Hoja 4, nº 7: $f_{(X,Y)}(x,y) = c\, e^{-(x^2-xy+y^2)/2}$

1) Hallar c. Como $\int_{\mathbb{R}^2} f_{(X,Y)}(x,y)\,dx\,dy = 1$, esta igualdad determina el valor de c.

Completamos el cuadrado: $x^2 - xy + y^2 + \frac{x^2}{4} - \frac{x^2}{4}$

$= \left(\frac{x^2}{4} + y^2 - xy\right) + \frac{3x^2}{4} = \left(\frac{x}{2} - y\right)^2 + \frac{3x^2}{4}$.

$$f_X(x) = \int_{-\infty}^{\infty} c \exp\left(-\frac{1}{2}\left[\frac{x}{2} - y\right]^2\right) \exp\left(-\frac{3x^2}{8}\right) dy$$

Como $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(y - \frac{x}{2}\right)^2\right) dy = 1$ (integramos la densidad de una v.a. $\sim N\left(\frac{x}{2}, 1\right)$)

$$f_X(x) = c \exp\left(-\frac{3x^2}{8}\right)\sqrt{2\pi} = c\sqrt{2\pi}\, \exp\left(-\frac{1}{2}\left(\frac{x}{2/\sqrt{3}}\right)^2\right)$$

Como $\frac{1}{\frac{2}{\sqrt{3}}\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{2/\sqrt{3}}\right)^2\right) dx = 1$ $\left(\sim N\left(0, \frac{2}{\sqrt{3}}\right)\right)$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = c\sqrt{2\pi}\,\frac{2}{\sqrt{3}}\sqrt{2\pi} = c\,\frac{4\pi}{\sqrt{3}}, \quad \boxed{c = \frac{\sqrt{3}}{4\pi}}$$

$$f_{(X,Y)}(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2-xy+y^2}{2}}$$

$f_X(x) = \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-\frac{1}{2}\frac{3x^2}{4}}$, $f_Y(y) = \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-\frac{1}{2}\frac{3y^2}{4}}$ por simetría.

$X, Y \sim N\left(0, \frac{2}{\sqrt{3}}\right)$, $f_{(X,Y)}(x,y) \neq f_X(x) f_Y(y)$.

No son independientes.