these estimates for $A(x, y; iu)$ and $B(x, y; iu)$ with (12), we obtain the required estimates for $r_\varepsilon^{p,\, iu}$ in the region $G \cap D^\eta$.

We now prove (ii). We claim that if $(x, y) \in G \cap (D^\eta)^c$, then $|x-y|^2 \geqslant \eta$. Indeed,

$$|x-y| \geqslant \min(1, |x+y|^{-1}) \geqslant \min(1, \eta\, |x-y|^{-1}).$$

Then either $\eta\, |x-y|^{-1} > 1$, so that $|x-y| > 1$, or $\eta\, |x-y|^{-1} \leqslant 1$, so that $|x-y| > \eta\, |x-y|^{-1}$, i.e., $|x-y|^2 \geqslant \eta$, as required to prove the claim. Now, $\sigma \geqslant \eta$ and $a\sigma = (\cos \phi_p)\, |x-y|^2/4 > \eta(\cos \phi_p)/4$. Therefore, we may apply Lemma 4.2(i) (with $\kappa = \eta(\cos \phi_p)/4$, $v = iu - d/2$ and $\delta = \eta$) to estimate the absolute value of the integral in (13) and (with the same values of κ and δ, but with $v = 1 - d/2$) to estimate the last integral in (14). We obtain that

$$|A(x, y; iu)| \leqslant Ce^{-\phi_p u}\, |x-y|^{-2} \times e^{-(\cos \phi_p)\, |x+y|\, |x-y|/2 - (1-1/2\eta)^2 \cos \phi_p\, |x-y|^2/2},$$

and that

$$|B(x, y; iu)| \leqslant C(1+u)\, e^{-\phi_p u}\, |x-y|^{-2} \times e^{-(\cos \phi_p)\, |x+y|\, |x-y|/2 - (1-1/2\eta)^2 \cos \phi_p\, |x-y|^2/2}.$$

By combining these estimates with (12), we obtain the required estimates for $r_\varepsilon^{p,\, iu}$ in the region $G \backslash D^\eta$.

The proof of the proposition is complete. ∎

The next proposition gives a condition which implies that an integral operator with kernel supported in the global region G is bounded on $L^p(\gamma)$. A related result, due to S. Pérez, is in [P, p. 71].

If $E \subset \mathbb{R}^d \times \mathbb{R}^d$, we denote by E_x its x-section, i.e., the set $\{ y \in \mathbb{R}^d : (x, y) \in E \}$.

PROPOSITION 4.4. *Suppose that* $1 < p < \infty$, $|1/r - 1/2| \leqslant |1/p - 1/2|$, $0 < \eta < 1$, $\mu > 0$ *and* $m: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ *is measurable. The following hold*:

(i) *if for every* $(x, y) \in D^\eta$

$$|m(x, y)| \leqslant C \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}} \left(1 + \frac{|x-y|^{3/2}}{|x+y|^{1/2}}\right) e^{(|x|^2 + |y|^2)/2 - |1/p - 1/2| \, |x-y| \, |x+y|},$$

then the integral operator $\mathcal{M}_1$ *defined by*

$$\mathcal{M}_1 \phi(x) = \int_{G_x \cap D_x^\eta} m(x, y) \, \phi(y) \, d\gamma(y) \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

extends to a bounded operator on $L^r(\gamma)$;

(ii) *if for every* $(x, y) \in (D^\eta)^c$

$$|m(x, y)| \leqslant C \frac{e^{-\mu |x-y|^2}}{|x-y|^2} e^{(|x|^2 + |y|^2)/2 - |1/p - 1/2| \, |x-y| \, |x+y|},$$

then the integral operator $\mathcal{M}_2$ *defined by*

$$\mathcal{M}_2 \phi(x) = \int_{G_x \setminus D_x^\eta} m(x, y) \, \phi(y) \, d\gamma(y) \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

extends to a bounded operator on $L^r(\gamma)$.

Proof. We fix r such that $|1/r - 1/2| \leqslant |1/p - 1/2|$. Let $\mathcal{U}_r: L^r(\mathbb{R}^d) \to L^r(\gamma)$ denote the invertible isometry defined by

$$\mathcal{U}_r f = \gamma_0^{-1/r} f \qquad \forall f \in L^r(\mathbb{R}^d).$$

We prove (i). We need to show that the operator $\mathcal{U}_r^{-1} \mathcal{M}_1 \mathcal{U}_r$, whose kernel with respect to Lebesgue measure is $(\gamma_0^{1/r} \otimes \gamma_0^{1/r'}) \, m\chi_G$, extends to a bounded operator on $L^r(\mathbb{R}^d)$. We define $q_r: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$q_r(x, y) = \left|\frac{1}{r} - \frac{1}{2}\right| \, ||x|^2 - |y|^2| - \left|\frac{1}{p} - \frac{1}{2}\right| \, |x+y| \, |x-y|.$$

Note that

$$\gamma_0(x)^{1/r} \, \gamma_0(y)^{1/r'} \, e^{(|x|^2 + |y|^2)/2 - |1/p - 1/2| \, |x-y| \, |x+y|} \leqslant e^{q_r(x, y)}.$$

Since $q_r \leqslant q_p$, our hypotheses imply that

$$\gamma_0^{1/r}(x)\,|m(x,y)|\,\gamma_0^{1/r'}(y) \leqslant C\,\frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\left(1+\frac{|x-y|^{3/2}}{|x+y|^{1/2}}\right)e^{q_p(x,y)}$$
$$\forall (x,y) \in D^\eta.$$

We claim that

$$\sup_{x\in\mathbb{R}^d}\int_{G_x \cap D_x^\eta} \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\left(1+\frac{|x-y|^{3/2}}{|x+y|^{1/2}}\right)e^{q_p(x,y)}\,dy < \infty. \tag{15}$$

If the claim holds, then by symmetry (15) holds with the role of x and y interchanged, so that

$$\sup_{x\in\mathbb{R}^d}\int_{G_x \cap D_x^\eta} \gamma_0^{1/r}(x)\,|m(x,y)|\,\gamma_0^{1/r'}(y)\,dy$$
$$+\sup_{y\in\mathbb{R}^d}\int_{G_y \cap D_y^\eta} \gamma_0^{1/r}(x)\,|m(x,y)|\,\gamma_0^{1/r'}(y)\,dx < \infty.$$

Hence $\mathcal{U}_r^{-1}\mathcal{M}_1\mathcal{U}_r$ extends to a bounded operator on $L^1(\mathbb{R}^d)$ and on $L^\infty(\mathbb{R}^d)$. By interpolation, $\mathcal{U}_r^{-1}\mathcal{M}_1\mathcal{U}_r$ extends to a bounded operator on $L^r(\mathbb{R}^d)$, as required.

To complete the proof of (i), it remains to prove (15). We denote by $B(z,r)$ the Euclidean ball centered at z and with radius r. It is straightforward to check that for every x in $\mathbb{R}^d\setminus\{0\}$ the set D_x^η is the ball $B(((1+\eta^2)/(1-\eta^2))\,x,\ (2\eta/(1-\eta^2))\,|x|)$. Thus, if $(x,y)\in D^\eta$

$$|y| \leqslant \frac{1+\eta}{1-\eta}\,|x|. \tag{16}$$

Moreover, $x+y$ is in the ball centered at $(2/(1-\eta^2))\,x$ and of radius $(2\eta/(1-\eta^2))\,|x|$, so that

$$\frac{2}{1+\eta}\,|x| \leqslant |x+y| \leqslant \frac{2}{1-\eta}\,|x|. \tag{17}$$

Note that $G_x \cap D_x^\eta$ is nonempty if and only if $|x|>c$ for some positive c. It is easy to check that there exists a constant $a>0$ such that for every x in $\mathbb{R}^d$ we have that $G_x \subseteq \{y\in\mathbb{R}^d : |y-x| \geqslant a/(1+|x|)\}$.

We treat the cases where $d=1$ and $d>1$ separately.

If $d=1$, then $q_p=0$. In view of the remarks above

$$
\begin{aligned}
\int_{G_x \cap D_x^\eta} & (|x+y|^{-1/2}\,|x-y|^{-3/2} + |x+y|^{-1})\,\mathrm{d}y \\
& \leqslant C\,|x|^{-1/2} \int_{G_x \cap D_x^\eta} |x-y|^{-3/2}\,\mathrm{d}y + C\,|x|^{-1} \int_{G_x \cap D_x^\eta} \mathrm{d}y \\
& \leqslant C\,|x|^{-1/2} \int_{a/(1+|x|)}^{(1+\eta)/(1-\eta)\,|x|} r^{-3/2}\,\mathrm{d}r + C\,|x|^{-1} \int_{a/(1+|x|)}^{(1+\eta)/(1-\eta)\,|x|} \mathrm{d}r \\
& \leqslant C,
\end{aligned}
$$

as required.

Suppose now that $d>1$. We need to estimate q_p on $G_x \cap D_x^\eta$. By combining (16) and (17), we obtain that

$$
\begin{aligned}
||y|^2 - |x|^2| + |x-y|\,|x+y| &\leqslant |x-y|\,(|x| + |y| + |x+y|) \\
&\leqslant \frac{4}{1-\eta}\,|x-y|\,|x|.
\end{aligned}
$$

If $x \neq 0$, let $\pi_x \colon \mathbb{R}^d \to \mathbb{R}^d$ denote the orthogonal projection onto the hyperplane of $\mathbb{R}^d$ orthogonal to x. Since $||y|^2-|x|^2|^2 - |x-y|^2\,|x+y|^2 = -4\,|x|^2\,|\pi_x(y)|^2$, we see that

$$
\begin{aligned}
||y^2| - |x|^2| - |x-y|\,|x+y| &= -4\,\frac{|x|^2\,|\pi_x(y)|^2}{||y|^2-|x|^2| + |x-y|\,|x+y|} \\
&\leqslant (\eta-1)\,\frac{|x|\,|\pi_x(y)^2|}{|x-y|}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{G_x \cap D_x^\eta} & \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\,e^{q_p(x,\,y)}\,\mathrm{d}y \\
& \leqslant C\,|x|^{d/2-1} \int_{G_x \cap D_x^\eta} \frac{e^{(\eta-1)\,|x|\,|\pi_x(y)|^2/|x-y|}}{|x-y|^{d/2+1}}\,\mathrm{d}y.
\end{aligned}
$$

We pass to polar coordinates around x, i.e., we write $y = x + r\omega$, where r is in $\mathbb{R}^+$ and $|\omega|=1$; the right hand side in the last inequality is bounded by

$$
C\,|x|^{d/2-1} \int_{S^{d-1}} \mathrm{d}\sigma(\omega) \int_{a/(1+|x|)}^{(1+\eta)/(1-\eta)\,|x|} r^{d/2-1} \exp\left((\eta-1)\,\frac{|x|\,|\pi_x(x+r\omega)|^2}{r}\right) \frac{\mathrm{d}r}{r}.
$$

We observe that $|\pi_x(x+r\omega)|^2 = r^2\,|\pi_x(\omega)|^2$, change variables by letting $|x|\,|\pi_x(\omega)|^2\,r = v$ in the inner integral, and obtain for $d>2$

$$\begin{aligned}
&\int_{G_x \cap D_x^\eta} \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\, e^{q_p(x,\,y)}\,\mathrm{d}y \\
&\leqslant C \int_{S^{d-1}} \mathrm{d}\sigma(\omega)\, |\pi_x(\omega)|^{2-d} \int_{a\,|x|\,|\pi_x(\omega)|^2/(1+|x|)}^{(1+\eta)/(1-\eta)\,|x|^2\,|\pi_x(\omega)|^2} v^{d/2-1}\, e^{(\eta-1)\,v}\, \frac{\mathrm{d}v}{v} \\
&\leqslant C \int_{\{|\pi_x(\omega)|<1/|x|\}} |\pi_x(\omega)|^{2-d}\,(|x|\,|\pi_x(\omega)|)^{d-2}\,\mathrm{d}\sigma(\omega) \\
&+ C \int_{\{|\pi_x(\omega)|\geqslant 1/|x|\}} |\pi_x(\omega)|^{2-d}\,\mathrm{d}\sigma(\omega) \\
&\leqslant C,
\end{aligned}$$

since here $|x|>c$. For $d=2$ we get

$$\begin{aligned}
&\int_{G_x \cap D_x^\eta} \frac{|x+y|^{d/2-1}}{|x-y|^{d/2+1}}\, e^{q_p(x,\,y)}\,\mathrm{d}y \\
&\leqslant C \int_{S^{d-1}} \mathrm{d}\sigma(\omega) \int_{a\,|x|\,|\pi_x(\omega)|^2/(1+|x|)}^{(1+\eta)/(1-\eta)\,|x|^2\,|\pi_x(\omega)|^2} e^{(\eta-1)\,v}\, \frac{\mathrm{d}v}{v} \\
&\leqslant C \int_{S^{d-1}} \mathrm{d}\sigma(\omega) \int_{c\,|\pi_x(\omega)|^2}^{\infty} e^{(\eta-1)\,v}\, \frac{\mathrm{d}v}{v} \\
&\leqslant C \int_{S^{d-1}} \log\left(1+\frac{1}{|\pi_x(\omega)|}\right) \mathrm{d}\sigma(\omega) \\
&\leqslant C.
\end{aligned}$$

Analogously, we may prove that

$$\sup_{x\in\mathbb{R}^d} \int_{G_x \cap D_x^\eta} \frac{|x+y|^{d/2-3/2}}{|x-y|^{d/2-1/2}}\, e^{q_p(x,\,y)}\,\mathrm{d}y < \infty,$$

as required to finish the proof of (15) and of (i).

We now prove (ii). By arguing as in the proof of (i), we may reduce the problem to showing that

$$\sup_{x\in\mathbb{R}^d} \int_{G_x \setminus D_x^\eta} \frac{1}{|x-y^2|}\, e^{q_p(x,\,y)-\mu\,|x-y|^2}\,\mathrm{d}y < \infty. \tag{18}$$

We observe that $|x-y| \geqslant c$ for y in $G_x \backslash D_x^\eta$. Since $q_p \leqslant 0$, (18) is easily proved.

The proof of the proposition is now complete. ∎

5. ANALYSIS OF $J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})$

In Lemma 5.1 below we prove estimates for $|||J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})|||_2$ when $\mathscr{R}w \geqslant 0$. We denote by $j_\varepsilon^{p,w}$ the kernel of $J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})$. In Proposition 5.2, we shall prove that the distribution $j_\varepsilon^{p,w}$ agrees off the diagonal with the function $r_\varepsilon^{p,w}$ from Section 4. It follows that $j_\varepsilon^{p,w}$ is locally integrable for $\operatorname{Re} w > 0$, and if $\operatorname{Re} w = 0$ its singular support is contained in the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$. The main result concerning $J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})$ is Proposition 3.1.

LEMMA 5.1. *Suppose that $1<p<2$ and that $N \in \mathbb{R}^+$. Then there exists C such that for every $\varepsilon \in \mathbb{R}^+$ and for every w with $0 \leqslant \mathscr{R}w \leqslant N$*

$$|||J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})|||_2 \leqslant C \frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(1+w)|}.$$

Moreover, let u be in $\mathbb{R}\backslash\{0\}$. Then $J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})$ converges to $J^{p,iu}(\mathscr{L}+\varepsilon\mathscr{I})$ in the strong operator topology of $L^2(\gamma)$ as w tends to iu in $\mathbf{S}_{\pi/2}$.

Proof. Note that $\arg z \geqslant \phi_p$ and $|z| \leqslant C$ for every z in α_p^*. Thus, $|z^w| \leqslant |z|^{\operatorname{Re} w} e^{-\phi_p \operatorname{Im} w}$, and we deduce from (8) that for $\lambda > 0$

$$|J^{p,w}(\lambda)| \leqslant C \frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(1+w)|} \left(e^{-\operatorname{Re} z_p \lambda} + \lambda \int_{\alpha_p} |z|^{\operatorname{Re} w} e^{-\lambda \operatorname{Re} z} \, |\mathrm{d}z| \right).$$

We claim that

$$\lambda \int_{\alpha_p} |z|^{\operatorname{Re} w} e^{-\lambda \mathscr{R} z} \, |\mathrm{d}z| \leqslant C \frac{\lambda}{(1+\lambda)^{1+\operatorname{Re} w}}.$$

Indeed,

$$\lambda \int_{\alpha_p} |z|^{\operatorname{Re} w} e^{-\lambda \operatorname{Re} z} \, |\mathrm{d}z| \leqslant C \lambda \int_0^1 t^{\operatorname{Re} w} e^{-\lambda t} \, \mathrm{d}t,$$

and considering separately the cases $\lambda \leqslant 1$ and $\lambda > 1$, one easily verifies the claim.

Thus, there exists C such that for every w with $0 \leqslant \mathscr{R}w \leqslant N$

$$|J^{p,w}(\lambda)| \leqslant C \frac{e^{-\phi_p \operatorname{Im} w}}{|\Gamma(1+w)|} \left(e^{-\operatorname{Re} z_p \lambda} + \frac{\lambda}{(1+\lambda)^{1+\operatorname{Re} w}} \right) \qquad \forall \lambda > 0.$$

The required estimate for $|||J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})|||_2$ follows from this by spectral theory.

We already know that $J^{p,w}(\lambda)$ is holomorphic in w near iu, and a routine computation shows that $J^{p,w}(\lambda) - J^{p,iu}(\lambda)$ is uniformly bounded as w tends to iu within $\mathbf{S}_{\pi/2}$.

$$\begin{aligned}&\|J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})f - J^{p,iu}(\mathscr{L}+\varepsilon\mathscr{I})f\|_2^2 \\ &\quad = \sum_{n=1}^{\infty} |J^{p,w}(n+\varepsilon) - J^{p,iu}(n+\varepsilon)|^2 \, \|\mathscr{P}_n f\|_2^2 \to 0,\end{aligned}$$

as required to finish the proof of the lemma. ∎

Proposition 5.2. *Suppose that $1<p<2$ and that $\varepsilon \in \mathbb{R}^+$.*

(i) *If $\mathscr{R}w > 0$, then the distribution $j_\varepsilon^{p,w}$ is the locally integrable function $r_\varepsilon^{p,w}$.*

(ii) *If $u \in \mathbb{R}\backslash\{0\}$ and $\Phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, then*

$$\begin{aligned}\langle (1 \otimes \gamma_0)\, j_\varepsilon^{p,iu} \Phi \rangle &= J^{p,iu}(\varepsilon) \int_{\mathbb{R}^d} \Phi(x,x)\,\mathrm{d}x \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\Phi(x,y) - \Phi(x,x))\, r_\varepsilon^{p,iu}(x,y)\,\mathrm{d}x\,\mathrm{d}\gamma(y).\end{aligned}$$

In particular, $j_\varepsilon^{p,iu}$ agrees with $r_\varepsilon^{p,iu}$ off the diagonal.

Proof. We first prove (i). For every pair of functions ϕ and ψ in $L^2(\gamma)$

$$\begin{aligned}(J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})\,\phi, \psi) &= \sum_{n=0}^{\infty} J^{p,w}(n+\varepsilon)(\mathscr{P}_n\phi, \psi) \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(w)} \int_{\alpha_p} z^w e^{-(n+\varepsilon)z} \frac{\mathrm{d}z}{z} (\mathscr{P}_n\phi, \psi),\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\gamma)$. Since $\sum_{n=0}^{\infty} |(\mathscr{P}_n\phi, \psi)| \leqslant \|\phi\|_2 \, \|\psi\|_2$ and

$$\left| \frac{1}{\Gamma(w)} \int_{\alpha_p} z^w e^{-(n+\varepsilon)z} \frac{\mathrm{d}z}{z} \right| \leqslant C(w) \int_{\alpha_p} |z|^{\operatorname{Re} w - 1} \, |\mathrm{d}z| < \infty,$$

we may interchange the order of summation and integration to get

$$\begin{aligned}(J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})\,\phi,\psi) &= \frac{1}{\Gamma(w)}\int_{\alpha_p}\frac{\mathrm{d}z}{z}\,z^w e^{-\varepsilon z}\sum_{n=1}^{\infty} e^{-nz}\,(\mathscr{P}_n\phi,\psi)\\ &= \frac{1}{\Gamma(w)}\int_{\alpha_p}\frac{\mathrm{d}z}{z}\,z^w\,e^{-\varepsilon z}\iint_{\mathbb{R}^d\times\mathbb{R}^d} h_z(x,y)\\ &\quad\times\phi(y)\,\psi(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(y).\end{aligned}$$

Suppose now that ϕ and ψ are in $C_c^\infty(\mathbb{R}^d)$. From formula (7) for the Mehler kernel, we deduce that

$$\begin{aligned}\sup_{z\in\alpha_p^*}\|h_z(x,\cdot)\|_{L^1(\gamma)} &= \sup_{\zeta\in[0,\,e^{i\phi_p}/2]}\|h_{\tau(\zeta)}(x,\cdot)\|_{L^1(\gamma)}\\ &\leqslant Ce^{|x|^2/2}\sup_{\zeta\in[0,\,e^{i\phi_p}/2]}\int_{\mathbb{R}^d}|\zeta|^{-d/2}\\ &\quad\times e^{-\cos\phi_p\,|x-y|^2/4\,|\zeta|}\,e^{-|y|^2/2}\,\mathrm{d}y\\ &\leqslant Ce^{|x|^2/2}.\end{aligned}$$

Thus, by Hölder's inequality,

$$\begin{aligned}&\iint_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\alpha_p}|z^w\,e^{-\varepsilon z}h_z(x,y)\,\phi(y)\,\psi(x)|\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(y)\,\frac{|\mathrm{d}z|}{|z|}\\ &\quad\leqslant\|\phi\|_\infty\int_{\mathbb{R}^d}\mathrm{d}\gamma(x)\sup_{z\in\alpha_p^*}\|h_z(x,\cdot)\|_1\,|\psi(x)|\int_{\alpha_p}|z^w|\,\frac{|\mathrm{d}z|}{|z|}\\ &\quad\leqslant C\,\|\phi\|_\infty\int_{\mathbb{R}^d}\mathrm{d}\gamma(x)\,e^{|x|^2/2}\,|\psi(x)|\int_{\alpha_p}|z^w|\,\frac{|\mathrm{d}z|}{|z|}\\ &\quad<\infty.\end{aligned}$$

Therefore, we may interchange the order of integration and obtain that

$$(J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})\,\phi,\psi)=\iint_{\mathbb{R}^d\times\mathbb{R}^d} r_\varepsilon^{p,w}(x,y)\,\phi(y)\,\psi(y)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(y),$$

and finally

$$J^{p,w}(\mathscr{L}+\varepsilon\mathscr{I})\,\phi(x)=\int_{\mathbb{R}^d} r_\varepsilon^{p,w}(x,y)\,\phi(y)\,\mathrm{d}\gamma(y),$$

as required.

We now prove (ii), and start by continuing $j_\varepsilon^{p,w}$ analytically to the half-plane $\operatorname{Re} w > -1/2$. If $\mathscr{R}w > 0$ and Φ is in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we may write $\langle (1 \otimes \gamma_0) j_\varepsilon^{p,w}, \Phi \rangle_{\mathbb{R}^{2d}}$ as

$$(19) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} r_\varepsilon^{p,w}(x, y)(\Phi(x, y) - \Phi(x, x))\, dx\, d\gamma(y)$$

$$+ \iint_{\mathbb{R}^d \times \mathbb{R}^d} r_\varepsilon^{p,w}(x, y)\, \Phi(x, x)\, dx\, d\gamma(y).$$

Since

$$|\Phi(x, y) - \Phi(x, x)| \leqslant C\, |x - y|,$$

it follows from the pointwise estimates for $r_\varepsilon^{p,w}$ proved in Proposition 4.1 that the first integral is absolutely convergent for $\operatorname{Re} w > -1/2$ and defines an analytic function there.

To continue analytically the second integral, observe that by the estimates in Proposition 4.1(i) and its proof, for $\operatorname{Re} w > 0$

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\alpha_p} |z^w e^{-\varepsilon z} h_z(x, y)\, \Phi(x, x)|\, dx\, d\gamma(y)\, \frac{|dz|}{|z|}$$

$$\leqslant C\, e^{-\phi_p \operatorname{Im} w} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{|x|^2/2} \frac{|\Phi(x, x)|}{|x - y|^{d - 2 \operatorname{Re} w}}\, e^{-|y|^2/2}\, dx\, dy.$$

In this double integral, we integrate first in y and obtain a continuous function of x, which is clearly in $L^1(\mathbb{R}^d)$. Therefore, we may interchange the order of integration in the second integral in (19) by Fubini's theorem, integrate first in y, use the fact that $\int_{\mathbb{R}^d} h_z(x, y)\, d\gamma(y) = 1$, and obtain that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} r_\varepsilon^{p,w}(x, y)\, \Phi(x, x)\, dx\, d\gamma(y) = J^{p,w}(\varepsilon) \int_{\mathbb{R}^d} \Phi(x, x)\, dx.$$

The right hand side here has an analytic continuation to $\operatorname{Re} w > -1$.

Thus $j_\varepsilon^{p,w}$ can be continued to $\mathscr{R}w > -1/2$. In particular, $\langle (1 \otimes \gamma_0) j_\varepsilon^{p,w}, \Phi \rangle_{\mathbb{R}^{2d}}$ tends to the right-hand side of the formula in (ii), as $w \to iu$, $\operatorname{Re} w > 0$.

The convergence in the strong operator topology from Lemma 5.1 implies that for Φ of the form $\psi \otimes \phi$ with $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$, this limit is

$$\langle (1 \otimes \gamma_0)\, j_\varepsilon^{p,iu}, \psi \otimes \phi \rangle$$

But two distributions in $\mathbb{R}^d \times \mathbb{R}^d$ which coincide on all tensor products $\psi \otimes \phi$ are equal.

Now (ii) follows, and the lemma is proved. ∎

We now prove Proposition 3.1, which we restate for the reader's convenience.

PROPOSITION 3.1. *Suppose that $1 < p < 2$. Then there exists C such that*

$$|||J^{p,\, iu}(\mathscr{L} + \varepsilon \mathscr{I})|||_p \leqslant C(1+u)^{5/2}\, e^{\phi_p^* u} \qquad \forall \varepsilon \in (0, 1] \qquad \forall u \in \mathbb{R}^+.$$

Proof. In this proof, C will denote a positive constant independent of ε in $(0, 1]$. Let φ be a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ which vanishes off L, is equal to 1 in

$$\left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \leqslant \frac{1}{2\,(1+|x|+|y|)}\right\},$$

and satisfies the estimate

$$|\nabla_x \varphi(x, y)| + |\nabla_y \varphi(x, y)| \leqslant C\, |x-y|^{-1}. \tag{20}$$

By Proposition 4.3 and Proposition 4.4 we see that the integral operator with kernel $(1-\varphi)\, j_\varepsilon^{p,\, iu}$ (with respect to the Gauss measure) is bounded on $L^r(\gamma)$ for $|1/r - 1/2| \leqslant |1/p - 1/2|$, and its operator norm is bounded by

$$C\,(1+u)^2\, \frac{e^{-\phi_p u}}{|\Gamma(iu)|} \qquad \forall \varepsilon \in (0, 1] \qquad \forall u \in \mathbb{R}^+.$$

Moreover, $J^{p,\, iu}(\mathscr{L} + \varepsilon \mathscr{I})$ is bounded on $L^2(\gamma)$ by spectral theory (Lemma 5.1), and

$$|||J^{p,\, iu}(\mathscr{L} + \varepsilon \mathscr{I})|||_2 \leqslant C\, \frac{e^{-\phi_p u}}{|\Gamma(1+iu)|} \qquad \forall \varepsilon \in (0, 1] \qquad \forall u \in \mathbb{R}^+.$$

Therefore, the integral operator with kernel $\varphi j_\varepsilon^{p,\, iu}$ (with respect to the Gauss measure) is bounded on $L^2(\gamma)$, and its operator norm is bounded by

$$C e^{-\phi_p u} \left(\frac{(1+u)^2}{|\Gamma(iu)|} + \frac{1}{|\Gamma(1+iu)|} \right) \qquad \forall \varepsilon \in (0, 1] \qquad \forall u \in \mathbb{R}^+.$$

The kernel of the same integral operator with respect to Lebesgue measure is $(1 \otimes \gamma_0)\, \varphi j_\varepsilon^{p,\, iu}$. We show that this kernel satisfies standard estimates.

From Proposition 4.1(i) we deduce that for every (x, y) in L such that $x \neq y$,

$$\gamma_0(y)\, \varphi(x, y)\, |j_\varepsilon^{p,\, iu}(x, y)| \leqslant C \frac{e^{-\phi_p u}}{|\Gamma(iu)|}\, e^{(|x|^2 - |y|^2)/2}\, |x-y|^{-d}.$$

Since $e^{|x|^2 - |y|^2}$ is uniformly bounded above for (x, y) in L, we may conclude that

$$\gamma_0(y)\, \varphi(x, y)\, |j_\varepsilon^{p,\, iu}(x, y)| \leqslant C \frac{e^{-\phi_p u}}{\Gamma(iu)}\, |x-y|^{-d}.$$

The gradient of $(1 \otimes \gamma_0)\, \varphi j_\varepsilon^{p,\, iu}$ with respect to y is the sum of three terms: $(1 \otimes \nabla_y \gamma_0)\, \varphi j_\varepsilon^{p,\, iu}$, $(1 \otimes \gamma_0)\, \nabla_y \varphi j_\varepsilon^{p,\, iu}$ and $(1 \otimes \gamma_0)\, \varphi\, \nabla_y j_\varepsilon^{p,\, iu}$. By Proposition 4.1(ii) the absolute value of the last term is bounded by

$$C \frac{e^{-\phi_p u}}{|\Gamma(iu)|}\, |x-y|^{-d-1} \qquad \forall (x, y) \in L, \qquad x \neq y.$$

Since

$$|y| \leqslant |x-y|^{-1} \qquad \forall (x, y) \in L,$$

we deduce from Proposition 4.1(i) that

$$\begin{aligned} |\nabla_y \gamma_0(y)|\, |\varphi j_\varepsilon^{p,\, iu}(x, y)| &= 2\, |y \gamma_0(y)|\, |\varphi j_\varepsilon^{p,\, iu}(x, y)| \\ &\leqslant 2\, |x-y|^{-1}\, |\gamma_0(y)\, \varphi j_\varepsilon^{p,\, iu}(x, y)| \\ &\leqslant C \frac{e^{-\phi_p u}}{|\Gamma(iu)|}\, |x-y|^{-d-1}. \end{aligned}$$

By (20), the second term satisfies similar estimates.

A trivial modification of the above argument shows that

$$|\nabla_x [(1 \otimes \gamma_0)\, \varphi j_\varepsilon^{p,\, iu}]\, (x, y)| \leqslant C \frac{e^{-\phi_p u}}{|\Gamma(iu)|}\, |x-y|^{-d-1}.$$

We deduce from [GMST, Theorem 3.7] that the integral operator with kernel $\varphi j_\varepsilon^{p,\, iu}$ with respect to the Gauss measure is bounded on $L^p(\gamma)$ and its $L^p(\gamma)$ operator norm is bounded by a constant times the sum of its $L^2(\gamma)$ operator norm and the constants appearing in the standard estimates, i.e., by

$$C\, e^{-\phi_p u} \left(\frac{(1+u)^2}{|\Gamma(iu)|} + \frac{1}{|\Gamma(1+iu)|} \right) \qquad \forall \varepsilon \in (0, 1] \qquad \forall u \in \mathbb{R}^+.$$

Thus, we may conclude that

$$|||J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p \leqslant C e^{-\phi_p u}\left(\frac{(1+u)^2}{|\Gamma(iu)|}+\frac{1}{|\Gamma(1+iu)|}\right)$$

$$\forall \varepsilon \in (0,1] \qquad \forall u \in \mathbb{R}^+.$$

From this and the asymptotics for the Γ-function, we deduce that

$$|||J^{p,\,iu}(\mathscr{L}+\varepsilon\mathscr{I})|||_p \leqslant C(1+u)^{5/2}\, e^{(\pi/2-\phi_p)\,u} \qquad \forall \varepsilon \in (0,1] \qquad \forall u \in \mathbb{R}^+,$$

as required. ▮

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