Se asume siempre que estamos trabajando en un espacio de probabilidad $(\Omega, \mathcal{A}, P)$. We also assume that the notation makes sense, so for instance, if we are talking about a martingale, there is a filtration, etc..

1) Prove that if $f \in L^1$, then $\{E(f|\mathcal{A}_n)\}_{n\geq 0}$ is U.I.. You may use standard measure theoretic results without proof. You may, or may not, use Chebyshev-Markov.

2) Let X be a submartingale and let T, S be stopping times. Prove that for all $n \geq 0$, $E(X_{T\wedge n}) \leq EX_n$. Prove that if $S \geq T$, then for all $n \geq 0$, $E(X_{T\wedge n}) \leq E(X_{S\wedge n})$.

3) There are several versions of Wald's identity on random sums. Sometimes it is assumed that all the variables involved are independent, as in, for instance Ex. 2, H.W.2, and the conclussion is that $E(S) = E(W)E(X_1)$, with the obvious notation. Here is a martingale variant: Prove that if $\{X_n\}_{n\geq 0}$ is a sequence of i.i.d.r.v. in L^1, $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$ and T is an integrable stopping time with respect to $\{\mathcal{A}_n\}_{n\geq 0}$, then the random sum $S_T(w) := \sum_{n=0}^{T(w)} X_n(w)$ satisfies $E(S_T) = E(T)E(X_1)$.

The total variation is one of the many ways to define a distance on the space of probability measures. In the following problems you are asked to prove some basic facts about the total variation when $\Omega = \mathbb{Z}$.

4) Sean μ y ν medidas de probabilidad en $\mathbb{Z}$. Definimos la distancia entre μ y ν, dada por la variación total, como $\|\mu - \nu\|_{TV} := \sum_{k=-\infty}^{\infty} |\mu(k) - \nu(k)|$. Probar que $\|\mu - \nu\|_{TV} = 2\sup_{A\subset\mathbb{Z}} |\mu(A) - \nu(A)|$. Sugerencia: Considerar $A = \{\mu \geq \nu\}$.

5) Probar que $\|\mu_1 * \mu_2 - \nu_1 * \nu_2\|_{TV} \leq \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|_{TV}$, donde la convolución se define como $P * Q(n) = \sum_{k=-\infty}^{\infty} P(n-k)Q(k)$.

6) Probar que $\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|_{TV} \leq \|\mu_1 - \nu_1\|_{TV} + \|\mu_2 - \nu_2\|_{TV}$.

7) Probar que $\|\mu_1 * \cdots * \mu_n - \nu_1 * \cdots * \nu_n\|_{TV} \leq \sum_{k=1}^{n} \|\mu_k - \nu_k\|_{TV}$. Sugerencia. Usar inducción.