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Multiplicity of Solutions for the p-Laplacian

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Preface

This lectures notes are a review of some contributions to the theory of Quasilinear Elliptic Equations with a reaction term. We study the existence, multiplicity, positivity of solutions, bifurcation, etc., with respect to the relation between the **growth** and **shape** of the zero order reaction term.

We can propose a quite general class of elliptic problems involving the *p-laplacian*, for instance,

$$(P)\begin{cases} -\Delta_p u \equiv -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = F'(u) \quad \text{in } \Omega, \\ u|_{\delta\Omega} = 0, \end{cases}$$

where $1 < p < N$, Ω is a regular bounded domain in $I\!R^N$, and

F1) F' is increasing, $F(0) = 0$, F regular,

F2) $|F'(u)| \le C(1 + |u|^r)$, for some $r > 0$.

If we consider $p^* = \dfrac{Np}{N-p}$, the Sobolev theorem gives the inclusion

$$W_0^{1,p}(\Omega) \subset L^{p^*}.$$

We say that the problem (P) is *subcritical*, *critical* or *supercritical* if, $r < p^* - 1$, $r = p^* - 1$ or $r > p^* - 1$ respectively. If $p \ge N$ the problem (P) is always subcritical.

This type of problems has been studied extensively for $p = 2$ in many contexts. Thus in Riemannian Geometry the critical case for $p = 2$ in the Yamabe problem, (see [93]). In Astrophysics there appears some supercritical problem, (see [32] for instance). When $F(u) = \lambda e^u$, we have the model of Frank-Kamenetstkii for solid ignition, (see [48]).

In recent years there has been an increasing interest in looking at these problems in the more general framework of quasilinear elliptic equations for which (P) is a model case. In general, problem (P) can be seen as the stationary counterpart of evolution equations with nonlinear diffusion. Here it is necessary to say that the growth of the second member is important to get the blow-up of the solutions of the evolution problems, but it is clearly insufficient to conclude other properties, such that global existence or stability of the *equilibrium states*. For such sort of results it is necessary to know more deeply the behaviour of the associated elliptic problem, which depends on the growth and also on other properties of the zero order term.

The subcritical case is more or less easy, and we refer to [51] and the references there. The generality of problem (P) is enough to have a nontrivial project in our hands. The difficulty of considering problem (P) in its full generality can be understood by the following well known facts.

1. If Ω is a bounded starshaped domain in $I\!R^N$, $N > 2$, $p = 2$ and $F(u) = \lambda u^{\frac{N+2}{N-2}}$, $\lambda > 0$, problem (P) has no positive solution, as was remarked by Pohozaev in his famous paper [71].

2. Under the same hypotheses about the domain, if we assume $1 < p < N$ and $F(u) = \lambda u^{p^*-1}$, $\lambda > 0$, Pucci and Serrin show that problem (P) has no positive solution. See [73].

3. It is very easy to prove that if $\Omega = \{x \in I\!R^N | 0 < a < |x| < b\}$ and the same hypotheses as in (1) or (2), problem (P) has positive solutions. In fact, if we look for radial solutions, it is an easy one-dimensional problem. See for instance [61].

4. The behaviour of the problem in (3) when $a \to 0$, is not evident. The answer to this question was given by Bahri-Coron, [19]. More precisely: *If some fundamental group of the domain is non trivial then problem* (P) *with* $p = 2$ *and the zero order term* $F(u) = \lambda u^{\frac{N+2}{N-2}}$, $\lambda > 0$, *has a nontrivial solution.*

   At this level it seems that the existence of nontrivial solution of (P) depends strongly on the topology of the domain.

   This idea is not totally exact.

5. In [40] and [36], independently, it is obtained that there are contractible domains for which the problem in item (1) has a nontrivial solution. In particular, this fact means that the existence result depends deeply on the geometry of the domain and not only on the topology.

The rough consequence from the remarks above is the *strong dependence of the results on the domain.*

On the other hand we have as starting point the seminal result by Brezis and Nirenberg in [29] that we can summarize as follows.

Theorem *(Brezis-Nirenberg) Consider the problem*

$$(LP) \begin{cases} -\Delta u = \lambda |u|^{q-2} u + |u|^{\frac{4}{(N-2)}} u & in\ \Omega, \\ u|_{\delta\Omega} = 0, \end{cases}$$

where Ω *is a bounded domain in* $I\!R^N$ *and* $\lambda > 0$. *Then,*

a) If $q = 2$ *and* $N \geq 4$, *there exists a positive solution of (LP), for all* $0 < \lambda < \lambda_1$, λ_1 *being the first eigenvalue of the laplacian in* Ω.

b) If $q = 2$ and $N = 3$, there exists λ_0 depending on Ω such that the problem (LP) admits a positive solution if $0 < \lambda_0 < \lambda < \lambda_1$. Moreover if Ω is a ball then the problem has a positive soltution if and only if $\lambda_1/4 < \lambda < \lambda_1$, λ_1 first eigenvalue of the laplacian.

c) If $2 < q < \frac{2N}{N-2}$ and $N \geq 4$, the problem (LP) admits a positive solution for all $\lambda > 0$. In dimension $N = 3$ there exists positive solution for large λ.

The Theorem above means that the action of the term $\lambda|u|^{q-2}u$ breaks down the nonexistence obstruction found by Pohozaev [71]. Therefore, Pohozaev's result depends not only on the growth of the second member in the equation and on the geometry of the domain, but also on the precise expression of F.

We will emphasize this last fact in these notes and we will obtain results that depends in a deep way on the shape of F.

In first place we deal with the *typical eigenvalue problems* in Chapter 1; we will study existence, asymptotic behaviour, properties and, as a consequence, some results on bifurcation. Chapter 2 will be devoted mainly to study the influence of the shape of the zero order term in the behaviour of the elliptic problems with respect to multiplicity of solutions and bifurcation properties.

Chapter 3 is devoted to a critical problem related with a Hardy inequality. It seems that it can be considered as a starting point for a more extensive research. It shows how the dependence on x of the second member, (*non autonomous case*) changes the concept of growth to be considered.

General references for different aspect on the topics considered here are the books, [6], [12], [17], [20], [26], [34], [59], [63], [67], [76], [81], [83], [91], [92] and [94]. But some general results that will be used often, are collected in several appendix at the end of the lectures notes.

* * *

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Chapter 1

Eigenvalue problem

1.1 Introduction

This chapter will be devoted to the typical eigenvalue problem for the p-Laplacian, namely, the problem,

$$\begin{cases} -\Delta_p u \equiv -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) &= \lambda |u|^{p-2}u, \quad x \in \Omega, \\ u &= 0 \qquad\qquad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. We look for nontrivial solutions, i.e., $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}$, $u \neq 0$, verifiying the problem in a weak sense.

We call this case the typical eigenvalue problem because if we have a solution (u, λ) to problem (1.1), then for all $\alpha \in \mathbb{R}$, $(\alpha u, \lambda)$ is a solution. With different homogeneity in the second member we get a *curve of solutions* for each fixed solution to the problem with $\lambda = 1$, simply by a scaling, namely, if $u \neq 0$ is solution of $-\Delta_p u = |u|^{q-2}u$ then if $\alpha > 0$, $v = \alpha u$ verifies the same equation with $\lambda = \alpha^{p-q}$.

1.2 Existence of eigenvalues by minimax techniques

In this section we study the existence of a sequence of eigenvalues for problem (1.1). The result that we will explain can be found in [51]. The method used in the proof is an adaptation of the methods in [4], i.e., the Lusternik-Schnirelman theory.

Consider

$$B : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}, \qquad \text{defined by} \quad B(u) = \frac{1}{p}\int_\Omega |u|^p dx, \tag{1.2}$$

that is the potential of

$$b : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) \qquad \text{defined by} \qquad b(u) = |u|^{p-2}u. \tag{1.3}$$

It is easy to check that:

1. By the compactness of the Sobolev inclusion, b is compact and uniformly continuous on bounded sets.
2. As a consequence B is compact.
3. b is odd and B is even.

The idea is to obtain critical points of $B(u)$ on the manifold

$$\mathcal{M} = \{u \in W_0^{1,p}(\Omega) \,|\, (\frac{1}{p}) \int_\Omega |\nabla u|^p dx = \alpha \,\}. \tag{1.4}$$

For each $u \in W_0^{1,p}(\Omega) - \{0\}$ we find $\lambda(u) > 0$ in such a way that $\lambda(u)u \in \mathcal{M}$, i.e.,

$$\lambda(u) = \left(\frac{p\alpha}{\int_\Omega |\nabla u|^p dx} \right)^{1/p} \tag{1.5}$$

hence $\lambda : W_0^{1,p}(\Omega) - \{0\} \longrightarrow (0, \infty)$ is even and bounded on subsets in the complementary of the origin. Moreover the derivative of λ is

$$\langle \lambda'(u), v \rangle = -\frac{1}{p} \alpha^{\frac{1}{p}} \left(\frac{1}{p} \int_\Omega |\nabla u|^p dx \right)^{-(p+1)/p} \langle -\Delta_p u, v \rangle, \tag{1.6}$$

for all $v \in W_0^{1,p}(\Omega)$ where $\langle \cdot, \cdot \rangle$ denotes the duality product. Therefore,

$$\langle -\Delta_p u, v \rangle = 0 \quad \text{implies} \quad \langle \lambda'(u), v \rangle = 0,$$

$$\langle -\Delta_p u, u \rangle = p\alpha, \quad \text{if} \quad u \in \mathcal{M}$$

and, for all $u \in W_0^{1,p}(\Omega) - \{0\}$,

$$\langle -\Delta_p u, u \rangle = \left(\frac{1}{\lambda(u)} \right)^p \langle -\Delta_p(\lambda(u)u), \lambda(u)u \rangle = \frac{\alpha p}{(\lambda(u))^p}$$

The next step is to construct a flow in $\mathcal{M}$ related to B and the corresponding deformation lemma that allows us to apply the *mini-max* theory. (See [76]). By direct computation we get the derivative of $B(\lambda(u)u)$ in $u \in \mathcal{M}$, that according to (1.6), becomes

$$D(u) = b(u) - \frac{\langle b(u), u \rangle}{\langle -\Delta_p u, u \rangle} (-\Delta_p u) \in W^{-1,p'}(\Omega). \tag{1.7}$$

We need the tangent component of $D(u)$ to $\mathcal{M}$. Then we will use the duality mapping

$$J : W^{-1,p'}(\Omega) \longrightarrow W_0^{1,p}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

which gives the possibility to consider $J(Du) \in W_0^{1,p}(\Omega)$ and gives its tangent component. It is well known that J verifies for all $f \in W^{-1,p'}(\Omega)$,

i) $\langle f, J(f)\rangle = ||f||^2_{W^{-1,p'}(\Omega)}$

ii) $||J(f)||_{W_0^{1,p}(\Omega)} = ||f||_{W^{-1,p'}(\Omega)}$.

(See for instance [26]). Taking into account the properties of J and the previous comments we define

$$T : \mathcal{M} \longrightarrow W_0^{1,p}(\Omega), \text{ by } T(u) = J(D(u)) - \frac{\langle -\Delta_p u, J(D(u))\rangle}{-\langle \Delta_p u, u\rangle} u \tag{1.8}$$

and then, $T(u)$ is the tangent component that we are looking for, because

$$\langle -\Delta_p u, T(u)\rangle = 0 \quad \text{if} \quad u \in \mathcal{M},$$

and also by (1.6)

$$\langle \lambda'(u), T(u)\rangle = 0 \quad \text{if} \quad u \in \mathcal{M}.$$

Moreover, by definition, T is odd, uniformly continuous and bounded on $\mathcal{M}$; in particular we can find a $\gamma_0 > 0$, $t_0 > 0$, such that

$$||u + tT(u)|| \geq \gamma_0, \quad \text{for all} \quad (u, t) \in \mathcal{M} \times [-t_0, t_0].$$

As a consequence we define the flow

$$\begin{cases} H : \mathcal{M} \times [-t_0, t_0] \longrightarrow \mathcal{M} \\ (u, t) \longrightarrow H(u, t) = \lambda(u + tTu)(u + tTu). \end{cases} \tag{1.9}$$

Then H verifies:

i) Is odd as function of u, namely, $H(u, t) = -H(-u, t)$ for fixed t.

ii) Is uniformly continuous.

iii) $H(u, 0) = u$ for $u \in \mathcal{M}$.

The next result shows that H define trajectories on $\mathcal{M}$ for which the functional B is increasing. This is the relevant property of H in order to get a deformation result.

Lemma 1.2.1 *Let H be defined by (1.9). Then there exists*

$$r : \mathcal{M} \times [-t_0, t_0] \to I\!R$$

such that $\lim_{\tau \to 0} r(u, \tau) = 0$ *uniformly on* $u \in \mathcal{M}$, *and:*

$$B(H(u,t)) = B(u) + \int_0^t \{||Du||^2 + r(u,s)\}ds$$

for all $u \in \mathcal{M}$, $\quad t \in [-t_0, t_0]$.

Proof. We have $< B'(u), v >=< b(u), v >$ and $H(u,0) = u$ hence

$$B(H(u,t)) = B(u) + \int_0^t \langle b(H(u,s)), \frac{\partial}{\partial s} H(u,s)\rangle ds. \tag{1.10}$$

But

$$\begin{array}{l} \frac{\partial}{\partial s} H(u,s) = \frac{\partial}{\partial s}(\lambda(u + sT(u))(u + sT(u)) = \\ \langle \lambda'(u + sT(u)), T(u)\rangle(u + sT(u)) + \lambda(u + sT(u))T(u) = \\ \langle \lambda'(u + sT(u)) - \lambda'(u), T(u)\rangle(u + sT(u)) + \\ (\lambda(u + sT(u)) - \lambda(u))\, T(u) + \lambda(u)T(u). \end{array}$$

If we call,

$$\begin{array}{l} R(u,s) = \\ \langle \lambda'(u + sT(u)) - \lambda'(u), T(u)\rangle(u + sT(u)) + \\ (\lambda(u + sT(u)) - \lambda(u))\, T(u) \end{array}$$

we have $\frac{\partial}{\partial s} H(u,s) = R(u,s) + T(u)$. Now, because T is bounded and λ and λ' are uniformly continuous,

$$\lim_{s \to 0} R(u,s) = 0, \quad \text{uniformly in} \quad u \in \mathcal{M}.$$

Hence (1.10) becomes

$$\begin{array}{rl} B(H(u,t)) & = B(u) + \int_0^t < b(H(u,s)), R(u,s) + T(u) > ds = \\ & B(u) + \int_0^t \langle b(u), T(u)\rangle + r(u,s) ds, \end{array}$$

where

$$r(u,s) = \langle b(u), R(u,s)\rangle + \langle b(H(u,s)) - b(u), R(u,s) + T(u)\rangle.$$

and also $r(u,s) \to 0$ uniformly in $u \in \mathcal{M}$, as $s \to 0$. But

$$\langle b(u), T(u)\rangle = \langle D(u), J(D(u))\rangle = ||Du||^2_{W^{-1,p'}(\Omega)},$$

and we conclude. ■

We can now formulate the necessary deformation result. For $\beta > 0$, consider the level sets,

$$\Lambda_b = \{\, u \in \mathcal{M} \,|\, B(u) \geq \beta \,\} \tag{1.11}$$

then

Lemma 1.2.2 *Let $\beta > 0$ be fixed and assume that there exists an open set $U \subset \mathcal{M}$, such that for some constants $0 < \delta$, $0 < \rho < \beta$, the condition*

$$||D(u)||_{W^{-1,p'}(\Omega)} \geq \delta, \ \textit{if}\ u \in V_\rho = \{u \in \mathcal{M} |\, u \notin U, \ |B(u) - \beta| \leq \rho\}$$

is verified. Then, there exists $\varepsilon > 0$, and an operator H_ε odd and continuous such that

$$H_\varepsilon(\Lambda_{\beta-\varepsilon} - U) \subset \Lambda_{\beta+\varepsilon}.$$

Proof. With t_0 and $r(u, s)$ as in the Lemma 1.2.1, consider $t_1 \in (0, t_0]$ such that for $s \in [-t_1, t_1]$

$$|r(u, s)| \leq \frac{1}{2}\delta^2, \quad \text{for all} \quad u \in \mathcal{M}.$$

Then, for $u \in V_\rho$ and $t \in [0, t_1]$,

$$\begin{aligned} B(H(u,t)) \ &= B(u) + \textstyle\int_0^t \{||Du||^2 + r(u, s)\} ds \\ &\geq B(u) + \textstyle\int_0^t (\delta^2 - \frac{1}{2}\delta^2) ds = B(u) + \frac{1}{2}\delta^2 t. \end{aligned}$$

Choosing $\varepsilon = \min\{\rho, \frac{1}{4}\delta^2 t_1\}$. If $u \in V_\rho \cap \Lambda_{\beta-\varepsilon}$, we have $|B(u) - \beta| \leq \rho$ and then

$$B(H(u, t_1)) \geq B(u) + 2\varepsilon \geq \beta + \varepsilon.$$

By Lemma 1.2.1, fixed $u \in V_\rho$, $B(H(u, \cdot))$ is increasing in some interval $[0, s_0) \subset [0, t_1)$. Then for $u \in V_\varepsilon = \{u \in \mathcal{M} |\, u \notin U, \, |B(u) - \beta| \leq \varepsilon\}$ the functional

$$t_\varepsilon(u) = \min\{t \geq 0 \,|\, B(H(u, t)) = \beta + \varepsilon\}.$$

is well defined and verifies:

i) $0 < t_\varepsilon(u) \leq t_1$,
ii) $t_\varepsilon(u)$ is continuous in V_ε.

Define

$$H_\varepsilon : \Lambda_{\beta-\varepsilon} - U \to \Lambda_{\beta+\varepsilon}$$

by

$$H_\varepsilon(u) = \begin{cases} H(u, t_\varepsilon(u)), \text{ if } u \in V_\varepsilon, \\ u, \text{ if } u \in \Lambda_{\beta-\varepsilon} - \{U \cup V_\varepsilon\}. \end{cases}$$

In this way H_ε is odd and continuous. ■

We will prove the existence of a sequence of critical values and critical points by using a *mini-max* argument on the class of sets defined below. For each $k \in I\!N$ consider,

$$\mathcal{C}_k = \{\, C \subset \mathcal{M} \,|\, C \text{ compact }, C = -C \;\gamma(C) \geq k\}, \tag{1.12}$$

where γ is the *genus* defined in the Appendix. The main result on existence of eigenvalues is the following

Theorem 1.2.3 *Let $\mathcal{C}_k$ be defined by (1.12) and let β_k be defined by*

$$\beta_k = \sup_{C \in \mathcal{C}_k} \min_{u \in C} B(u). \tag{1.13}$$

Then, $\beta_k > 0$, and there exists $u_k \in \mathcal{M}$ such that $B(u_k) = \beta_k$, and u_k is a solution to problem (1.1) for $\lambda_k = \dfrac{\alpha}{\beta_k}$.

Proof. The inequality $\beta_k > 0$ is a consequence of $\gamma(\mathcal{M}) = +\infty$, namely, for all $k > 0, \mathcal{C}_k \neq \emptyset$, and given $C \in \mathcal{C}_k$, we have $\min\limits_{u \in C} B(u) > 0$, hence $\beta_k > 0$ for all k. Let k be a fixed positive integer. In the general hypotheses there exists a sequence $\{u_j\} \subset \mathcal{M}$ such that

$$\begin{aligned} &a)\, B(u_j) \to \beta_k, \\ &b)\, D(u_j) \to 0. \end{aligned} \tag{1.14}$$

In fact if not, there must exists $\delta, \rho > 0$, such that

$$||Du|| \geq \delta \quad \text{ if } u \in \{\, u \in \mathcal{M} \,|\, |B(u) - \beta_k| \leq \rho\}.$$

We can assume $\delta < \beta_k$ and then by the previous deformation Lemma 1.2.2 with $U = \emptyset$, there exists $\varepsilon > 0$ and a odd continuous mapping H_ε, such that

$$H_\varepsilon(\Lambda_{\beta_k - \varepsilon}) \subset \Lambda_{\beta_k + \varepsilon}.$$

From (1.13) there exists $C_\varepsilon \in \mathcal{C}_k$ such that $B(u) \geq \beta_k - \varepsilon$ en C_ε; namely, $C_\varepsilon \subset \Lambda_{\beta_k - \varepsilon}$. Then, $B(u) \geq \beta_k + \varepsilon$ in $H_\varepsilon(C_\varepsilon)$.

Now $\gamma(H_\varepsilon(C_\varepsilon)) \geq k$, because $\gamma(C_\varepsilon) \geq k$, hence $H_\varepsilon(C_\varepsilon) \in \mathcal{C}_k$, that is a contradiction with the definition of β_k.

In this way we find a sequence, $\{u_j\}$, verifying (1.14) and then, for some subsequence, $u_j \rightharpoonup u_k$ in $W_0^{1,p}(\Omega)$. Since B is compact, $B(u_j) \to \beta_k$ and by continuity $D(u_k) = 0$, so we find

$$-\Delta_p u_k = \lambda_k |u_k|^{p-2} u_k, \quad \text{with} \quad \lambda_k = \frac{\alpha}{\beta_k}$$

■

To finish this section, we will prove that the sequence of critical values $\{\beta_k\}$ is infinite, namely that we have a true sequence of eigenvalues.

Lemma 1.2.4 *Let β_k be defined in (1.13). Then $\lim_{k\to\infty} \beta_k = 0$.*

(As a consequence $\lambda_k = \alpha\beta_k^{-1} \to \infty$ as $k \to \infty$).

PROOF.

Consider $\{E_j\}$ sequence of linear subspaces in $W_0^{1,p}(\Omega)$ such that

i) $E_k \subset E_{k+1}$,

ii) $\overline{\mathcal{L}(\cup E_k)} = W_0^{1,p}(\Omega)$,

iii) $\dim E_k = k$.

Define

$$\tilde{\beta_k} = \sup_{C\in\mathcal{C}_k} \min_{u\in C\cap E_{k-1}^c} B(u) \tag{1.15}$$

where E_k^c is the linear and topological complementary of E_k.

Obviously $\tilde{\beta_k} \geq \beta_k > 0$. So it is sufficient to prove that $\lim_{k\to 0} \tilde{\beta_k} = 0$.

Now, if for some positive constant, $\gamma > 0$, $\tilde{\beta_k} > \gamma > 0$ for all $k \in I\!N$, then for each $k \in I\!N$ there exists $C_k \in \mathcal{C}_k$ such that

$$\tilde{\beta_k} > \min_{u\in C_k\cap E_{k-1}^c} B(u) > \gamma,$$

and then there exists $u_k \in C_k \cap E_{k-1}^c$ such that

$$\tilde{\beta_k} > B(u_k) > \gamma.$$

In this way $\{u_k\} \subset \mathcal{M}$, $B(u_k) > \gamma > 0$ for all $k \in I\!N$, hence for some subsequence,

$$\begin{aligned} u_k &\rightharpoonup v \text{ weakly in } W_0^{1,p}(\Omega),\\ u_k &\to v \text{ in } L^p(\Omega). \end{aligned}$$

As a consequence $b(v) > \gamma$ and this is a contradiction because $u_k \in E_{k-1}^c$, implies $v = 0$. ■

In the next section we will give a more precise result about the asymptotic behaviour of the sequence $\{\beta_k\}$.

1.3 Asymptotic behaviour

Consider the sequence of eigenvalues

$$\lambda_k = \alpha\beta_k^{-1}, \text{ where } \beta_k = \sup_{F\in\mathcal{C}_k(\Omega)} \inf_{u\in F} \int_\Omega |u|^p dx, \quad k \in I\!N \tag{1.16}$$

which has be found in the previous section.

We obtain the following result that gives the asymptotic behaviour of the sequence of eigenvalues obtained above. The case $p = 2$ is classical.

Theorem 1.3.1 *The sequence $\{\lambda_k\}_{k\in I\!N}$ of eigenvalues of $-\Delta_p$ defined by (1.16), verifies that there exist two constants, $c(\Omega)$, $C(\Omega)$, depending only on the domain Ω, such that*

$$c(\Omega)k^{p/N} \le \lambda_k \le C(\Omega)k^{p/N}, \quad \textit{for all} \quad k \in I\!N, \tag{1.17}$$

where N is the dimension.

STEP 1. DIMENSION $N = 1$. A fundamental role in proving the estimate is played by the following Bernstein's Lemma, which gives the inequality,

$$\lambda_k \le Ck^p.$$

Lemma 1.3.2 *Let $S(x)$ be a n-degree trigonometric polynomial*

$$S(x) = \sum_{j=-n}^{n} a_j e^{i\pi j}.$$

Then:

$$\left(\int_{-1}^{1} |S'(x)|^p dx\right)^{\frac{1}{p}} \le n\left(\int_{-1}^{1} |S(x)|^p dx\right)^{\frac{1}{p}}, \quad p \ge 1$$

(See [94]).

The reverse inequality is obtained in the following one-dimensional result which proof uses Fourier Analysis techniques, and completes the case of dimension $N = 1$.

Lemma 1.3.3 *Let E_k be the linear space spanned by*

$$\{\sin(\pi jx) | 1 \le j \le k\},$$

and E_k^c its topological complement in the Sobolev space $W_0^p(0,1)$. Then, there exists a constant $C > 0$ such that

$$||u||_p \le C\frac{1}{k}||u'||_p$$

for each $u \in E_k^c$.

Proof. Consider

$$g(t) = \begin{cases} t, \text{ si } -1 \leq t \leq 1. \\ 1/t, \text{ si } |t| > 1. \end{cases}$$

Then $g(t) = \widehat{w}(t)$, namely, is the Fourier transform of $w(t) \in L^1(I\!R)$; in fact $|w(t)| \leq C(1+t^2)^{-1}$ and

$$\lim_{t\to 0^+} w(t) \neq \lim_{t\to 0^-} w(t).$$

We show the previous assertions on w. We have by definition

$$w(x) = 2\int_0^\infty g(t)\sin(tx)dt = 2\int_0^1 t\sin(tx)dt + 2\int_1^\infty \frac{\sin(xt)}{t}dt.$$

Integrating by parts we get

$$w(x) = 2\frac{\sin x}{x^2} - \frac{2}{x}\int_1^\infty \frac{\cos(xt)}{t^2}dt,$$

and integrating by parts a second time we get for $|x|$ large ,

$$|w(x)| \approx \frac{c}{|x|^2}.$$

For $x > 0$ small, by the change of variable $s = xt$ we have,

$$w(x) = \frac{2}{x^2}\int_0^x s\sin s ds + 2\int_x^\infty \frac{\sin s}{s}ds \to \pi, \quad x \to 0.$$

Because w is odd we get that w is discontinuous at 0, and moreover that it satisfies in the whole real line

$$|w(t)| \leq \frac{C}{(1+t^2)}.$$

Define $w_k(t) = kw(kt)$, then $||w_k||_1 = ||w||_1 = C < \infty$ and

$$g(\frac{t}{k}) = \widehat{w_k}(t).$$

For $k \geq 1$ we consider the 2-periodization of $w_k(t)$ i.e.,

$$\widetilde{w_k}(t)) = \sum_{j=-\infty}^{\infty} w_k(t+2j).$$

It is clear that $||\widetilde{w_k}||_{L^1(-1,1)} = C$ for all $k \geq 1$ and, moreover, the Fourier series of $\widetilde{w_k}$ is

$$\widetilde{w_k}(x) = \sum_{j=-\infty}^{\infty} \hat{w}_k(j)e^{\pi ijx},$$

and, by definition, the Fourier coefficients of $\widetilde{w_k}$ for $|j| \geq k > 1$ are

$$\widehat{\widetilde{w}}_k(j) = g(\frac{j}{k}) = \frac{k}{j}.$$

Call $\Omega = (0,1)$. Then if $u \in W_0^{1,p}(\Omega)$ and $\bar{u}$ is the odd extension of u to $(-1,1)$, we get that if

$$\bar{u}(x) = \sum_j a_j e^{ij\pi x},$$

then its derivative is,

$$\bar{u}'(x) = i\pi \sum_j j a_j e^{ij\pi x} = \pi \sum_j a_j' e^{ij\pi x}.$$

Now, if $u \in E_k^c$ we get that

$$\bar{u}(x) = \sum_j a_j e^{ij\pi x} = \frac{\pi}{k} \sum_{|j|\geq k} g(\frac{j}{k}) a_j' e^{ij\pi x},$$

that gives the following identity

$$\frac{k}{\pi}\widehat{u}(j) = g(\frac{j}{k})\widehat{u'}(j),$$

which is equivalent to

$$\frac{k}{\pi}u = \widetilde{w_k} * u',$$

where $*$ means the integral of convolution. As conclusion,

$$\frac{k}{\pi}||u||_p \leq ||\tilde{w}_k||_1||u'||_p \equiv c||u'||_p$$

With this estimate we get,

$$\lambda_k \geq ck^p$$

■

Step 2. Dimension $N > 1$ and $\Omega = (0,1)^N$. The next step is to show the case $N > 1$ in the particular case of a product of N intervals that, by scaling, we can assume that $\Omega = (0,1)^N$. We will consider

$$\{e_{\overline{k}} = \sin(k_1\pi x_1)\sin(k_2\pi x_2)...\sin(k_N\pi x_N), k_i \in I\!N, i = 1, ..., N\}$$

a basis of $W_0^{1,p}(\Omega)$. If on the lattice $I\!N^N$ we take the complement of the cube Q_k, of edge k, we are considering $E_{k^N}^c$, the complement of the k^N-dimensional linear space E_{k^N} spanned for the functions with frequencies in such cube.

In fact, we consider for $u \in E^c_{k^N}$,

$$u = \sum_{\bar{m}} e_{\bar{m}}, \;\; b_m = 0 \;\; \text{if } \bar{m} = (m_1, ..., m_N), \;\; m_i \leq k, \quad i = 1, ...N.$$

Then we prove the corresponding result

Lemma 1.3.4 *There exists a constant $C > 0$, such that if $u \in E^c_{k^N}$,*

$$||u||_p \leq \frac{C}{k} ||\nabla u||_p$$

PROOF. If $u \in E^c_{k^N}$, there exists $b_m \neq 0$ for some $m = (m_1, ..., m_N)$ with $m_j > k$, for some $j = 1, ..., N$.

Then by Lemma 1.3.3 we get

$$\int |u(x_1, ..., x_j, ..., x_N)|^p dx_j \leq \frac{C}{k^p} \int |\frac{\partial u}{\partial x_j}(x_1, ..., x_N)|^p dx_j.$$

The reverse inequality, as above, is an application of Bernstein inequalities. So we conclude. ■

Assume the linear subspaces in $W^{1,p}_0(\Omega)$ spanned by the basis $\{e_{\bar{k}}\}$ defined above and ordered as follows

$$i) E_{k^N} = \mathcal{L}\{e_{\bar{l}}\}, \text{ where } \bar{l} \in Q_k.$$
$$ii) \text{ If } k^N < j < k^{N+1}, \text{ then } E_j \subset E^c_{k^N} \cap E_{k^{N+1}}.$$

In this way the result in Lemma 1.3.4 gives us the result in the Theorem 1.3.1 for an interval. To read correctly the result put in the horizontal axis the subspace dimension and as ordinate the constant of the inequality, we have for extrapolation the function $f(x) = \dfrac{C}{x^{1/N}}$. Hence we get,

Lemma 1.3.5 *Given Ω, a product of N one dimensional intervals in $\mathbb{R}^N$, there exists a constant $C > 0$, such that*

$$||u||_p \leq \frac{C}{K^{1/N}} ||\nabla u||_p, \;\; \textit{for all } u \in E^C_K,$$

where the linear spaces E_K are defined and ordered as above.

Finally, the case of a general bounded domain can be proved by comparison. More precisely, consider $Q' \subset \Omega \subset Q$, Q, Q' cubes in $\mathbb{R}^N$.

By definition we have $\beta_k(Q) \geq \beta_k(\Omega) \geq \beta_k(Q')$ and then

$$\lambda_k(Q) \leq \lambda_k(\Omega) \leq \lambda_k(Q').$$

Remark 1.3.6 *The main open problem is to show that the sequence that we have found contains all the eigenvalues. A partial answer to this problem is given in [57], for dimension $N = 1$ and in [43] for the radial case, that we will explain below.*

1.4 Isolation of the first eigenvalue

Let us consider the problem ($1 < p < \infty$):

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u, \\ u \in W_0^{1,p}(\Omega), u \not\equiv 0. \end{cases} \tag{1.18}$$

Then according to the previous section, the first eigenvalue is given by

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla w|^p \, dx : w \in W_0^{1,p}(\Omega), \int_\Omega |w|^p \, dx = 1 \right\}.$$

The main result in this section is the following Theorem.

Theorem 1.4.1 *The first eigenvalue of $-\Delta_p$ is isolated and simple.*

Remark 1.4.2 *The problem*

$$\begin{cases} -\Delta_p u = \lambda V(x) |u|^{p-2} u, \\ u \in W_0^{1,p}(\Omega), u \not\equiv 0. \end{cases} \tag{1.19}$$

with $V \in L^\infty$ and Ω with smooth boundary was studied in [15] and was extended in [66] to general domains. Often the case $V \in L^q$ with $q < \infty$ is useful for the applications.

We can find similar results if $V(x) \geq 0$, $V(x) \in L^q(\Omega)$, and $|\{x \in \Omega : V(x) > 0\}| \neq 0$, where $q \geq 1$ if $p > N$, $q > 1$ if $p = N$, $q > N/p > 1$ otherwise. The details of this extension can be found in [2].

Without these hypotheses the behaviour is different and we will give an example in the last Chapter of this lectures notes.

The proof of Theorem 1.4.1 follows closely the arguments in [66] and is divided in several Lemmas.

Lemma 1.4.3 *λ_1 is an eigenvalue, and every eigenfunction u_1 corresponding to λ_1 does not changes sign in Ω: either $u_1 > 0$ or $u_1 < 0$.*

Proof. Directly from the proof of existence of the first eigenvalue we conclude that there exists a positive eigenfunction: if v is an eigenfuction also $u = |v|$ is a solution of the minimization problem and then an eigenfunction. By the strong maximum principle $|v| > 0$ and then u has constant sign. ■

Lemma 1.4.4 *i) If $p \geq 2$ then*

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)\frac{|\xi_2 - \xi_1|^p}{2^p - 1}, \tag{1.20}$$

for all $\xi_1, \xi_2 \in I\!R^N$.

ii) If $p < 2$, then

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)\frac{|\xi_2 - \xi_1|^2}{(|\xi_2| + |\xi_1|)^{2-p}}, \tag{1.21}$$

for all $\xi_2, \xi_2 \in I\!R^N$.
(C(p) is a constant depending only on p).

(See the proof in [66]).

Lemma 1.4.5 *λ_1 is simple, i.e. if u, v are two eigenfunctions corresponding to the eigenvalue λ_1, then $u = \alpha v$ for some α.*

Proof. As [15] and [66] point out, if the functions $\eta_1 = u - v^p u^{1-p}$ and $\eta_2 = v - u^p v^{1-p}$ could be considered as admisible test function in

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \eta_i\rangle dx = \lambda_1 \int_\Omega |u|^{p-2} u \eta_i, \quad i = 1, 2, \tag{1.22}$$

then the result would be obtained by direct calculations.

In sufficiently regular domains it is possible to assume this hypothesis by a convenient use of Hopf lemma, otherwise, it is necessary to regularize the candidates to test functions. Following the proof in [66] we consider for $\varepsilon > 0$, $u_\varepsilon = u + \varepsilon$, $v_\varepsilon = v + \varepsilon$ and then

$$\eta_1 = \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}} \quad \text{and} \quad \eta_2 = \frac{v_\varepsilon^p - u_\varepsilon^p}{v_\varepsilon^{p-1}}.$$

Hence the gradient of η_1 is

$$\nabla \eta_1 = \left(1 + (p-1)(\frac{v_\varepsilon}{u_\varepsilon})^p\right)\nabla u_\varepsilon - p(\frac{v_\varepsilon}{u_\varepsilon})^{p-1}\nabla v_\varepsilon$$

and by symmetry $\nabla \eta_2$ is obtained in a similar way. We substitute in the equations (1.22) and adding we get:

$$\lambda_1 \int_\Omega \left(\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p) dx =$$

$$\int_\Omega \left(\left(1 + (p-1)(\frac{v_\varepsilon}{u_\varepsilon})^p \right) |\nabla u_\varepsilon|^p + \left(1 + (p-1)(\frac{u_\varepsilon}{v_\varepsilon})^p \right) |\nabla v_\varepsilon|^p \right) dx -$$

$$p \int_\Omega \left((\frac{v_\varepsilon}{u_\varepsilon})^{p-1} |\nabla u_\varepsilon|^{p-2} + (\frac{u_\varepsilon}{v_\varepsilon})^{p-1} |\nabla v_\varepsilon|^{p-2} \right) \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle dx = \tag{1.23}$$

$$\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)(|\nabla \log u_\varepsilon|^p - |\nabla \log v_\varepsilon|^p) dx -$$

$$-p \int_\Omega v_\varepsilon^p |\nabla \log u_\varepsilon|^{p-2} \langle \nabla \log u_\varepsilon, \nabla \log v_\varepsilon - \nabla \log u_\varepsilon \rangle dx -$$

$$-p \int_\Omega u_\varepsilon^p |\nabla \log v_\varepsilon|^{p-2} \langle \nabla \log v_\varepsilon, \nabla \log u_\varepsilon - \nabla \log v_\varepsilon \rangle dx,$$

that it is not positive by the convexity of the function $f(s) = s^p$. Moreover the Lebesgue convergence theorem implies that

$$\lim_{\varepsilon \to 0^+} \int_\Omega \left(\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{u^{p-1}}{u_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p) dx = 0. \tag{1.24}$$

Now, if $p \geq 2$, by the estimate (1.20) applied to $\xi_1 = \nabla \log u_\varepsilon$, $\xi_2 = \nabla \log v_\varepsilon$ and viceversa we get

$$0 \leq c(p) \int_\Omega \left(\frac{1}{v_\varepsilon^p} + \frac{1}{u_\varepsilon^p} \right) |v_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon|^p dx \leq$$
$$-\lambda_1 \int_\Omega \left(\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{u^{p-1}}{u_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p) dx.$$

According to (1.24) we conclude that $v \nabla u = u \nabla v$ in Ω, and then $u = \alpha v$. Similarly, if $1 < p < 2$ we use the estimate (1.21) for the same functions as above and we get

$$0 \leq c(p) \int_\Omega (v_\varepsilon + u_\varepsilon)^p (u_\varepsilon v_\varepsilon)^p \frac{|v_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon|^2}{(v_\varepsilon |\nabla u_\varepsilon| + u_\varepsilon |\nabla v_\varepsilon|)^{2-p}} dx \leq$$
$$-\lambda_1 \int_\Omega \left(\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p) dx,$$

that finished the proof as in the case $p \geq 2$. ■

To finish the study of the isolation of the first eigenvalue, we extend slightly the results by Anane [15] and [66] to the case of unbounded potential. We have the following nodal result.

Lemma 1.4.6 *If w is an eigenfunction corresponding to the eigenvalue λ, $\lambda > 0$, $\lambda \neq \lambda_1$, then w changes sign in Ω: $w^+ \not\equiv 0$, $w^- \not\equiv 0$ and*

$$|\Omega^-| \geq (\lambda\, ||V||_q\, C^p)^{\sigma},$$

where $\Omega^- = \{x \in \Omega : w(x) < 0\}$, $\sigma = -2q'$ if $p \geq N$, $\sigma = -\dfrac{qN}{qp-N}$ if $1 < p < N$ and λ_1 is the first eigenvalue for the p-laplacian with weight V in Ω.

Proof. We have by definition $\lambda_1 \leq \lambda$ Let u, w be two eigenfunctions corresponding to λ_1 and λ respectively. If w does not change sign, by using the same arguments as above we obtain

$$\int_\Omega \left[\lambda_1 V(x)\left(\frac{w}{w_\varepsilon}\right) - \lambda V(x)\left(\frac{u}{u_\varepsilon}\right)\right](u_\varepsilon^p - w_\varepsilon^p)dx \leq 0,$$

then

$$(\lambda_1 - \lambda)\int_\Omega V(x)(u^p - w^p)\, dx \leq 0$$

and we arrive to a contradiction if $\lambda > \lambda_1$ by taking kw. Then w is positive implies $\lambda = \lambda_1$.

Moreover, if we take w^- as a test function, we get

$$||\nabla w^-||_p^p = \lambda \int_\Omega V(w^-)^p\, dx \leq \lambda\, ||V||_q\, ||\,(w^-)^p\,||_\alpha\, |\Omega^-|^{1/\beta},$$

with $\dfrac{1}{q} + \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Now we are considering two cases:

1. $p \geq N$. By Sobolev's inequality

$$||\,(w^-)^p\,||_\alpha = ||w^-||_{\alpha p}^p \leq C||\nabla w^-||_p^p, \quad \alpha > 1.$$

   Thus, if we take $\alpha = \beta = 2q'$ then

$$|\Omega^-| \geq (\lambda\, ||V||_q\, C^p)^{-2q'}.$$

2. $1 < p < N$. We take $\alpha = \dfrac{N}{N-p}$, $\beta = \dfrac{qN}{qp-N}$ $(||(w^-)^p||_\alpha = ||w^-||_{p^*}^p$, where $p^* = \dfrac{Np}{N-p})$. By Sobolev's inequality

$$||\nabla w^-||_p^p \leq \lambda\, ||V||_q\, ||\,w^-\,||_{p^*}^p,$$

$$|\Omega^-|^{\frac{qp-N}{qN}} \leq \lambda \, ||V||_q \, C^p \, ||\nabla w^-||_p^p \, |\Omega^-|^{\frac{qp-N}{qN}}.$$

Hence

$$|\Omega^-| \geq (\lambda \, ||V||_q \, C^p)^{-\frac{qN}{qp-N}}.$$

■

The isolation of λ_1 is studied to finish this section.

Lemma 1.4.7 *λ_1 is isolated; that is, λ_1 is the unique eigenvalue in $[0, a]$ for some $a > \lambda_1$.*

Proof. Let $\lambda \geq 0$ be an eigenvalue and v be the corresponding eigenfunction. By the definition of λ_1 (it is the infimum) we have $\lambda \geq \lambda_1$. Then, λ_1 is left-isolated.

We are now arguing by contradiction. We assume there exists a sequence of eigenvalues (λ_k), $\lambda_k \neq \lambda_1$ which converges to λ_1. Let (u_k) be the corresponding eigenfunctions with $||\nabla u_k||_p = 1$. We can therefore take a subsequence, denoted again by (u_k), converging weakly in $W_0^{1,p}$, strongly in $L^p(\Omega)$ and almost everywhere in Ω to a function $u \in W_0^{1,p}$. Since $u_k = -\Delta_p^{-1}(\lambda_k V |u_k|^{p-2} u_k)$, the subsequence (u_k) converges strongly in $W_0^{1,p}$, and subsequently u is the eigenfunction corresponding to the first eigenvalue λ_1 with norm equals to 1. Hence, by applying the Egorov's Theorem ([26] Th. IV.28), (u_k) converges uniformly to u in the exterior of a set of arbitrarily small measure. Then, there exists a piece of Ω of arbitrarily small measure in which exterior u_k is positive for k large enough, obtaining a contradiction with the conclusion of Lemma 1.4.6 ■

1.5 Eigenvalues in the radial case

In the case of the nonlinear Sturm Liouville problem, $N = 1$, the corresponding eigenvalue problem is

$$\begin{cases} -(|u'|^{p-2}u')' &= \lambda |u|^{p-2} u \\ u(0) &= 0 = u(1), \end{cases} \tag{1.25}$$

and can be solved explicitely getting the following results (see [43] and [57]):

1) Problem (1.25) has nontrivial solution if and only if

$$\lambda = \lambda_k = (\pi_p)^p k^p,$$

where

$$\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{dt}{(1 - s^p/(p-1))^{1/p}}$$

2) Fixed $k \in I\!N$ there exists an unique solution v_k to problem (1.25) for $\lambda = \lambda_k$, such that $||v_k|| = 1$ and $v_k \in S_k^+$. Moreover

$$v_k(x) = (-1)^m v_1(kx - m), \quad m/k < x \leq (m+1)/k,$$

where v_1 is the normalized positive solution corresponding to λ_1.

In this way problem (1.25) is a typical nonlinear eigenvalue problem, and all of its eigenvalues are simple.

Hence the set of solutions of (1.25) is

$$\mathcal{S} = \{(\mu v_k, \lambda_k) \,|\, \mu \in I\!R - \{0\},\, k \in I\!N\}.$$

We will study in some detail the N-dimensional radial eigenvalue problem and we follow the paper [43] where the results are obtained in a elementary way. In particular we recover the nonlinear Sturm-Liouville eigenvalue problem for $N = 1$.

Consider the eigenvalue problem,

$$\left\{ \begin{array}{rcl} \mathcal{A}_p u &=& r^{N-1}\left(\lambda |u(r)|^{p-2} u(r)\right) \\ u(1) &=& 0, \\ u'(0) &=& 0, \end{array} \right. \tag{1.26}$$

where $\mathcal{A}_p u = -(r^{N-1}|u'|^{p-2}u')'$.

We will use the following results.

Lemma 1.5.1 *The Cauchy problem*

$$\left\{ \begin{array}{rcl} \mathcal{A}_p u &=& r^{N-1}|u(r)|^{p-2}u(r), \\ u(0) &=& a, \\ u'(0) &=& 0, \end{array} \right. \tag{1.27}$$

has a unique $C^{1,\nu}$ solution $u = u_a$ defined in $[0,\infty)$. Moreover, u_a has infinitely many zeros in $[0,\infty)$.

Proof. To be brief we call $\phi_p(s) = |s|^{p-2}s$. Then the equation in problem (1.27) can writen as

$$-(r^{N-1}\phi_p(u'))' = r^{N-1}\phi_p(u) \tag{1.28}$$

Then a solution to (1.27) must satisfy

$$u(r) = a - \int_0^r \phi_{p'}\left(\frac{1}{s^{N-1}}\int_0^s \phi_p(u(t))dt\right)ds \equiv \mathcal{L}_p(u)(r), \quad \frac{1}{p}+\frac{1}{p'}=1,$$

namely u must be a fixed point for the operator $\mathcal{L}_p$. It is very easy to check that for small $\delta > 0$, $\mathcal{L}_p$ verifies the hypotheses of the Schauder's fixed point theorem. Then we have local solution to problem (1.27).

We prove the uniqueness following the ideas in [47] where a more general situation is also considered. If u and v are solutions to 1.27, assume that $|u'(r)| \le |v'(r)|$ in some interval. We call

$$w(r) = \phi_p(u') - \phi_p(v'),$$

then w satisfies the equation,

$$w' + \frac{N-1}{r}w = \psi(r), \qquad w(0) = 0,$$

where

$$\psi(r) = \phi_p(v) - \phi_p(u) \equiv |v(r)|^{p-2}v(r) - |u(r)|^{p-2}u(r),$$

then

$$|w(r)| = |\int_0^r \psi(t)(\frac{t}{r})^{N-1}dt| \le \frac{r}{N}\sup_{t\in[0,r]}|\psi(t)|,$$

and because u and v leaves in the positive real axe we get a local Lipschitz constant for the second member, namely

$$|\psi(r)| \le M|u(r)-v(r)|,$$

and then

$$|\psi(r)| \le M\int_0^r |u'(t)-v'(t)|.$$

But

$$|u'(r)-v'(r)| \le \frac{1}{\inf_{|u'|\le\gamma\le|v'|}\phi_p'(\gamma)}|w(r)|.$$

Moreover for a solution

$$\phi_p(u'(r)) = -\int_0^r |u(t)|^{p-2}u(t)(\frac{t}{r})^{N-1}dt,$$

then

$$\phi_p(\gamma) > \phi_p(|u'|) = \int_0^r |u(t)|^{p-2}u(t)(\frac{t}{r})^{N-1}dt \ge a^{p-1}\frac{r}{2N},$$

because ϕ_p is increasing. Moreover $\phi_p'(\gamma) \geq |\phi_p(\gamma)|^\mu$, $\mu = 0$ if $1 < p \leq 2$ and $\mu = 1$ if $p > 2$ and locally. Then we conclude that

$$\phi_p'(\gamma) \geq |\phi_p(\gamma)|^\mu \geq (a^{p-1}\frac{r}{2N})^\mu$$

and then

$$|\psi(r)| = M\int_0^r |u'(t) - v'(t)|dt \leq M\int_0^r (\frac{2N}{a^{p-1}t})^\mu |w(t)|dt.$$

In this way we obtain

$$|w(r)| \leq Cr\int_0^r \frac{|w(t)|}{t^\mu}dt,$$

Calling $y(r) = \dfrac{|w(r)|}{r}$, we get

$$|y(r)| \leq C\int_0^r \frac{|w(t)|}{t^\mu}dt = C\int_0^s \frac{|w(t)|}{t^\mu}dt + C\int_s^r \frac{|y(t)|}{t^{\mu-1}}dt,$$

for all $s \in [0, r]$, and by Gronwall inequality

$$|y(r)| \leq C\int_0^s \frac{|w(t)|}{t^\mu}dt\, \mathrm{e}^{(C\int_s^r \frac{dt}{t^{\mu-1}})},$$

letting $s \to 0$ we get $w(r) = 0$.

To see that the solution is defined in $[0, \infty)$, multiply the equation by u' and integrate by parts. Then we arrive to the inequality

$$|u'(r)|^p + |u(r)|^p \leq K(a)\left(1 + \int_0^r (|u'(s)|^p + |u(s)|^p)ds\right), \; r \in [0, a]$$

and by the Gronwall inequality we conclude that

$$|u'(r)|^p + |u(r)|^p \leq e^{Kr}.$$

Classical arguments imply that u is defined in the whole interval $[0, \infty)$.

Finally, we will prove that u is oscillatory. We proceed by contradiction. Assume that u solution of

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}u^{p-1} = 0 \tag{1.29}$$

and that for some $r_0 > 0$ u does not vanish on $[r_0, \infty)$. We follow arguments in [69]. Since u^{p-1} is positive for $r > r_0$,

$$(r^{N-1}|u'|^{p-2}u')' < 0 \qquad \text{for } r > r_0$$

and $r^{N-1}|u'|^{p-2}u'$ is a decreasing function.

In particular, since $r^{N-1}|u'|^{p-2}u' < 0$ as $u(r) > 0$, we have $u' < 0$ for $r > 0$, that is, $u(s) > u(r)$ for $s < r$. By integrating (1.29) from r_0 to r we obtain,

$$r^{N-1}|u'(r)|^{p-2}u'(r) - r_0^{N-1}|u'(r_0)|^{p-2}u'(r_0) = -\int_{r_0}^{r} s^{N-1}u^{p-1}(s)ds \leq$$
$$-u^{p-1}(r)\int_{r_0}^{r} s^{N-1}ds = -\frac{r^N - r_0^N}{N}u^{p-1}(r).$$

By integrating again from $s_0 = 2r_0$ to r ($\geq s_0$)

$$\log\frac{u(s_0)}{u(r)} \geq C(r^{\frac{p}{p-1}} - s_0^{\frac{p}{p-1}}),$$

that is

$$u(r) \leq u(s_0)e^{(C(s_0^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}))}.$$

But $u' < 0$ and $r^{N-1}|u'|^{p-2}u'$ is a decreasing function, so $r^{N-1}|u'|^{p-2}u' \leq -C$ for large r. By integrating this inequality from r to ∞, we arrive to a contradiction with the decay of u above, since

$$u(r) \geq C_1\, r^{-\frac{N-p}{p-1}}.$$

So we conclude. ■

A direct consequence is the following result.

Corollary 1.5.2 *Let $u \neq 0$ a solution of the equation*

$$\mathcal{A}_p u(r) = r^{N-1}|u(r)|^{p-2}u(r),\ r \in (0,1)$$

such that $u(r_0) = 0$, then $u'(r_0) \neq 0$.

As a consequence all zeros of u are simple.

Theorem 1.5.3 *The problem (1.26) has nontrivial solution if and only if λ belongs to an increasing sequence $\{\lambda_k(p)\}$. Moreover,*

1. *For each eigenvalue $\lambda_k(p)$ any solution takes the form $\alpha v_k(r)$ with $\alpha \in I\!R$; namely the multiplicity of each eigenvalue is 1. Moreover v_k has exactly $(k-1)$ simple zeros.*

2. *Each $\lambda_k(p)$ depends continously on p.*

Proof. Let u be the solution to the Cauchy problem

$$\left\{\begin{array}{rcl} \mathcal{A}_p u &=& r^{N-1}|u(r)|^{p-2}u(r), \\ u(0) &=& 1, \\ u'(0) &=& 0, \end{array}\right. \tag{1.30}$$

and $0 < \nu_1(p) < \nu_2(p) < ... < \nu_k(p) < ...$ the zeros of u, that are simple and, according to Lemma 1.5.1, $\lim_{k\to\infty} \nu_k(p) = \infty$. We define

$$\begin{cases} \lambda_k = (\nu_k(p))^p, & k \in I\!N \\ v_k(r) = u(\nu_k(p)r), & k \in I\!N. \end{cases} \tag{1.31}$$

In this way (λ_k, v_k) are solution to problem (1.26), namely, λ_k is an eigenvalue and v_k the corresponding eigenfunction and moreover v_k has exactly $k-1$ zeros in $(0,1)$.

We need to prove that they are all the eigenvalues to problem (1.26). Assume $\mu > 0$ is an eigenvalue and w the corresponding eigenfunction. By the uniqueness result in Lemma 1.5.1 we have

$$w(r) = w(0)u(\mu^{\frac{1}{p}} r)$$

and because $w(1) = 0$ then we have $\mu = \lambda_k(p)$ for some $k \in I\!N$ and as a conclusion we get, $w(r) = w(0)v_k(r)$.

Now, the solution u to the Cauchy problem (1.30) verifies

$$u(r) = 1 - \int_0^r \phi_{p'}\left(\frac{1}{s^{N-1}} \int_0^s \phi_p(u(t))dt\right) ds \equiv \mathcal{L}_p(u)(r), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

so is continuous in p in the sense of the uniform convergence on bounded intervals. According to this fact and taking into account that for p fixed the zeros of u are simple, we conclude the continuous dependence of the eigenvalues on p. ■

1.6 Bifurcation for the p-Laplacian

Let $\Psi_{p,\lambda}$ be defined in

$$X = \{\, v \in C^1(\Omega) \,:\, v(x) = v(|x|),\, v(1) = 0\}$$

by

$$\Psi_{p,\lambda}(u) = u - \mathcal{A}_p^{-1}(r^{N-1}\lambda|u|^{p-2}u).$$

It is no difficult to show that $\Psi_{\lambda,p}$ is a nonlinear compact perturbation of the identity.

Fixed $k \in I\!N$ we call

$$S_k^+ = \{v \in X \,|\, v(r) > 0, \text{ if } r \in [0, \varepsilon) \text{ and has exactly } (k-1) \text{ simple zeros}\}.$$

In this way if $S_k^- = -S_k^+$ both are open sets in X.

Taking into account the behaviour of the inverse operator of the eigenvalue problem (1.25), we can use the extension of the Leray-Schauder degree definided in [31], in order to perform a homotopic deformation in p as is used in [42], to prove the following result.

Lemma 1.6.1 *Let* $\{\lambda_k(p)\}_{k\in I\!N}$ *be the sequence of eigenvalues of (1.25). Consider* $\lambda \neq \lambda_k(p)$, $k \in I\!N$, *then*

$$deg(\Phi_{p,\lambda}, B_\varepsilon(0), 0) = (-1)^\beta,\ \forall \varepsilon > 0,$$

where $\beta = \#\{\lambda_k(p) \,|\, \lambda_k(p) < \lambda\}$

Proof. If $\lambda < \lambda_1(p)$, $deg(\Phi_{p,\lambda}, B_\varepsilon(0), 0) = 1$ by the variational characterization of the first eigenvalue. Assume that $\lambda > \lambda_1(p)$ and $\lambda_k(p) < \lambda < \lambda_{k+1}(p)$. Since the eigenvalues depend continously on q, there exists a continous function

$$\nu : (1, \infty) \longrightarrow I\!R$$

such that $\lambda_k(q) < \nu(q) < \lambda_{k+1}(q)$ and $\lambda = \nu(p)$. Define

$$\mathcal{T}(q, u) = \Phi_{q,\nu(q)}(u) = u - \mathcal{A}_q^{-1}(\nu(q)|u|^{q-2}u).$$

It is easy to show that $\mathcal{T}(q, u)$ is a compact map such that for all $u \neq 0$, by definition of $\nu(q)$, $\mathcal{T}(q, u) \neq 0$, for all $q \in [p_0, \infty)$. Hence the invariance of the degree under homotopies and the classical result for $p = 2$ imply (recall $\lambda = \nu(p)$)

$$deg(\Phi_{p,\lambda}, B_\varepsilon(0), 0) = deg(\Phi_{2,\tilde{\lambda}(2)}, B_\varepsilon(0), 0) = (-1)^k.$$

■

Remark 1.6.2 *It is important to point out that the main point in this proof is the continuity of the eigenvalues with respect to* p.

As an application we found the following variation of the classical Rabinowitz bifurcation theorem:

Consider the problem

$$\begin{cases} \mathcal{A}_p u = r^{N-1}(\lambda|u|^{p-2}u + g(r, u)) \\ u'(0) = 0, \\ u(1) = 0, \end{cases} \tag{1.32}$$

where for some $\alpha > p$,

$$\lim_{u\to 0} \frac{g(r, u)}{u^\alpha} = M > 0$$

uniformly in r.

Theorem 1.6.3 *From each eigenvalue* λ_k *of problem (1.25) it bifurcates an unbounded continuum* $\mathcal{C}_k$ *of solutions to problem (1.32), with exactly* $k - 1$ *simple zeros.*

PROOF. We will be sketchy. The same argument as in the Rabinowitz theorem, [74], taking into account Lemma 1.6.1, gives that either $\mathcal{C}_k$ is unbounded, either it touches $(\lambda_j, 0)$. But the last alternative is impossible because if there exists $(\mu_m, u_m) \to (\lambda_j, 0)$ when $m \to \infty$, $u_m \neq 0$ and $u_m \notin S_j^+$, then we could consider $v_m = \dfrac{u_m}{||u_m||}$, and v_m should be a solution of problem,

$$\mathcal{A}_p v_m = r^{N-1}(\mu_m v_m + \frac{g(r, u_m)}{||u_m||^{p-1}}),$$

and by the properties of g it is easily seen that

$$\lim_{u \to 0} \frac{g(r, u)}{||u_m||^{p-1}} = 0.$$

By the compactness of $\mathcal{A}_p^{-1}$ we obtain that for some convenient subsequence $v_m \to w \neq 0$ as $m \to \infty$. Now w verifies the equation $\mathcal{A}_p w = r^{N-1}\lambda_j |w|^{p-2} w$ and $||w|| = 1$. Hence $w \in S_j^+$ which is an open set in X (observe that any zero of these solutions must be simple), and as a consequence for some m large enough, $v_m \in S_j^+$, and this is a contradiction. ■

Chapter 2

Multiplicity of solutions: Growth versus Shape

2.1 Introduction

In this chapter the model problem that we will consider is the following,

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\Omega \subset I\!\!R^N$ is a smooth bounded domain, $1 < q$, $1 < p < r$, $\lambda > 0$ and, perhaps, with some restrictions on λ and q that we will precise in each situation.

We define $p^* = \dfrac{Np}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \geq N$. We have different kind of problems:

A) With respect to r

1. $r < p^*$ subcritical case.
2. $r = p^*$ critical case.
3. $r > p^*$ supercritical case.

This *classification* takes into account mainly the growth.

Case 1 is classical and can be studied by critical point methods. We refer to [51] for the details. See also [70]. In case 2, one of the main difficulties in the application of the critical point methods is the lack of compactness. Sometimes a local Palais-Smale property can be recovered and this allows us to solve the problem under convenient conditions. We will explain the main ideas and methods. Case 3 is different because the study from the point of view of the critical point theory has no sense in general. Nevertheless, recently some results are obtained in [33] by using the critical point techniques. Also in this case, it is impossible to have a general result on regularity (by regularity we mean $\mathcal{C}^{1,\alpha}$).

B) With respect to the relation between q and p.

1. $1 < q = p$.
2. $p^* > q > p$.
3. $1 < q < p$.

The two first cases can be seen in [51], [53] and [58] and the references therein. The last case has precedents in [53] and in [11] in the case $p = 2$.

C) With respect to λ.

In [11] a more deep study of the case $1 < q < p$ is performed. This study is of *global character*. The meaning of global here deals with respect to λ. Namely we have the last classification of the problems to be considered as:

1. Small $\lambda > 0$, i.e., local behaviour.
2. All $\lambda > 0$, i.e., global behaviour.

2.2 First results

We will explain two types of results that show the influence of the shape of the zero order term.

First of all, we are interested in obtaining results about the ordering of solutions. Some previous results can be found in [49] and in [39] for other nonlinearities. We obtain below this type of result in an elementary way as a consequence of the following lemma by Adimurthi-Yadava [1].

Lemma 2.2.1 *Let $0 \le \rho_1(x) \le \rho_2(x)$ be two functions in $L^r(\Omega)$ defined on Ω, a bounded domain with smooth boundary, where $r \ge 1$ if $p > N$, $r > N/p$ if $p \le N$ and $\rho_1 \not\equiv 0$. Let $\lambda_1(\rho_i)$ be the first eigenvalue corresponding to*

$$\left\{ \begin{array}{rcl} -\Delta_p \varphi & = & \lambda \rho_i(x) \varphi^{p-1}, \quad x \in \Omega, \ i = 1, 2 \\ \varphi & = & 0 \quad on\ \partial\Omega. \end{array} \right. \tag{2.2}$$

Then, either $\lambda_1(\rho_1) < \lambda_1(\rho_2)$ or $\lambda_1(\rho_1) = \lambda_1(\rho_2)$ and the corresponding eigenfunctions φ_1, φ_2 vanish in $K = \{x \in \Omega \ : \ \rho_2(x) > \rho_1(x)\}$.

Proof. It is clear that $\lambda_1(\rho_1) \le \lambda_1(\rho_2)$. Let us suppose that $\lambda_1(\rho_1) = \lambda_1(\rho_2)$. Since $\varphi_1, \varphi_2 > 0$ and

$$\lambda_1(\rho_1) = \inf \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega \rho_1 u^p\,dx} = \frac{\int_\Omega |\nabla \varphi_1|^p\,dx}{\int_\Omega \rho_1 \varphi_1^p\,dx},$$

$$\lambda_1(\rho_2) = \inf \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega \rho_2 u^p\,dx} = \frac{\int_\Omega |\nabla \varphi_2|^p\,dx}{\int_\Omega \rho_2 \varphi_2^p\,dx} \le \frac{\int_\Omega |\nabla \varphi_1|^p\,dx}{\int_\Omega \rho_2 \varphi_1^p\,dx},$$

we get

$$\int_\Omega (\rho_2 - \rho_1)\varphi_1^p\,dx \le 0.$$

Then, we conclude that $\varphi_1 \equiv 0$ in K. Therefore

$$\int_\Omega \rho_1 \varphi_1^p\,dx = \int_\Omega \rho_2 \varphi_1^p\,dx \quad \text{i.e.} \quad \frac{\int_\Omega |\nabla \varphi_1|^p\,dx}{\int_\Omega \rho_2 \varphi_1^p\,dx} = \lambda_1(\rho_2).$$

Thus $\varphi_2 = \mu\varphi_1$ and $\varphi_2 \equiv 0$ in K. ■

As a consequence we can prove the following general result:

Lemma 2.2.2 *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous function verifying that $\dfrac{f(s,\lambda)}{s^{p-1}}$ is nondecreasing and*

$$\frac{f(s,\lambda)}{s^{p-1}} \in L^\infty_{loc}([0,\infty)).$$

Assume that u_1 and u_2 are bounded solutions of

$$\begin{cases} -\Delta_p u &= f(u,\lambda) \quad x \in \Omega \subset I\!R^N,\ \lambda > 0 \\ u(x) &= 0 \qquad\qquad x \in \partial\Omega, \end{cases} \tag{2.3}$$

such that $0 < u_1 \le u_2$ $(0 < u_2 \le u_1)$. Then $u_1 \equiv u_2$

Proof. If $0 < u_1 \le u_2$, then

$$-\Delta_p u_i = f(u_i,\lambda) = \frac{f(u_i,\lambda)}{u_i^{p-1}} u_i^{p-1} \equiv \rho_i(x) u_i^{p-1} \quad \text{in } \Omega, \quad i = 1,2.$$

with $\rho_1(x) \le \rho_2(x)$ by hypothesis, hence u_1 and u_2 are eigenfunctions for the eigenvalue $\lambda_1(\rho_1) = \lambda_1(\rho_2) \equiv 1$ of the p-Laplacian with weights ρ_1 and ρ_2, respectively. By Lemma 2.2.1, u_1 and u_2

vanish in $\{\rho_1(x) < \rho_2(x)\}$. By Hopf's Lemma, $u_1, u_2 > 0$ in Ω; so, $\{\rho_1(x) < \rho_2(x)\} = \emptyset$, i.e., $u_1 \equiv u_2$. ■

Other application of Lemma 2.2.1 is the following uniqueness result for a radial problem obtained in [1].

Theorem 2.2.3 *Consider the problem*

$$\left\{\begin{array}{rcl} \mathcal{A}_p u &=& r^{N-1}\left(\lambda|u|^{p-2}u + |u|^{\alpha-2}u\right) \\ u(1) &=& 0, \\ u'(0) &=& 0, \end{array}\right. \tag{2.4}$$

where $\mathcal{A}_p u = -(r^{N-1}|u'|^{p-2}u')'$, $\lambda > 0$ *and* $1 < p < \alpha < p^*$. *Then the problem (2.4) has a unique positive solution.*

(See the proof in [1].)

The second elementary result that we study here was obtained in [24] and deals with the existence of positive solution to the following problem,

$$\left\{\begin{array}{l} -\Delta_p u = |u|^{r-2}u + \lambda|u|^{q-2}u \\ u|_{\delta\Omega} = 0, \end{array}\right. \tag{2.5}$$

where we assume:

$$(\mathbf{H}) \quad 1 < q < p < r \text{ and } \lambda > 0.$$

Notice that the growth of the zero order term can be arbitrarily large, but there are a *subdiffusive* part, in the sense that $1 < q < p$, that produces the behaviour described in the following result.

Theorem 2.2.4 *Consider the problem under the hypothesis* (H). *Then there exist* $\lambda_0 > 0$ *such that for* $0 < \lambda \leq \lambda_0$, *(2.5) has a positive solution.*

Proof. The proof of the theorem is organized in several steps.

Step 1. We find a supersolution to problem (2.5).

Let v be the solution to the Dirichlet Problem

$$\left\{\begin{array}{l} -\Delta_p u = \lambda + 1 \\ u|_{\delta\Omega} = 0. \end{array}\right.$$

Then $0 < v < K$ in Ω. By simplicity of writing we call

$$F(u) = |u|^{r-2}u + \lambda|u|^{q-2}u.$$

Define $\overline{u}(x) = Tv(x)$ where T is a constant that will be chosen in such a way that

$$-\Delta_p \overline{u} \geq F(TM) \geq F(\overline{u})$$

where $M = \max\{1, \|v\|_\infty\}$

Now $-\Delta_p \overline{u} = T^{p-1}(\lambda + 1)$ and

$$F(\overline{u}) \equiv \lambda T^{q-1} v^{q-1} + T^{r-1} v^{r-1} \leq \lambda T^{q-1} M^{q-1} + T^{r-1} M^{r-1}.$$

Then, it is sufficient to find T such that

$$(\lambda + 1) \geq \lambda T^{q-p} M^{q-1} + T^{r-p} M^{r-1}.$$

We call

$$\phi(T) = c_1 \lambda T^{q-p} + c_2 T^{r-p},$$

with $c_1 = M^{q-1}$, $c_2 = M^{r-1}$. Then

$$\lim_{T \to 0^+} \phi(T) = \lim_{T \to \infty} \phi(T) = \infty,$$

because $q - p < 0 < r - p$; then ϕ attains a minimum in $[0, \infty)$. By elementary computations we have that

$$\phi'(T) \equiv -c_3 \lambda T^{q-p-1} + c_4 T^{r-p-1} = 0 \text{ in } T_0 = c_5 \lambda^{\frac{1}{r-q}},$$

where $c_5 = M^{-1}(r-q)^{-1}(p-q)$. To finish it is sufficient that,

$$\phi(T_0) \leq \lambda + 1,$$

that is

$$c_6 \lambda^{\frac{r-p}{r-q}} < \lambda + 1,$$

and here $c_6 = c_{p,q} M^{p-1}$. Moreover M depends on λ in the correct way because

$$||v||_\infty \leq c|\Omega|^{\left(\frac{1}{N} - \frac{1}{r(p-1)}\right)} (\lambda + 1)^{\frac{1}{p-1}}$$

where $r > \frac{N}{p-1}$; then there exists λ_0 such that for $0 < \lambda < \lambda_0$ and $1 < q < p < N$, $\overline{u}(x) = T_0 v$ is a supersolution of problem (2.5).

Step 2. Construction of a subsolution to (2.5).

Consider ϕ_1, positive eigenfunction of the eigenvalue problem,

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u \\ u|_{\delta\Omega} = 0, \end{cases}$$

such that $\|\phi_1\|_\infty = 1$. We define $\underline{u}(x) = t\phi_1(x)$.Then $\underline{u}$ is a subsolution if t is small enough because

$$\lambda_1 t^{p-1}\phi_1^{p-1} \le \lambda t^{q-1}\phi_1^{p-1} \le \lambda t^{q-1}\phi_1^{q-1} + t^{r-1}\phi_1^{r-1}.$$

Now fixed the supersolution $\overline{u}$, i.e. T, for t small enough, we get

$$-\Delta_p \underline{u} = t^{p-1}\lambda_1\phi_1^{q-1} \le t^{p-1}\lambda_1 \le -\Delta_p \overline{u}$$

Then by the weak comparison principle $\underline{u} \le \overline{u}$.

End of the proof. By the classical iteration method we get a solution between the subsolution and supersolution. ■

Theorem 2.2.5 *Assume that Ω has a smooth boundary. There exists $\Lambda > 0$ such that (2.5) has no positive solution for $\lambda > \Lambda$.*

Proof. Consider $\phi_1 > 0$ eigenfunction corresponding to the first eigenvalue λ_1, namely,

$$\begin{cases} -\Delta_p \phi_1 & = \lambda_1 |\phi_1|^{p-2}\phi_1 \\ \phi_1|_{\delta\Omega} & = 0. \end{cases}$$

The regularity of the boundary of Ω implies that the solutions of the eigenvalue problem are $\mathcal{C}^{1,\alpha}$ in the closure of Ω. If we assume that (2.5) has a positive solution u for all λ then by Hopf Lemma there exist a $t > 0$ depending on λ such that $t^{\frac{1}{p-1}}\phi_1 \le u$ in Ω.

We call $\psi = t^{\frac{1}{p-1}}\phi_1$. Pick $\varepsilon > 0$ such that for $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$ the eigenvalue problem has no nontrivial solution. The existence of such ε is a consequence of the result of isolation of the first eigenvalue of the p-laplacian studied in Section 1.4.

If λ is such that $\lambda_1 + \varepsilon \le b_{p,q}\lambda^{\frac{r-p}{r-q}}$ where $b_{p,q}$ verifies

$$b_{p,q}\lambda^{\frac{r-p}{r-q}} u^{p-1} \le \lambda u^{q-1} + u^{r-1}$$

then

$$\begin{aligned} &-\Delta_p \psi = \lambda_1 \psi^{p-1} \le \mu\psi^{p-1} \le \mu u^{p-1} \le \\ &(\lambda_1 + \varepsilon)u^{p-1} \le b_{p,q}\lambda^{\frac{r-p}{r-q}} u^{p-1} \le \lambda u^{q-1} + u^{r-1} = -\Delta_p u \end{aligned}$$

But then the problem

$$\begin{cases} -\Delta_p u & = \mu |u|^{p-2}u \\ u|_{\delta\Omega} & = 0, \end{cases}$$

with $\mu \in (\lambda_1, \lambda + \varepsilon)$ must have nontrivial solution and this is a contradiction. ■

We will consider the following subcritical problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{\alpha-2}u \\ u|_{\delta\Omega} = 0 \end{cases} \tag{2.6}$$

with $1 < q < p < \alpha < p^*$. The associated energy functional is

$$J(u) = \frac{1}{p}\int_\Omega |\ u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{\alpha}\int_\Omega |u|^\alpha dx.$$

If λ is small enough it is possible to show that the solution to problem (2.6), found in the Theorem 2.2.4, verifies that $J(u) < 0$. Also it is easy to show that at least for small $\lambda > 0$ we can apply the *Mountain Pass Theorem* and get the following result.

Theorem 2.2.6 *There exist a λ_0 such that problem (2.6) has at least two positive solutions for $0 < \lambda < \lambda_0$.*

Comparing the results in this section we notice that the *shape* of the zero order term plays a very important role, namely:

1. If $p = q$ the growth is important for existence, and the positive solutions cannot be ordered.

   In the radial case the subcritical problem has a *unique* solution.

2. If $1 < q < p$ there exists solution without restriction in the growth, the subcritical problem has a second positive solution that is ordered with the *minimum solution*.

In the next section we will obtain different results that involve the interplay of shape and growth.

2.3 Critical Problems: The Palais-Smale condition

The study of existence of solutions to the critical problem (2.1), i.e., the critical case $r = p^*$, is done by looking for critical points of the funcional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}, \quad \text{if} \quad q \neq p,$$

or, if $p = q$, by homogeneity, looking for the minimun in the variational problem with constraints,

$$I_\lambda = \inf\{\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx \,|\, ||u||_{p^*}^{p^*} = 1,\, u \in W_0^{1,p}(\Omega)\}.$$

Then the idea is to consider *approximate critical points* and pass to the limit. (See appendix). The main difficulty is to have compactness because the Sobolev embedding is not compact.

The concept of Palais-Smale sequence gives form to the meaning of *approximate critical point*.

We say that $\{u_j\}_{j\in I\!N} \subset W_0^{1,p}(\Omega)$ *is a Palais-Smale sequence if:*

a) $J(u_j) \to c$, $j \to \infty$; *b)* $J'(u_j) \to 0$ *in* $W^{-1,p'}(\Omega)$, $j \to \infty$.

We call to the constant c *energy level of the sequence.*

We say that J verifies *the Palais-Smale condition at the level* c if any Palais-Smale sequence at the level c, admits a strongly convergent subsequence .

Under our hypothesis on J, every Palais -Smale sequence is *bounded* in $W_0^{1,p}(\Omega)$, but because of the lack of compactness it is not possible to conclude strong convergence for any subsequence. The solution of this difficulty depends on a deep analysis of the behavior of the lack of compactness. More precisely, assume that we have a bounded sequence $\{u_j\}_{j\in I\!N} \subset W_0^{1,p}(\Omega)$. Taking a convenient subsequence we can assume that:

i) $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.

ii) $u_j \to u$ almost everywhere and in L^r with $1 \le r < p^*$.

iii) $|u_j|^{p^*} \rightharpoonup d\nu$ weakly-$*$.

iv) $|\nabla u_j|^p \rightharpoonup d\mu$ weakly-$*$.

where ν and μ are finite measures. We can extend the functions u_j by 0 to $I\!R^N$. To precise the ideas, assume initially that $u \equiv 0$. By the Sobolev inequality we have

$$\Big(\int_{I\!R^N} |\phi u_j|^{p^*} dx\Big)^{\frac{1}{p^*}} S^{\frac{1}{p}} \le \Big(\int_{I\!R^N} |\nabla(\phi u_j)|^p dx\Big)^{\frac{1}{p}}, \quad \text{for all} \quad \phi \in \mathcal{C}_0^\infty,$$

where S is the best constant in the Sobolev inclusion, i.e.,

$$S = \inf\{||\nabla u||_p^p \mid ||u||_{p^*} = 1,\ \nabla u \in L^p\}.$$

From iii) we obtain,

$$\lim_{j\to\infty} \Big(\int_{I\!R^N} |\phi u_j|^{p^*} dx\Big)^{\frac{1}{p^*}} = \Big(\int_{I\!R^N} |\phi|^{p^*} d\nu\Big)^{\frac{1}{p^*}},$$

moreover, by ii) and

$$|\|\nabla(\phi u_j)\|_p - \|\phi \nabla u_j\|_p| \le \|u_j \nabla\phi\|_p,$$

we obtain that

$$\lim_{j\to\infty} \int_{I\!R^N} |\nabla(\phi u_j)|^p dx = \int_{I\!R^N} |\phi|^p d\mu,$$

then we conclude that

$$(\int_{\mathbb{R}^N} |\phi|^{p^*} d\nu)^{\frac{1}{p^*}} S^{\frac{1}{p}} \le (\int_{\mathbb{R}^N} |\phi|^p d\mu)^{\frac{1}{p}}, \quad \text{for all} \quad \phi \in \mathcal{C}_0^\infty .$$

The last inequality is a *Reverse Hölder Inequality* from which we get the following representation result due to P.L. Lions. (See [64] and appendix.)

Lemma 2.3.1 *Let μ, ν be two non-negative and bounded measures on $\overline{\Omega}$, such that for $1 \le p < r < \infty$ there exists some constant $C > 0$ such that*

$$(\int_\Omega |\varphi|^r d\nu)^{\frac{1}{r}} \le C(\int_\Omega |\varphi|^p d\mu)^{\frac{1}{p}} \quad \forall \varphi \in \mathcal{C}_0^\infty .$$

Then, there exists $\{x_j\}_{j\in I} \subset \overline{\Omega}$ and $\{\nu_j\}_{j\in I} \subset (0,\infty)$, where I is finite, such that:

$$\nu = \sum_{j\in I} \nu_j \delta_{x_j} \quad , \quad \mu \ge C^{-p} \sum_{j\in I} \nu_j^{\frac{p}{r}} \delta_{x_j},$$

where δ_{x_j} is the Dirac mass at x_j.

In the general case, i.e., when $u \ne 0$, we can obtain the same conclusion for $v_j = u_j - u$, and then to study the absolutely continous part. We have the following P.L. Lions Lemma. (See appendix).

Lemma 2.3.2 *Let $\{u_j\}$ be a weakly convergent sequence in $W_0^{1,p}(\Omega)$ with weak limit u, and such that:*

i) $|\nabla u_j|^p \to \mu$ weakly-$*$ in the sense of measures.

ii) $|u_j|^{p^*} \to \nu$ weakly-$*$ in the sense of the measures.

Then, for some finite index set I we have:

$$\begin{cases} 1) \quad \nu = |u|^{p^*} + \sum_{j\in I} \nu_j \delta_{x_j} \quad , \quad \nu_j > 0 \\ 2) \quad \mu \ge |\nabla u|^p + \sum_{j\in I} \mu_j \delta_{x_j} \quad , \quad \mu_j > 0 \quad , \quad x_j \in \overline{\Omega} \\ 3) \quad \nu_j^{\frac{p}{p^*}} \le \dfrac{\mu_j}{S}. \end{cases}$$

(See [64] and [65]).

The representation obtained in Lemma 2.3.2, allows us to conjecture how small must be the *energy level* for which the singular part of the measures must be 0, and then a local Palais-Smale condition holds. The calculus of how small must be the energy level is relatively easy.

Assume $\{v_j\}_{j\in I\!N} \subset W_0^{1,p}(\Omega)$ a Palais-Smale sequence for the energy level c, i.e.:

$$a)\ J(v_j) \to c; \quad b)\ J'(v_j) \to 0, \ \text{ in } \ W^{-1,p'}(\Omega).$$

Then the sequence is bounded, and we can apply the Lemma 2.3.2 to some subsequence, i.e., we obtain,

i) $|v_j|^{p^*} \rightharpoonup d\nu = |v|^{p^*} + \sum_{j\in I} \nu_j \delta_{x_j}$ weak-$*$.

ii) $|\nabla v_j|^{p^*} \rightharpoonup d\mu \geq |\nabla v|^{p^*} + \sum_{j\in I} \mu_j \delta_{x_j}$ weak-$*$.

iii) Moreover, $\mu_j/S \geq \nu_j^{p/p^*}$.

Localizing each singularity by a test function ϕ with support in $B(x_j, \varepsilon)$, and by $i), ii)$ applied to ϕv_j we have,

$$\int_\Omega \phi d\nu + \lambda \int_\Omega |v|^q \phi dx - \int_\Omega \phi d\mu = \lim_{j\to\infty} \int_\Omega |\nabla v_j|^{p-2} v_j \langle \nabla v_j, \nabla \phi \rangle dx.$$

Using the weak convergence, Hölder inequality and taking limits for $\varepsilon \to 0$, we get that necessarily $\nu_j = \mu_j$. Then taking into account iii), either $\mu_j = \nu_j \geq S^{N/p}$ or $\mu_j = \nu_j = 0$.

Then if we assume $q > p$, and there exists a Dirac mass, necessarily we obtain,

$$\begin{array}{l} c = \lim_{j\to\infty} J(v_j) = \lim_{j\to\infty} (J(v_j) - \langle J'(v_j), v_j \rangle) \geq \\ \frac{1}{N} \int\limits_\Omega |v|^{p^*} dx + \lambda(\frac{1}{p} - \frac{1}{q}) ||v||_q^q + \frac{1}{N} S^{N/p}, \end{array}$$

as a conclusion:

If there exists a Dirac mass, $c \geq \frac{1}{N} S^{\frac{N}{p}}$.

In a similar way if $1 < q < p$ we have:

If there exists a Dirac mass, $c \geq \dfrac{1}{N} S^{\frac{N}{p}} - K\lambda^\beta$, *where* $\beta = \dfrac{p^*}{p^* - q}$ *and* K *depends on* p, q, N *and* Ω.

As a consequence of the calculus above and the continuity of the inverse of the p-laplacian, we get the following Lemma.

Lemma 2.3.3 *Let* $\{v_j\} \subset W_0^{1,p}(\Omega)$ *be a Palais-Smale sequence for* J,

$$\begin{array}{lll} J(v_j) & \to c & \\ J'(v_j) & \to 0 \quad in \quad W^{-1,p'}(\Omega), & \frac{1}{p} + \frac{1}{p'} = 1 \end{array}$$

Then, we have:

1. *If* $p < q < p^*$, *and* $c < \frac{1}{N} S^{\frac{N}{p}}$, *there exists a subsequence* $\{v_{j_k}\} \subset \{v_j\}$, *strongly convergent in* $W_0^{1,p}(\Omega)$.

2. *If* $1 < q < p$, *and* $c < \frac{1}{N} S^{\frac{N}{p}} - K\lambda^{\beta}$, *where* $\beta = \dfrac{p^*}{p^* - q}$ *and* K *depends on* p, q, N *and* Ω, *then there exists a subsequence* $\{v_{j_k}\} \subset \{v_j\}$, *strongly convergent in* $W_0^{1,p}(\Omega)$.

Outline of the proof. From the previous arguments, we have that the singular part of ν is identically zero. Therefore, there exists a subsequence $\{v_{j_k}\} \subset \{v_j\}$, such that

i) $v_{j_k} \to v$ almost everywhere; *ii)* $\int\limits_\Omega |v_{j_k}|^{p^*} dx \to \int\limits_\Omega |v|^{p^*} dx$.

So we conclude that $v_{j_k} \to v$ strongly in L^{p^*}. (See [28]).

As a consequence

$$\lambda |v_{j_k}|^{q-2} v_{j_k} + |v_{j_k}|^{p^*-2} v_{j_k} \to \lambda |v|^{q-2} v + |v|^{p^*-2} v; \text{ in } W^{-1,p'}(\Omega)$$

and then the continuity of the inverse of the p-laplacian from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$ implies the theorem. (See [53] and appendix for more details).

Now assume that $p = q$. If we have a minimizing sequence for I_λ and $u_j \to u$ in $W_0^{1,p}(\Omega)$, by Sobolev inequality,

$$\int\limits_\Omega |\nabla u_j|^p dx - \lambda \int\limits_\Omega |u_j|^p dx \geq S - \lambda \int\limits_\Omega |u_j|^p dx,$$

and then $I_\lambda \geq S - \lambda ||u||_p^p$. As a conclusion:

A sufficient condition to get a nontrivial solution, $u \neq 0$, *is that* $I_\lambda < S$.

In fact we can prove compactness also with the same hypothesis. More precisely Lemma 2.3.2 gives the following result.

Lemma 2.3.4 *If* $\{u_j\}_{j \in \mathbb{N}} \subset W_0^{1,p}(\Omega)$ *is a minimizing sequence for* $I_\lambda < S$, *then, for some subsequence,* $u_j \to u \neq 0$ *in* $W_0^{1,p}(\Omega)$

(See [51]).

Then Lemmas 2.3.3 and 2.3.4 give the energy level for which we get the compactness. In the following sections we explain when it is possible to reach this critical level with a Palais-Smale sequence.

2.4 Local results on multiplicity

In this section we will study local results, namely, for λ small.

2.4.1 The critical case and $p^* > q > p$

We emphasize the dependence of J on λ writting the action functional as J_λ. We assume in first place that $q > p$. If we consider the function $g_\lambda(t) = J_\lambda(tv_0)$, we find that it attains its maximum for $t > 0$ and $\lim\limits_{t\to\infty} g(t) = -\infty$.

It is not difficult to prove that J_λ verifies the hypotheses of the Mountain Pass Theorem.

Lemma 2.4.1 *Let X be a Banach space and $J \in C^1(X, \mathbb{R})$ a functional. Let us assume that there exists $r, R > 0$, such that:*

i) $J(u) > r$, $\forall u \in X$ with $\|u\| = R$.

ii) $J(0) = 0$, and $J(w_0) < r$ for some $w_0 \in X$, with $\|w_0\| > R$.

Let us define $\Gamma = \{g \in C([0,1]; X) : g(0) = 0, g(1) = w_0\}$, and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)).$$

Then, there exists a sequence $\{u_j\} \subset X$, such that $J(u_j) \to c$, and $J'(u_j) \to 0$ in X' (dual of X).

(See [13], [17] or appendix for the proof). As a consequence of the results in Section 2.3 , it is sufficient to show that

$$\sup_{t\geq 0} J_\lambda(tv_0) < \frac{1}{N} S^{\frac{N}{p}}, \quad \text{for some} \quad v_0 \in W_0^{1,p}(\Omega).$$

Now, if $\lambda = 0$ then $\sup_{t\geq 0} J_0(tv_0) \geq \frac{1}{N} S^{\frac{N}{p}}$. The minimizers of Sobolev inclusion in $\mathbb{R}^N$ are,

$$U_\varepsilon(x) = (\varepsilon + c_{p,N}|x|^{p/(p-1)})^{\frac{p-N}{p}} \quad \varepsilon > 0.$$

(See [84]). Define $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$, $v_\varepsilon = u_\varepsilon||u_\varepsilon||_{p^*}^{-1}$ for some convenient cuttoff function ϕ. By definition we get

$$\lim_{\varepsilon\to 0} \sup_{t\geq 0} J_0(tv_\varepsilon) = \frac{1}{N} S^{\frac{N}{p}}.$$

The idea is that the perturbation term $\lambda|u|^{q-2}u$ provides a energy level from below the critical constant.

Now, fixed $w_0 \in W_0^{1,p}(\Omega)$ we can compute that

$$\lim_{\lambda\to\infty} (\sup_{t\geq 0} J_\lambda(tw_0)) = 0.$$

Then,

There exists λ_0 such that for $\lambda > \lambda_0$ the problem (2.1) has a nontrivial solution, obtained as critical point of J_λ by the Mountain Pass Lemma.

But if we want to find a solution for all $\lambda > 0$, according with the property of minimization which define U_ε, it seems natural to study if

$$\sup_{t\geq 0} J_\lambda(tv_\varepsilon) < \frac{1}{N} S^{\frac{N}{p}}.$$

We have the following estimates:

$$\|\nabla v_\varepsilon\|_p^p = S + O(\varepsilon^{\frac{N-p}{p}})$$

and

$$||v_\varepsilon||_\alpha^\alpha = \begin{cases} C_1 \varepsilon^{\frac{p-1}{p}(N-\alpha\frac{(N-p)}{p})} + o(\varepsilon^{\frac{p-1}{p}(N-\alpha\frac{(N-p)}{p})}) & \text{if} \quad \alpha > p^*(1-\frac{1}{p}) \\ C_1 \varepsilon^{\frac{N-p}{p^2}\alpha} |\log\varepsilon| + o(\varepsilon^{\frac{N-p}{p^2}\alpha} |\log\varepsilon|) & \text{if} \quad \alpha = p^*(1-\frac{1}{p}) \\ C_1 \varepsilon^{\frac{N-p}{p^2}\alpha} + o(\varepsilon^{\frac{N-p}{p^2}\alpha}) \text{ if } \alpha < p^*(1-\frac{1}{p}), \end{cases}$$

where the change of behaviour in the last estimate is due to the integrability or not integrability of U_ε^α in $I\!R^N$. For $\varepsilon > 0$ consider $g_\varepsilon(t) = J_\lambda(tv_\varepsilon)$. We can estimate directly the value t_ε for which,

$$g_\varepsilon(t_\varepsilon) = \max_{t\geq 0} g_\varepsilon(t),$$

getting that

$$\int_\Omega |\nabla v_\varepsilon|^p dx \geq t_\varepsilon^{p^*-p} > \frac{S}{2}.$$

Therefore,

$$g_\varepsilon(t_\varepsilon) \leq \frac{1}{N} S^{\frac{N}{p}} + C\varepsilon^{\frac{N-p}{p}} - \frac{\lambda}{q}(\frac{S}{2})^{\frac{q}{p^*-p}} \int_\Omega v_\varepsilon^q dx,$$

and then

$$g_\varepsilon(t_\varepsilon) < \frac{1}{N} S^{\frac{N}{p}} \quad \text{if} \quad C\varepsilon^{\frac{N-p}{p}} - \frac{\lambda}{q}(\frac{S}{2})^{\frac{q}{p^*-p}} \int_\Omega v_\varepsilon^q dx < 0,$$

that is equivalent to

$$\frac{N-p}{p} > \frac{p-1}{p}(N - q\frac{(N-p)}{p}).$$

As a conclusion we have the following result.

Theorem 2.4.2 *a) If $p < q < p^*$, there exists $\lambda_0 > 0$ such that the problem (2.1) has a nontrivial solution $\forall\lambda \geq \lambda_0$.*

b) If $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, then there exists a nontrivial solution of the problem (2.1), $\forall\lambda > 0$.

(See [53] for details)

Remark 2.4.3 *If we assume $p^2 \leq N$, part b) in Theorem 2.4.2 applies for $p < q < p^*$. This result corresponds for $p = 2$ to dimension $N \geq 4$. (See [29]). For $N = 3$ the existence of solution for all $\lambda > 0$ in this case cannot be obtained.*

A similar calculation in the case $q = p$ allows us to obtain a sufficient condition for which $I_\lambda < S$. The result is formulated in the following way.

Theorem 2.4.4 *Assume $p^2 \leq N$. Then for $0 < \lambda < \lambda_1$, first eigenvalue of the p-laplacian, we have $0 < I_\lambda < S$, and then the problem 2.1 with $q = p$ and $r = p^*$ has nontrivial solution.*

(See [51] for details).

We would like to emphasize the fact that *the result in the case $q \geq p$ depends strongly on the dimension.*

Remark 2.4.5 *To obtain positive solution as critical points, we can consider a functional with the term in u^α substitued by $(u_+)^\alpha$, where u_+ is the positive part of u.*

2.4.2 The critical case and $1 < q < p$

Results in the critical case, $r = p^*$ and $1 < q < p$ are quite different. The results of existence of a positive solution are independent of the dimension as we see in section 2.2. Also we can prove the existence of two positive solutions at least under some hypotheses.

Theorem 2.4.6 *Given problem (2.1), with $1 < q < p$ and $r = p^*$, there exists $\lambda_0 > 0$, such that, for $0 < \lambda < \lambda_0$, there exists infinitely many solutions. Moreover at least one of them is positive. For λ large enough no positive solution exists.*

Remark 2.4.7 *We find no restrictions on the dimensions and moreover we get infinitely many solutions, perhaps with change of sign.*

We explain briefly the methods of the proof of this statement.

Consider as in the previous section $g_\lambda(t) = J_\lambda(tv_0)$. If λ is small enough, $g_\lambda(t)$ attains a local minimum and a local maximum in $t > 0$, for $0 < \lambda < \lambda_0$ because, $1 < q < p$.

By Sobolev inequality we obtain

$$J_\lambda(u) \geq h(\|\nabla u\|_p).$$

where,

$$h(x) = \frac{1}{p}x^p - \frac{1}{p^* S^{\frac{p^*}{p}}} x^{p^*} - \frac{\lambda}{q} C_{p,q} x^q,$$

and $C_{p,q}$ is a constant. We can choose $\lambda_0 > 0$ such that, if $0 < \lambda \leq \lambda_0$, h attains its nonnegative maximum. Consider R_0 and R_1 in such a way that $h(R_0) = 0$, $h(R_1) = 0$, (the point R, where h attains its maximum, verifies $R_0 \leq R \leq R_1$). We make the following truncation of the functional J_λ. Take $\tau : I\!R^+ \to [0, 1]$, nonincreasing and $\mathcal{C}^\infty$, such that $\tau(x) = 1 \quad \text{if} \quad x \leq R_0$, $\tau(x) = 0$ if $x \geq R_1$.

Let $\varphi(u) = \tau(\|\nabla u\|_p)$. We consider the truncated functional

$$\tilde{J}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}\varphi(u) - \frac{\lambda}{q}\int_\Omega |u|^q.$$

As a consequence of Lemma 2.3.3 2) and the previous construction, it is not difficult to prove the following result.

Lemma 2.4.8

1. *$\tilde{J} \in \mathcal{C}^1(W_0^{1,p}(\Omega), I\!R)$, is even.*
2. *If $\tilde{J}_\lambda(u) \leq 0$, then $\|\nabla u\|_p < R_0$, and $\tilde{J}(v) = J_\lambda(v)$ for all v in a small enough neighbourhood of u.*
3. *There exists $\lambda_0 > 0$, such that, if $0 < \lambda < \lambda_0$, then $\tilde{J}$ verifies a local Palais-Smale condition for $c \leq 0$.*

We will use the *genus* of a symmetric set in $W_0^{1,p}(\Omega)$ defined in the appendix.

It is possible to prove the existence of level sets of $\tilde{J}$ with arbitrarily large genus, more precisely:

Lemma 2.4.9 *Given $n \in I\!N$, there is $\varepsilon = \varepsilon(n) > 0$, such that*

$$\gamma(\{u \in W_0^{1,p}(\Omega) : \tilde{J}(u) \leq -\varepsilon\}) \geq n.$$

PROOF. Fix n, let E_n be a n-dimensional subspace of $W_0^{1,p}(\Omega)$. We take $u_n \in E_n$, with norm $\|\nabla u_n\|_p = 1$. For $0 < \rho < R_0$, we have:

$$\tilde{J}(\rho u_n) = F(\rho u_n) = \frac{1}{p}\rho^p - \frac{1}{p^*}\rho^{p^*}{}_1|u|^{p^*} - \frac{\lambda}{q}\rho^q{}_1|u|^q$$

E_n is a space of finite dimension; so, all the norms are equivalent . Then, if we define:

$$\alpha_n = \inf\{\int_\Omega |u|^{p^*} : u \in E_n, \quad \|\nabla u_n\|_p = 1\} > 0$$
$$\beta_n = \inf\{\int_\Omega |u|^q : u \in E_n, \quad \|\nabla u_n\|_p = 1\} > 0$$

we have $\tilde{J}(\rho u_n) \leq \frac{1}{p}\rho^p - \frac{\alpha_n}{p^*}\rho^{p^*} - \frac{\lambda\beta_n}{q}\rho^q$, and we can choose ε (which depends on n), and $\eta < R_0$, such that $\tilde{J}(\eta u) \leq -\varepsilon$ if $u \in E_n$, and $\|\nabla u\|_p = 1$.

Let $S_\eta = \{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p = \eta\}$. $S_\eta \cap E_n \subset \{u \in W_0^{1,p}(\Omega) : \tilde{J}(u) \leq -\varepsilon\}$; therefore, by the properties of the genus,

$$\gamma(\{u \in W_0^{1,p}(\Omega) : \tilde{J}(u) \leq -\varepsilon\}) \geq \gamma(S_\eta \cap E_n) = n$$

(See appendix). ∎

The last result is an extension of [21] and permits to prove the existence of critical points by using a Lusternik-Schnirelmann argument. (See for instance [76]). We get the following results

Lemma 2.4.10 *Let*

$$\Sigma_k = \{C \subset W_0^{1,p}(\Omega) - \{0\} | \, C \text{ is closed}, C = -C, \quad \gamma(C) \geq k\}.$$

Let $c_k = \inf_{C\in\Sigma_k} \sup_{u\in C} \tilde{J}(u)$, $K_c = \{u \in W_0^{1,p}(\Omega) : \tilde{J}'(u) = 0, \tilde{J}(u) = c\}$, *and suppose* $0 < \lambda < \lambda_0$, *where* λ_0 *is the same that in Lemma 2.4.8.*

Then, if $c = c_k = c_{k+1} = ... = c_{k+r}$, $\quad \gamma(K_c) \geq r + 1$.

In particular, the c_k*'s are critical values of* $\tilde{J}$.

PROOF.

In the proof, we will use the Lemma 2.4.9, and a classical deformation lemma (see [76]).

For simplicity, we call $J^{-\varepsilon} = \{u \in W_0^{1,p}(\Omega) : J(u) \leq -\varepsilon\}$. By lemma 2.4.9, $\forall k \in I\!N$, $\quad \exists\varepsilon(k) > 0$ such that $\gamma(J^{-\varepsilon}) \geq k$.

Because J is continuous and even, $J^{-\varepsilon} \in \Sigma_k$; then, $c_k \leq -\varepsilon(k) < 0, \forall k$. But J is bounded from below; hence, $c_k > -\infty$ $\forall k$.

Let us assume that $c = c_k = ... = c_{k+r}$. Let us observe that $c < 0$; therefore, J verifies the Palais-Smale condition in K_c, and it is easy to see that K_c is a compact set.

If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U , $K_c \subset U$, such that $\gamma(U) \leq r$. (We can choose $U \subset J^0$, because $c < 0$).

By the deformation lemma, we have an odd homeomorphism $\eta : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$, for some $\delta > 0$. (Again, we must choose $0 < \delta < -c$, because J verifies the Palais-Smale condition on J^0 , and we need $J^{c+\delta} \subset J^0$). By definition,

$$c = c_{k+r} = \inf_{C \in \Sigma_{k+r}} \sup_{u \in C} J(u).$$

Then, there exists $A \in \Sigma_{k+r}$, such that $\sup_{u \in A} J(u) < c + \delta$; i.e., $A \subset J^{c+\delta}$, and

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta} \tag{2.7}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$.

Then, $\eta(\overline{A - U}) \in \Sigma_k$. And this contradicts (2.7); in fact,

$$\eta(\overline{A - U}) \in \Sigma_k \quad \text{implies} \quad \sup_{u \in \eta(\overline{A - U})} J(u) \geq c_k = c.$$

■

With the lemmas above we conclude Theorem 2.4.6. It is necessary only to point out that there exists at least a positive solution because $\tilde{J}$ has a global minimum and the solution where $\tilde{J}$ attains its minimum is positive. (See [53]).

With this result the fundamental idea is that all these solutions are related with the component of smaller degree of the zero order term, i.e., by $\lambda |u|^{q-2} u$. In fact this same result is obtained for the problem without the critical term. (See [51]). Moreover, in this subcritical problem, the positive solution is unique. (See [41] where some ideas by Brezis for $p = 2$ are generalized).

On the other hand, it is not difficult to show that the solutions obtained above tend to 0 when $\lambda \to 0$.

The previous remarks, make more relevant the problem of finding a second positive solution. Such a solution will be find by the Mountain Pass Theorem, and then, when $\lambda \to 0$ it is not too hard to prove that converges to a Dirac's delta. We can say in this sense that this second positive solution corresponds to the contribution of the critical term.

Theorem 2.4.11 *Assume that one of the following hypothesis holds.*

1. $\dfrac{2N}{N+2} < p < 3$, $1 < q < p$.

2. $p \geq 3$, $p > q > q_0 = p^* - \dfrac{2}{p-1}$.

Then there exists a constant $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then problem (2.1) with $q < p < r = p^*$ has at least two positive solutions.

Proof. The proof is much more delicate, then we give only some indications and refer to [54] for more details.

Step 1.- A local Palais-Smale condition.

Consider $c_0 = J_\lambda(u_0)$, minimum of the truncated functional. A detailed analysis gives us that for a Palais-Smale sequence, $\{u_j\} \subset W_0^{1,p}(\Omega)$, such that

i) $J_\lambda(u_j) \to c < c_0 + \dfrac{1}{N} S^{N/p}$ (S being the best Sobolev constant).

ii) $J'_\lambda(u_j) \to 0$,

and if λ small enough, then, there exists a strongly convergent subsequence.

Step 2.- The Mountain Pass Lemma hyphotesis.

It is possible to show that:

Given $v \in W_0^{1,p}(\Omega)$ such that $J_\lambda(v) < c_0$ there exists a sequence $\{u_j\} \in W_0^{1,p}(\Omega)$ such that

$$
\begin{aligned}
&i) J'_\lambda(u_j) \to 0\\
&ii) J_\lambda(u_j) \to c = \inf_{\gamma\in\mathcal{C}} \sup_{t\in[0,1]} J_\lambda(\gamma(t))
\end{aligned}
$$

where $\mathcal{C} = \{\gamma \in \mathcal{C}([0,1], W_0^{1,p}(\Omega)) \,:\, \gamma(0) = 0,\, \gamma(1) = v\}$.

Therefore, we must find $v \in W_0^{1,p}(\Omega)$ such that $c < c_0 + \dfrac{1}{N} S^{N/p}$.

Step 3.- The energy estimate.

We are looking for an estimate of the type

$$
\sup_{R>0} J_\lambda(u_0 + Ru_\varepsilon) < J_\lambda(u_0) + \frac{1}{N} S^{N/p},
$$

where u_ε is a convenient normalization of the Sobolev minimizer for ε small.

For such ε we choose $v = u_0 + Ru_\varepsilon$, with R large enough, such that $J_\lambda(v) < c_0$.

We indicate the proof in the case 1) of the theorem.

By using the estimates for the Sobolev minimizers, that u_0 is a solution of the problem, and convenient estimates for $J_\lambda(u_0 + v) - J_\lambda(u_0)$, it follows

$$
\begin{aligned}
&J_\lambda(u_0 + Ru_\varepsilon) \le\\
&J_\lambda(u_0) + \frac{R^p}{p} \int_\Omega |\nabla u_\varepsilon|^p dx - \frac{R^{p^*}}{p^*} \int_\Omega |u_\varepsilon|^{p^*} dx +\\
&C_1 R^\gamma \varepsilon^\beta - C_2 R^{p^*-1} \varepsilon^{\frac{N-p}{p}}, \quad \beta > \tfrac{N-p}{p}.
\end{aligned}
$$

For the parameters in the given range we have that the exponent of the negative error term, is smaller than those of the positive error term. This remark allows us to conclude after some estimates about the value of R_ε where the maximum is attained. (See [54] for all the details). ■

Remark 2.4.12

1. *$p > q_0$ implies the following restriction for the dimension:*
$$N > p\left(1 + \frac{p(p-1)}{2}\right).$$

2. *If $q \leq q_0$ the methods that we use in the proof don't work.*

3. $\dfrac{2N}{N+2} < p$ *is equivalent to* $p^* > 2$.

The obstruction in the dimension that appears in the second case of the theorem can be seen in the following example.

Example.-Assume $p = 4$, $N = 8$, $q = 2$ (and therefore $p^* = 8$). It is possible to compute the integrals in a explicit way (without any loss of accuracy in the estimates; we have an exact algebraic expression). More precisely, in this case the error term takes the form

$$J_\lambda(u_0 + Ru_\varepsilon) \leq$$
$$c_0 + \frac{R^4}{4}\int |\nabla U_1|^4 dx - \frac{R^8}{8}\int |U_1|^8 dx + C_1\varepsilon^{2/3} - C_2\varepsilon^{2/3} + o(\varepsilon^{2/3}),$$

and the constants depend on the L^∞ norms of u_0 and ∇u_0. Therefore, the sign of the coefficient of the main error term it is not clear.

Nevertheless, numerical experiments performed by Prof C. Simó, seem to indicate that if $q < p$ then there is no restriction, while if $p = q$, it appears the phenomena of the *bad* dimensions $p^2 < N$ (as in the case $p = 2$ and $N = 3$, see [29].)

2.5 Global results on multiplicity

In the work [11] by Ambrosetti, Brezis, and Cerami, it is proved the existence of a second positive solution in the semilinear case (p=2). Their result is global in the following sense:

Let $p = 2$, $1 < q < 2$, $2 < r \leq \frac{2N}{N+2}$ in problem 2.1 and

$$\Lambda = \sup\{\lambda \mid \textit{Problem}\,(2.1) \quad \textit{has a positive solution}\}.$$

For $\lambda \in (0, \Lambda)$, there is a second positive solution of problem (2.1).

This global result is obtained by the application of the results in [30]. (See also [39]).

There is also a multiplicity result by Tarantello [85], when the term u^{q-1} is replaced by an inhomogeneous term $f(x)$ with a convenient norm small enough.

Here we will study the case $p \neq 2$ in the radial case. As a byproduct of the method we obtain a different proof in the case $1 < p = 2 < (2+N)/(N-2)$

Consider the radial problem

$$\begin{cases} \mathcal{A}_p u &= r^{N-1} f_\lambda(u), \quad 0 \le r < 1, \\ u'(0) &= 0, \\ u(1) &= 0, \end{cases} \tag{2.8}$$

where

$$\mathcal{A}_p u = -(r^{N-1}|u'(r)|^{p-2}u'(r))', \; f_\lambda(u) = \lambda u^{q-1} + u^{\alpha-1},$$

and

$$1 < q < p < \alpha < p^*, \text{ with } p^* = \tfrac{Np}{N-p} \text{ if } p < N \text{ and } p^* = \infty \text{ otherwise.}$$

By a solution of (2.8) we mean hereafter $u \in C(0,1)$ which solves (2.8) weakly. In the sequel we shall make use (also without mentioning them explicitly) of the regularity results of [79], [44] and [87], namely that if $u \in C(0,1)$ is any weak solution of (2.8), then $u \in C^{1,\nu}$.

In section 1.5 we prove uniqueness of solution for the Cauchy Problem

$$\begin{cases} \mathcal{A}_p u &= r^{N-1}|u(r)|^{\alpha-2}u(r), \\ u(0) &= a, \\ u'(0) &= 0. \end{cases} \tag{2.9}$$

As an application of Lemma 1.6.1 we obtain the following uniqueness results.

Lemma 2.5.1 *For $\lambda = 0$, problem (2.8) has a unique positive solution v_0.*

Proof. Let u_1, u_2 be two solutions of (2.8) for $\lambda = 0$. Let, for example, $u_1(0) \ge u_2(0)$ and set

$$\mu = \frac{u_1(0)}{u_2(0)}, \quad R = \mu^{(p-a)/p}$$

and

$$v_2(r) = \mu u_2(\frac{r}{R}).$$

One checks that v_2 satisfies the Cauchy problem,

$$\begin{cases} \mathcal{A}_p v_2 &= r^{N-1}|v_2(r)|^{\alpha-2}v_2(r), \\ v_2(0) &= \mu u_2(0) = u_1(0), \\ v_2'(0) &= 0. \end{cases}$$

By Lemma 1.6.1 one deduces that $v_2(r) = u_1(r)$, namely

$$u_1(r) = \mu u_2(\frac{r}{R}).$$

As a consequence, $u_1(R) = \mu u_1(1) = 0$, where $R \leq 1$ because $\mu \geq 1$. Since $u_1 > 0$ for $r \in (0,1)$, it follows that $R = 1$, whence $\mu = 1$. Then $u_1(0) = u_2(0)$ and $u_1 \equiv u_2$ by the uniqueness for the Cauchy Problem. ■

Lemma 2.5.2 *For all $a > 0$ there exists a unique R_a such that*

$$\begin{cases} v_a(R_a) &= 0, \\ v_a'(R_a) &< 0, \end{cases}$$

where v_a is the solution of the Cauchy problem (2.9).

Proof. Let $a_0 \equiv v_0(0)$ and let

$$R_a = (\frac{a}{a_0})^{(p-\alpha)/p}$$

$$\tilde{u}(r) = \frac{a}{a_0}\, v_0(\frac{r}{R_a}).$$

Then a direct computation shows that

$$\begin{cases} \mathcal{A}_p \tilde{u} &= r^{N-1}\tilde{u}^{\alpha-1} \\ \tilde{u}(0) &= a, \\ \tilde{u}'(0) &= 0, \end{cases}$$

so $\tilde{u}(r) = v_a(r)$ by uniqueness, and hence $v_a(R_a) = \dfrac{a}{a_0}\, u_0(1) = 0$. Moreover by Hopf lemma $v_a'(R_a) < 0$. ■

As a consequence of the previous lemma we obtain the following Corollary which is an extension to our situation of the well known result by Gidas and Spruck, [56].

Corollary 2.5.3 *Let u be any solution of*

$$\begin{cases} \mathcal{A}_p u &= r^{N-1}|u|^{\alpha-2}u, \quad 0 \leq r < \infty, \\ u'(0) &= 0, \qquad u(r) \geq 0. \end{cases}$$

Then $u \equiv 0$.

Proof. If $a = u(0) = 0$ the result follows trivially by the uniqueness for the Cauchy Problem. Assume that $u > 0$. Then by Lemma 2.5.2 there exists R_a where $u(R_a) = 0$ and $u'(R_a) < 0$, so u changes sign, a contradiction. ■

Remark 2.5.4 *An elementary consequence of the uniqueness of the Cauchy Problem is that in the case $N = 1$ all the solutions to the problem (2.8) with $\lambda = 0$ are symmetric. The reader could check that if $N = 1$ the moving planes method also works. (See [50]).*

Now we obtain uniform bounds for the solutions of problem (2.8). In first place we have the radial version of the Theorem 2.2.5.

Lemma 2.5.5 *There exists $\tilde{\Lambda} > 0$ such that for $\lambda > \tilde{\Lambda}$, problem (2.8) has no positive solution.*

Set

$$\Lambda = \sup\{\lambda > 0 : \text{(2.8) has a positive (radial) solution}\}.$$

By the previous Lemma we know that $\Lambda < \infty$. We know also that $\Lambda > 0$ because for $\lambda > 0$ small enough we can find a positive solution: it suffices to use critical point theory, see [52], or by sub and supersolutions, see [24].

Now we get the uniform estimate in the L^∞-norm in the radial case.

Lemma 2.5.6 *There exists $C > 0$ such that*

$$||u||_\infty \leq C,$$

for all positive (radial) solutions of (2.8) and all $\lambda \in [0, \Lambda]$.

Proof. We argue by contradiction. Let u_n be a sequence of (radial) positive solutions of $\mathcal{A}_p u_n = r^{N-1} f_{\lambda_n}(u_n)$, ($\{\lambda_n\} \subset (0, \Lambda)$), such that $||u_n||_\infty \to \infty$. Without loss of generality we can assume $\lambda_n \to \lambda$.

It is easy to see that for any n the equation implies that u_n achieves its maximum at the point $r = 0$, i.e., $u_n(0) = ||u_n||_\infty$.

Now, we normalize by using the following rescaling:

$$w_n(r) = \mu_n^\beta u_n(\mu_n r)$$

where

$$\begin{cases} \mu_n &= u_n(0)^{-(\alpha-p)/p} \\ \beta &= \dfrac{p}{\alpha - p} > 0 \end{cases}$$

Obviously, $\lim_{n\to\infty} \mu_n = 0$.

Such a w_n satisfies

$$\begin{cases} \mathcal{A}_p w_n &= r^{N-1}\left(\lambda_n \mu_n^{\beta(p-q)+p} w_n^{q-1} + w_n^{\alpha-1}\right), \quad 0 < r < \mu_n^{-1}, \\ w_n(0) &= 1, \\ w_n(\mu_n^{-1}) &= 0. \end{cases}$$

Using Lemma 2.5.2, let R_1 be such that $v_1(R_1) = 0$ (hence $v_1(0) = 1$) and choose $\tilde{R} > R_1$. From the results of [44] and [87], it follows that w_n are uniformly bounded in $C^{1,\nu}(0, \tilde{R})$. Then one has that

$$w_n \to v$$

in $C^1(0, \tilde{R})$, up to a subsequence. The limit function v is a solution of (2.8) with $\lambda = 0$, $v(0) = 1$, such that $v(r) > 0$ in $0 < r < \tilde{R}$. This is a contradiction with the choice of $\tilde{R}$. ■

Remark 2.5.7 *The extension of the result in Lemma (2.5.6) for solutions of (2.8) on general domains and $p \neq 2$ is an open problem.*

We can state the main result in this section.

Theorem 2.5.8 *For all $\lambda \in (0, \Lambda)$ the problem (2.8) admits at least two positive solutions*

Proof. Fixed $\lambda_0 \in (0, \Lambda)$ we take $\mu \in (\lambda_0, \Lambda)$. Let $\overline{u}$ be a positive solution of (2.8) for $\lambda = \mu$. Now $\overline{u}$ is a strict supersolution to the problem (2.8) for $\lambda = \lambda_0$, because $\lambda_0 < \mu$. It is easy to check that $\underline{u} = \varepsilon\phi_1$ with $\varepsilon > 0$ small enough and $\phi_1 > 0$ a positive eigenfunction, verifies $\underline{u} < \overline{u}$ and is a strict subsolution of the problem (2.8) for $\lambda = \lambda_0$.

Let us set

$$X = \{\, v \in C^1(\Omega) \,:\, v(x) = v(|x|),\, v(1) = 0\}$$

and, for $\lambda \in (0, \Lambda)$, consider the map $\mathcal{K}_\lambda$ defined on X by setting:

$$\mathcal{K}_\lambda(v) = (\mathcal{A}_p)^{-1}(f_\lambda(v)).$$

It is well known (see for example [47]) that $\mathcal{K}_\lambda$ maps X into itself. Moreover

$$\underline{u} < \mathcal{K}_\lambda(\underline{u}),\, \overline{u} > \mathcal{K}_\lambda(\overline{u}).$$

So, if we define

$$\mathcal{X} = \{v \in X \,|\, \underline{u} \leq v \leq \overline{u}\},$$

we find that

$$\mathcal{K}_{\lambda_0} : \mathcal{X} \longrightarrow \mathcal{X}.$$

By the $C^{1\alpha}$ estimates in [44] and [87] already refered in the beginning of this section, it follows that $\mathcal{K}_\lambda$ is compact. In particular,

$$\overline{\mathcal{K}_{\lambda_0}(\mathcal{X})} \subset \mathcal{X}$$

is a compact set in X. The Schauder fixed point theorem implies that there exists a $u_0 \in \mathcal{X}$ such that $\mathcal{K}_{\lambda_0}(u_0) = u_0$, or, in others words, (2.8) has at least a solution in $\mathcal{X}$ for $\lambda = \lambda_0$. If u_0 is not the unique fixed point in $\mathcal{X}$ we have nothing to prove. Otherwise, u_0 can be obtained by iterations and then $\underline{u} < u_0 < \overline{u}$. So by the Hopf lemma we have that u_0 is in the interior of $\mathcal{X}$ (in the C^1 topology). Hence there exits $\varepsilon > 0$ such that $u_0 + \varepsilon\mathcal{B}_1 \subset \mathcal{X}$. Here $\mathcal{B}_1$ denote the unit ball in X.

To complete the proof we use topological degree arguments developed in [5]. In fact we have

$$\begin{aligned} &deg(I - \mathcal{K}_{\lambda_0}, u_0 + \varepsilon\mathcal{B}_1, 0) = \\ &i(\mathcal{K}_{\lambda_0}, u_0 + \varepsilon\mathcal{B}_1, E) = i(\mathcal{K}_{\lambda_0}, \mathcal{X}, \mathcal{X}) = 1, \end{aligned} \tag{2.10}$$

where we use the permanence and excision properties of the degree. (See [5] Chapter 3, specially the proof of the Schauder fixed point theorem for the last equality).

On the other hand, by Lemma 2.5.5, we know that for $\lambda > \Lambda$ the problem (2.8) has no positive solutions. By Lemma 2.5.6 we know that for all $\lambda \in [0, \Lambda]$, (2.8) has no positive solution such that

$$||u||_\infty \geq \rho > C.$$

So by the homotopy invariance of the Leray-Schauder degree we get

$$deg(I - \mathcal{K}_{\lambda_0}, \rho\mathcal{B}_1, 0) = deg(I - \mathcal{K}_{\Lambda+\delta}, \rho\mathcal{B}_1, 0) = 0.$$

Then by the excision property and (2.10) we infer

$$\begin{aligned} &deg(I - \mathcal{K}_{\lambda_0}, \rho\mathcal{B}_1 \setminus \{u_0 + \varepsilon\mathcal{B}_1\}, 0) = \\ &deg(I - \mathcal{K}_{\lambda_0}, \rho\mathcal{B}_1, 0) - deg(I - \mathcal{K}_{\lambda_0}, u_0 + \varepsilon\mathcal{B}_1, 0) = -1 \end{aligned} \tag{2.11}$$

Hence, $\mathcal{K}_{\lambda_0}$ has another fixed point $u_1 \in \rho\mathcal{B}_1 \setminus \{u_0 + \varepsilon\mathcal{B}_1\}$. ■

Remark 2.5.9 *The fact that Ω is a ball is used only to prove the uniform bounds of Lemma 2.5.6. If $p = 2$, such a priori estimate holds true for any bounded domain Ω, see [56]. As a consequence, the proof of Theorem 2.5.8 gives an alternative method to show the subcritical result in [11]. Actually, the minimal solution in [11] is a local minimum and has index $= 1$, while the solution found in [11] as mountain pass critical point has index $= -1$.*

Remark 2.5.10 *By using the a priori estimates of [56] and the degree theoretic results of [38], one can prove the preceding bifurcation result for the second order problem*

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + G'(u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.12}$$

where G is possibly not symmetric and Ω a bounded domain.

2.6 Multiple bifurcation

We will look for solutions of the following problem

$$\begin{cases} A_p u = \lambda h(u) + g(r,u), & 0 < r < 1 \\ u'(0) = 0, & \\ u(1) = 0, & \end{cases} \tag{2.13}$$

where $p > 1$,

$$A_p u = -\frac{1}{r^{N-1}}\left(r^{N-1}|u'|^{p-2}u'\right)'$$

denotes the radial p-Laplace operator, $\lambda \geq 0$ and $h \in C(\mathbb{R})$ and $g \in C(\mathbb{R}^+ \times \mathbb{R})$ satisfy the following conditions:

(H_1) $\lim_{s\to 0} \dfrac{h(s)}{|s|^{q-2}s} = 1$, with $1 < q < p$.

(H_2) $\lim_{s\to 0} \dfrac{g(r,s)}{|s|^{p-2}s} = 0$.

Hereafter all the properties of g are assumed to hold uniformly with respect to $r \in [0,1]$.
Let

$$X = \{u \in C^1(B_1) \,:\, u(x) = u(|x|),\ u(1) = 0\}$$

and set $E = \mathbb{R} \times X$. ($u \in X$ is regular radial function and then $u'(0) = 0$). In the sequel by a solution of (2.13) we mean a pair $(\lambda, u) \in E$, such that $u \in \mathcal{C}^{1,\alpha}$ and satisfies (2.13) in a weak sense. Our main bifurcation result is:

Theorem 2.6.1 *The point $(\lambda, u) = (0,0)$ is a bifurcation point for problem (2.13). More precisely, there are infinitely many unbounded continua (i.e. closed connected set) $\Gamma_k \subset E$ of solutions of (2.13) branching off from $(0,0)$, such that*

1) If $(\lambda, u) \in \Gamma_k$ *and* $\lambda > 0$ *then* $u \neq 0$.

2) If $(\lambda, u) \in \Gamma_k$, *then* u *has exactly* $k-1$ *simple zeros in the interval* $(0,1)$.

3) There exists a constant $\rho_o > 0$ *such that if* $\rho \in (0, \rho_o]$, *and* $(\lambda, u) \in \Gamma_k$ *with* $\|u\|_\infty = \rho$, *then* $\lambda > \lambda(\rho) > 0$.

As an inmediate consequence we get:

Corollary 2.6.2 *There exists* $\Lambda > 0$ *such that for all* $\lambda \in (0, \Lambda)$ *problem (2.13) has infinitely many solutions.*

Some of the preceding results hold in a greater generality, see Remark 2.6.7 below.

Theorem 2.6.1 cannot be proved using standard bifurcation techniques by linearization. Actually, even when $p = 2$ and the differential operator becomes the linear Laplacian Δ, the nonlinear term h has *infinite* derivative at $u = 0$. To overcome this problem we shall employ a limiting procedure. Define

$$h_\delta(s) = \begin{cases} \dfrac{h(\delta)}{\delta^{p-1}} |s|^{p-2} s & \text{if } |s| \leq \delta, \\ \\ h(s) & \text{if } |s| \geq \delta. \end{cases}$$

and consider the approximated problems

$$\left\{ \begin{array}{rcl} A_p u & = & \lambda h_\delta(u) + g(r,u), \quad 0 < r < 1, \\ u'(0) & = & 0, \\ u(1) & = & 0. \end{array} \right. \tag{2.14}$$

In order to study (2.14) we use the results from [43] studied in Sections 1.5 and 1.6 concerning to bifurcation from nonlinear eigenvalues of the problem

$$\left\{ \begin{array}{rcl} A_p u & = & \lambda |u|^{p-2} u, \quad 0 < r < 1 \\ u'(0) & = & 0, \\ u(1) & = & 0. \end{array} \right. \tag{2.15}$$

Hence, we have the following result.

Theorem 2.6.3 *For each positive integer* k *there exists an unbounded continuum* $S_k \subset E$ *of solutions* (λ, u) *of (2.14) branching off from* $(\mu_{k,\delta}, 0)$, *where,*

$$\mu_{k,\delta} = \frac{\delta^{p-1}}{h(\delta)} \cdot \mu_k,$$

and μ_k *is an eigenvalue of problem (2.15). Moreover if* $(\lambda, u) \in S_k$ *and* $u \neq 0$ *then* u *has exactly* $k-1$ *zeros in* $(0,1)$.

Remark 2.6.4 *The arguments in Section 1.6 permit to handle problem (2.14). More precisely, setting*

$$\mu_{k,\delta} = \frac{\delta^{p-1}}{h(\delta)} \cdot \mu_k,$$

one finds that from each $(\mu_{k,\delta}, 0)$ *branches off an unbounded continuum* $S_{k,\delta}$ *of solutions of (2.14) having exactly* $k-1$ *zeros in* $(0,1)$.

For $R > 0$ let T_R denote the ball of radius R in E. For each fixed k and R let $\Sigma_{k,\delta}$ denote the connected component of $S_{k,\delta} \cap T_R$ that contains $(\mu_{k,\delta}, 0)$. Let us point out that $\Sigma_{k,\delta}$ is nonempty because $S_{k,\delta}$ is unbounded.

We will use the following topological theoretical lemma from [92]:

Lemma 2.6.5 *Let* $\{\Sigma_n\}_{n \in I\!N}$ *be a sequence of connected sets in a complete metric space* E. *Assume that*

i) $\cup \Sigma_n$ *is precompact in* E.

ii) $\liminf \Sigma_n \neq \emptyset$.

Then $\limsup \Sigma_n$ *is not empty closed and connected.*

Here $\liminf \Sigma_n$, respectively $\limsup \Sigma_n$, denotes the set of all $x \in E$ such that any neighbourhood of x meets all but finitely many of Σ_n, respectively infinitely many Σ_n.

Take a sequence $\delta_n \to 0$ and let $\Sigma_{k,n} = \Sigma_{k,\delta_n}$. By classical a priori estimates the set $\cup \Sigma_{k,n}$ is precompact. Moreover, since $\mu_{k,\delta_n} \to 0$ as $\delta_n \to 0$, then $(0,0) \in \liminf \Sigma_{k,n}$. Then Lemma 2.6.5 applies to $\Sigma_n = \Sigma_{k,n}$ and hence $\Gamma_{k,R} := \limsup(\Sigma_{k,n}) = \limsup(S_{k,\delta_n} \cap T_R)$ is a not empty closed and connected set. Moreover it is clear that $\Gamma_{k,R}$ meets T_R for all $R > 0$.

We set $\Gamma_k = \cup_{R>0} \Gamma_{k,R}$ and show in the rest of the section that the Γ_k satisfy the properties stated in Theorem 2.6.1.

First of all, it follows directly by the preceding arguments that each Γ_k is an unbounded continuum in E and $(0,0) \in \Gamma_k$. It is also clear that any $(\lambda, u) \in \Gamma_k$ is a solution of (2.13). Next we give the proofs of Properties $1-2$ of Theorem 2.6.1.

Proof of 2.6.1-1). Let δ_0 be such that $\lambda h(\delta_0) \delta_0^{1-p} > \mu_1$ and consider $(\lambda, u) \in \Sigma_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$. Fixed $\epsilon > 0$ small, by assumptions $(H_1 - H_2)$ there exists $c = c(\lambda) > 0$ such that

$$\lambda h_\delta(s) + g(r,s) > (\mu_1 + \epsilon) s^{p-1}, \quad \forall\, s \in (0, c].$$

Hence, if $\|u\|_\infty \le c$, u satisfies

$$A_p u > (\mu_1 + \varepsilon) |u|^{p-1} u$$

and the usual iterative method yields the existence of a positive solution $u \in X$ of the eigenvalue problem

$$A_p u = (\mu_1 + \varepsilon)|u|^{p-1}u,$$

a contradiction, because $\mu_1+\varepsilon$ is not the first eigenvalue of (2.15). This shows that if $(\lambda, u) \in \Sigma_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$, then $\|u\|_\infty > c(\lambda)$. Passing to the limit as $\delta \to 0$ it follows that if $(\lambda, u) \in \Gamma_1$ then $\|u\|_\infty \geq c(\lambda)$.

When we consider Γ_k with $k > 1$ the argument is similar. If $(\lambda, u) \in \Sigma_{k,\delta}$ then there exists at least one interval I_k with length $1/k$ where u has constant sign. Therefore if we restrict ourselves to the interval I_k and we replace μ_1 by the first eigenvalue of (2.15) on the interval I_k, then we get the same contradiction as before.

Proof of 2.6.1-2). By 1) above, we know that if $(\lambda, u) \in \Gamma_k$ and $\lambda > 0$ then $u \neq 0$. Let u_n be such that $(\lambda_n, u_n) \in \Sigma_{k,n}$, with $\lambda_n \to \lambda$ and $u_n \to u$. First, let $k = 1$. Then $u_n > 0$ in $[0, 1)$. Hence $u \geq 0$ and the strong Maximum Principle implies that $u > 0$ in $[0, 1)$.

Next, take $k > 1$ and let $x_n < y_n$, $0 < x_n < y_n < 1$, be two consecutive zeros of u_n with $x_n \to \xi$ and $y_n \to \eta$. Obviously, $u(\xi) = u(\eta) = 0$. We claim that $\xi \neq \eta$. Otherwise, there exists a third sequence z_n such that $u_n'(z_n) = 0$ and $\lim z_n = \xi$. By the *Ascoli-Arzelà* theorem we find that u is a solution of

$$A_p u = \lambda h(u) + g(r, u),$$

such that $u(\xi) = u'(\xi) = 0$. If $\xi > 0$ this is in contradiction with the Hopf Lemma applied to the ball $|r| \leq \xi$. If $\xi = 0$ the contradiction arises from the strong maximum principle applied to a ball $|r| \leq \varepsilon$, where $u(r) \geq 0$. This shows that u has at least $k - 1$ zeros in $(0, 1)$. On the other side, if we suppose, as before, that, say, $u_n > 0$ in the open interval (x_n, y_n), then $u > 0$ in (ξ, η) and therefore u cannot have less than $k - 1$ zeros in $(0, 1)$, proving 2). ■

Before proving Property 3 we state the following Lemma

Lemma 2.6.6 *There exists $\rho_0 > 0$ such that any nontrivial solution u of*

$$\left\{ \begin{array}{rcl} A_p u &=& g(r, u), \quad 0 < r < 1 \\ u'(0) &=& 0, \\ u(1) &=& 0. \end{array} \right. \tag{2.16}$$

satisfies $||u||_\infty > \rho_o$.

Proof. If not, we can find a sequence $u_n \neq 0$ of solutions of (2.16) such that $\|u_n\|_\infty \to 0$. Setting $v_n = u_n ||u_n||_\infty^{-1}$ one finds that

$$A_p v_n = \frac{g(r, u_n)}{\|u_n\|_\infty^{p-1}}. \tag{2.17}$$

Using the compactness of A_p^{-1} we infer that, up to a subsequence, $v_n \to w$ in $C^{1,\nu}$, with $w \neq 0$. On the other side, from (2.17) it follows that

$$\int_0^1 |v_n'|^p r^{N-1} dr = \int_0^1 \frac{g(r,u_n)}{|u_n|^{p-2}u_n} v_n^p r^{N-1} dr.$$

According to (H_2) the last integral tends to 0, a contradiction. ∎

Proof of 2.6.1-3).

Arguing by contradiction, we suppose there exists a sequence $(\lambda_n, u_n) \in \Sigma_{k,n}$ such that $\lambda_n \to 0$, $u_n \to u$ and $||u_n||_\infty = \rho \le \rho_o$. Passing to the limit we find that $u \neq 0$ is a solution of (2.16) such that $\|u\|_\infty \le \rho_0$, in contradiction with Lemma 2.6.6. This completes the Proof of Theorem 2.6.1. ∎

Remark 2.6.7 (i) *The preceding arguments show that the existence of the continuum of positive solutions Γ_1 can be established for a problem with the same type of zero order term on a general bounded domain $\Omega \subset I\!R^N$.*

(ii) *The preceding bifurcation and multiplicity results hold true also for Sturm-Liouville equations*

$$-(a(x)u')' + b(x)u = \lambda h(u) + g(x,u),$$

with separated boundary conditions, provided $h(u) \simeq |u|^{q-2}u$, $q < 2$, at $u = 0$, and g is of higher order.

2.7 Further results

We can give in some cases the global behaviour of the branches obtained in the Section 2.6. More precisely we have the following results.

a) Subdiffusive nonlinearities.

Let us suppose that $h(s) = |s|^{q-2}s$ and $g(r,s)/|s|^{q-2}s \to \gamma < 0$ (possibly $-\infty$) as $|s| \to \infty$, uniformly in $r \in [0,1]$. In particular there exists $s_0 > 0$ such that $\lambda h(s) + g(r,s) < 0$ for any s such that $|s| > s_0$ and any $\lambda \ge 0$.

We claim that if (λ, u) is any solution of (2.13) with $\lambda \ge 0$ then $\|u\|_\infty \le s_0$. To prove this fact let $r_0 \in [0,1)$, respectively $r_1 \in (0,1]$, be the point where u attaints its maximum, respectively minimum. From the equation it readily follows that $\lambda h(u(r_0)) + g(r_0, u(r_0)) \ge 0$, respectively $\lambda h(u(r_1)) + g(r_1, u(r_1)) \le 0$. Therefore $u(r_0) \le s_0$, respectively $u(r_1) \ge -s_0$, as claimed. In addition if , for example, $g(r,s)s < 0$ for all $s \neq 0$ then $\Gamma_k \subset I\!R^+ \times \{u \in X : \|u\|_\infty \le s_0\}$ for all k and hence (2.13) has has infinitely many solutions for all $\lambda > 0$.

b) Equidiffusive nonlinearities. Consider the eigenvalue problem

$$\begin{cases} A_p u &= \lambda h(u), \quad 0 < r < 1 \\ u'(0) &= 0, \\ u(1) &= 0. \end{cases} \tag{2.18}$$

where h satisfies (H_1), $h > 0$ on $I\!R^+$ and

$$\lim_{s \to +\infty} \frac{h(s)}{s^{p-1}} = m_\infty > 0.$$

In such a case we can apply the results of [7], Section 4, to infer that Γ_1 blows up at infinity as $\lambda \to \lambda_\infty = \mu_1 \cdot m_\infty^{-1}$. Similarly, if $\dfrac{h(s)}{|s|^{p-2}s} \to m_\infty$ as $|s| \to \infty$, then one could also show that Γ_k blows up at infinity as $\lambda \to \mu_k \cdot m_\infty^{-1}$. Moreover, if $m_\infty = 0$ then, similarly as in the preceding point a), (2.18) has infinitely many solutions for all $\lambda > 0$.

c) Subcritical problems. Consider problem (2.13) and assume, in addition to $(H_1 - H_2)$, that g satisfies

(H_3) $g(r,s)s > 0 \quad \forall s \neq 0$;

(H_4) $g(r,s) \simeq |s|^{\alpha-2}s$ as $|s| \to \infty$, with $1 < q < p$ and $p < \alpha < p^*$, where $p^* = pN/(N-p)$ if $p < N$ and $p^* = +\infty$ if $p \geq N$.

In particular, the growth of g is said *subcritical*, in the sense of the Sobolev embedding. In the sequel, for the sake of simplicity, we will take $h(s) = |s|^{q-2}s$. We will also consider only solutions u such that $u(0) > 0$.

The main result in this section is :

Theorem 2.7.1 *Suppose that $(H_1 - H_2 - H_3 - H_4)$ hold. Then, for any $k \geq 1$ there exist $\Lambda_k > 0$ and $C_k > 0$ such that for every $(\lambda, u) \in \Gamma_k$ with $\lambda \geq 0$ one has*

$$\lambda \leq \Lambda_k, \quad \|u\|_\infty \leq C_k.$$

In particular for all $\lambda \geq 0$ small, problem (2.13) has infinitely many pairs *u_k, v_k of solutions with $k-1$ zeros.*

Remark 2.7.2 *The continua emanating from $(0,0)$ turn back and cross the axis $\{\lambda = 0\}$ at a solution of (2.13) for $\lambda = 0$ with $k-1$ zeros. In general, $C_k \to +\infty$ as $k \to +\infty$. We also suspect that $\Lambda_k \to +\infty$. If this is true, then (2.13) has infinitely many pairs of solutions for any $\lambda \geq 0$.*

The details can be found in [8] where it is described also the behaviour of the branch of positive solutions Γ_1^+ in the critical case. More precisely if

$$\left\{\begin{array}{rcl} A_p u &=& \lambda u^{q-1} + u^{\alpha-1},\ u(r) > 0, \text{ in } 0 < r < 1 \\ u'(0) &=& 0, \\ u(1) &=& 0. \end{array}\right. \tag{2.19}$$

we have

Theorem 2.7.3 *Let $p < N$ and suppose $1 < q < p < \alpha \leq p^*$ and that if $\alpha = p^*$ then either $p = 2$ or*

$$\frac{2N}{N+2} < p < 3 \quad \textit{or} \quad p \geq 3 \textit{ and } p > q > p^* - \frac{2}{p-1}. \tag{2.20}$$

holds.

(i) If $\alpha < p^*$ then Γ_1^+ is bounded.

(ii) If $\alpha = p^*$ then $\Gamma_1^+ \setminus \{0,0\} \subset]0, \Lambda_1^*] \times X$ and blows up at infinity as $\lambda \to 0^+$, in the sense sepecified above.

(iii) Eq. (2.19) has at least two (positive) solutions for all $0 < \lambda < \Lambda_1^*$.

(iv) Eq. (2.19) has one (positive) solution for all $0 \leq \lambda \leq \Lambda_1^*$, resp. $0 < \lambda \leq \Lambda_1^*$, when $\alpha < p^*$, resp. $\alpha = p^*$.

(v) Eq. (2.19) has no (positive) solution for $\lambda > \Lambda_1^*$.

The same problem in the supercritical case is considered in [3], where the existence of an unbounded radial solution is proved, and the behaviour of Γ_1^+ is discused in terms of the dimension.

Chapter 3

What means Growth?

3.1 Introduction

In this Chapter we analyze problems related to a critical potential, more precisely, problems related to the operator

$$\mathcal{L}_\lambda u \equiv -\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u.$$

We will study equations of the form $\mathcal{L}_\lambda u = F(x,u)$ under suitable hypotheses of regularity and growth for F.

We will see that if for instance $F(x,u) = u^{p^*-1}$, then the Dirichlet Problem in any bounded starshaped domain containing the origin, has no solution. This means that the term $-\dfrac{\lambda}{|x|^p}|u|^{p-2}u$ does not regularize as in the autonomous case, previously studied in Chapter 2.

We can also look to this kind of result from the point of view of the eigenvalue problem in a extreme case, namely, in the case where the potential $-\dfrac{\lambda}{|x|^p}$ belongs to L^r if and only if $r < \dfrac{N}{p}$, which is the complementary range of the classical results. (See for instance the classical book [63].)

More details and applications of this kind of ideas can be found in [55]. Roughly speaking all the new problems appear from the lack of compactness in the Hardy inequality that we will study in the next section.

3.2 Hardy inequality

The main point of this section is to discuss the following classical result, esentially dues to Hardy. (See [59]). By completeness we include the proof.

Lemma 3.2.1 *Assume $1 < p < N$, then if $u \in W^{1,p}(I\!\!R^N)$.*

1. $\dfrac{u}{|x|} \in L^p(I\!\!R^N)$.
2. *(Hardy Inequality)*
$$\int_{I\!\!R^N} \frac{|u|^p}{|x|^p} dx \le C_{N,p} \int_{I\!\!R^N} |\nabla u|^p dx$$
with $C_{N,p} = (\dfrac{p}{N-p})^p$.
3. *The constant $C_{N,p}$ is optimal.*

Proof.

Step 1. A density argument allows us to consider only smooth functions $u \in C_0^\infty(I\!\!R^N)$. Under this hypothesis we have the following identity

$$|u(x)|^p = -\int_1^\infty \frac{d}{d\lambda}|u(\lambda x)|^p d\lambda = -p\int_1^\infty u^{p-1}(\lambda x)\langle x, \nabla u(\lambda x)\rangle d\lambda \,.$$

By using Hölder inequality, it follows that

$$\begin{aligned}
&\int_{I\!\!R^N} \frac{|u(x)|^p}{|x|^p} dx = -p\int_1^\infty \int_{I\!\!R^N} \frac{u^{p-1}(\lambda x)}{|x|^{p-1}}\langle \frac{x}{|x|}, \nabla u(\lambda x)\rangle dx d\lambda = \\
&-p\int_1^\infty \frac{d\lambda}{\lambda^{N-1-p}} \int_{I\!\!R^N} \frac{u(y)^{p-1}}{|y|^{p-1}}\frac{\partial u(y)}{\partial r} dy = -\frac{p}{N-p}\int_{I\!\!R^N} \frac{u(y)^{p-1}}{|y|^{p-1}}\frac{\partial u(y)}{\partial r} dy \le \\
&\frac{p}{N-p}\Big(\int_{I\!\!R^N} \frac{|u(y)|^p}{|y|^p} dy\Big)^{(p-1)/p}\Big(\int_{I\!\!R^N} |\frac{\partial u(y)}{\partial r}|^p dy\Big)^{1/p}.
\end{aligned}$$

And then we conclude that

$$\int_{I\!\!R^N} \frac{u^p(x)}{|x|^p} dx \le (\frac{p}{N-p})^p \int_{I\!\!R^N} |\nabla u(x)|^p dx \,.$$

Step 2. Optimality of the constant . Following the idea of Hardy for the one dimensional case, we show that the best constant is $C_{N,p} = (\frac{p}{N-p})^p$.

Given $\varepsilon > 0$, take the radial function

$$U(r) = \begin{cases} A_{N,p,\varepsilon} & \text{if} \quad r \in [0,1], \\ A_{N,p,\varepsilon} r^{\frac{p-N}{p}-\varepsilon} & \text{if} \quad r > 1, \end{cases} \tag{3.1}$$

where $A_{N,p,\varepsilon} = p/(N - p + p\varepsilon)$, whose derivative is

$$U'(r) = \begin{cases} 0, & \text{if} \quad r \in [0,1], \\ -r^{-\frac{N}{p}-\varepsilon} & \text{if} \quad r > 1. \end{cases} \tag{3.2}$$

By direct computation we get

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{U^p(x)}{|x|^p} dx &= \int_B \frac{U^p(x)}{|x|^p} dx + \int_{\mathbb{R}^N - B} \frac{U^p(x)}{|x|^p} dx = \\ &= A^p_{N,p,\varepsilon} \omega_N \left(\int_0^1 r^{N-1-p} dr + \int_1^\infty r^{-(1+p\varepsilon)} dr \right) = \\ &= A^p_{N,p,\varepsilon} \omega_N \int_0^1 r^{N-1-p} dr + A^p_{N,p,\varepsilon} \int_{\mathbb{R}^N} |\nabla U(x)|^p dx, \end{aligned}$$

where ω_N is the measure of the $(N-1)$-dimensional unit sphere. We conclude by letting $\varepsilon \to 0$.
■

Corollary 3.2.2 *The same result is true in $W^{1,p}(B)$, where B is the unit ball in $\mathbb{R}^N$.*

Proof. The proof of the first step is the same. The argument of optimality in the case of the unit ball proceeds by approximation as follows. First, we remark that by the invariance under dilations the optimal constant has to be the same for any ball. Second, let B_R be a ball with large radius. We take as test function $v(x) = \psi(x)U(x)$ where U is one of the approximate optimizers explicitly given above and $\psi \in C_0^\infty(B_R)$ is a cutoff function which is identically 1 on B_{R-1} with $|\nabla\psi| \leq m$. It is easily seen that for $R \gg 1$ the influence of ψ in the calculation of Step 2 is negligible. ■

Remark 3.2.3 *Sometimes the Hardy inequality in the case $p = 2$ is known as uncertainty principle, see [46]. We can read the Hardy inequality saying that the embedding of $W^{1,p}(\mathbb{R}^N)$ in L^p with respect to the weight $|x|^{-p}$ is continuous. It is very easy, working a little bit more with the minimizers that we use in the proof, to see that the inclusion is non compact. This will be the cause of many of our difficulties.*

In the sequel, we will denote $\lambda_{N,p} = C^{-1}_{N,p}$.

It will be useful to compare the best constant in the Hardy inequality with the following approximating eigenvalue problems.

Theorem 3.2.4 *Consider $\lambda_1(n)$ the first eigenvalue to the problem*

$$\begin{cases} -\Delta_p \psi_1 = \lambda W_n(x) |\psi_1|^{p-2} \psi_1, \; x \in \Omega \subset \mathbb{R}^N, \\ \psi_1(x) = 0, \; x \in \partial\Omega. \end{cases} \tag{3.3}$$

where $W_n(x) = \min\{|x|^{-p}, n\}$.

Then $\lambda_1(n) \geq \lambda_{N,p}$, *and moreover* $\lim\limits_{n\to\infty} \lambda_1(n) = \lambda_{N,p}$.

Proof. The first inequality follows immediatly from the definition of the first eigenvalue by the Rayleigh quotient. Also, it is easy to see that $\{\lambda_1(n)\}$ is a nonincreasing sequence; then we have to prove that the limit cannot be bigger than $\lambda_{N,p}$. Assume by contradiction that $\lim\limits_{n\to\infty} \lambda_1(n) = \lambda_{N,p} + \rho$.

Then, we can choose $\phi \in W_0^{1,p}(\Omega)$ such that

$$\frac{\int\limits_\Omega |\nabla\phi|^p dx}{\int\limits_\Omega \phi^p |x|^{-p} dx} < \lambda_{N,p} + \rho/2.$$

But then $\lambda_1(n) \leq \dfrac{\int\limits_\Omega |\nabla\phi|^p dx}{\int\limits_\Omega \phi^p W_n(x) dx}$, and this is a contradiction because the last expression has to be smaller than $\lambda_{N,p} + \rho$ for n large. ■

3.3 The Dirichlet problem with singular potential

The first result in this section is an easy consequence of the Hardy inequality

Lemma 3.3.1 *Consider the nonlinear operator*

$$\mathcal{L}_\lambda u \equiv -\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u \tag{3.4}$$

in $W_0^{1,p}(\Omega)$. *Then*

1. *If* $\lambda \leq \lambda_{N,p}$, $\mathcal{L}_\lambda$ *is a positive operator.*

2. *If* $\lambda > \lambda_{N,p}$, $\mathcal{L}_\lambda$ *is unbounded from below.*

Proof. 1) It is obvious from the Hardy inequality . 2) An easy consequence of the optimality of the constant and a density argument is the existence of $\phi \in \mathcal{C}_0^\infty(\Omega)$ such that $\langle \mathcal{L}_\lambda \phi, \phi\rangle < 0$. We can assume that $||\phi||_p = 1$ and then by defining $u_\mu(x) = \mu^{N/p}\phi(\mu x)$ we have $||u_\mu||_p = 1$ and the homogeneity of the operator allows us to conclude that $\langle \mathcal{L}_\lambda u_\mu, u_\mu\rangle = \mu^p \langle \mathcal{L}_\lambda \phi, \phi\rangle < 0$. ■

Taking into account the previous result, in this section we will study the following problem

$$\left\{\begin{array}{rcl} \mathcal{L}_\lambda u & = & f(x) \in W^{-1,p'}(\Omega),\ x \in \Omega,\ \lambda < \lambda_{N,p},\ \dfrac{1}{p} + \dfrac{1}{p'} = 1 \\ u(x) & = & 0,\ x \in \partial\Omega, \end{array}\right. \tag{3.5}$$

where $\Omega \subset I\!R^N$ is a bounded domain. If $0 \notin \Omega$ then we have a classical problem with a bounded potential. So we will assume hereafter that $0 \in \Omega$.

Consider the energy functional,

$$J(u) = \int_\Omega F(x, u, \nabla u) dx$$

where $F(x, u, \xi) = \dfrac{1}{p}|\xi|^p - \dfrac{\lambda}{p}\dfrac{u^p}{|x|^p} - f(x)u$.

The classical results in the Calculus of Variations characterize the weak lower semicontinuity of J if $F(x, u, \bullet)$ is convex and F verifies a lower bound: positivity, or a lower estimate by a linear combination of ξ, etc. (See Tonelli [86], Serrin [78], De Giorgi [37], the book by Dacorogna [34] and the references therein). However, in our case these usual hypotheses are not fulfilled.

Variational approach.- The energy functional,

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \frac{u^p}{|x|^p} dx - \int_\Omega f u dx,$$

by the Hardy inequality, is continuous, Gateaux differentiable and coercive, namely, there exist constants $\gamma > 0$ and $c \in I\!R$ such that

$$J(u) \geq \gamma \int_\Omega |\nabla u|^p dx - c.$$

Hence by the *Variational Principle of Ekeland* we can find a sequence $\{u_n\}_{n\in I\!N}$ such that

$$J(u_n) \to \inf J, \text{ and } J'(u_n) \to 0, \text{ as } n \to \infty.$$

As usually we say that $\{u_n\}_{n\in I\!N}$ is a *Palais-Smale sequence*. (See [45] and Appendix). The coercivity of J implies the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$, so we have that for some subsequence:

i) $\nabla u_n \rightharpoonup \nabla u$ in L^p,

ii) u_n converges in L^p and a.e.,

iii) $\lambda \dfrac{u_n^{p-1}}{|x|^p}$ are bounded as Radon measures and converges weakly in L^1.

Under these hypotheses we can apply the convergence theorem by Boccardo and Murat in [25]. The results proved by Boccardo and Murat are much more general and we refer to the paper [25] for the details.

Lemma 3.3.2 *Let $\{u_n\}_{n\in\mathbb{N}}$ verifying the problems*

$$\begin{cases} -\Delta_p u_n = g_n + f_n, & in\ \ \Omega \\ u_n \in W_0^{1,p}(\Omega). \end{cases} \tag{3.6}$$

Assume that

1) $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega)$ *and* $u_n \to u$ *in* $L^p(\Omega)$.

2) $f_n \to f$ *in* $W^{-1,p'}(\Omega)$.

3) $g_n \rightharpoonup g$ *weak-$*$ in the sense of the measures.*

Then, $\nabla u_n \to \nu$ *in* $(L^q(\Omega))^N$ *for* $1 < q < p$.

Moreover if 3) *is replaced by*

3′) $g_n \rightharpoonup g$ *weak-$*$ in* $L^1(\Omega)$,

then

$$\mathcal{T}_k(u_n) \to \mathcal{T}_k(u) \quad in \quad W_0^{1,p}(\Omega)$$

for all $k > 0$, *where* $\mathcal{T}_k(s) = s$ *if* $|s| \leq k$ *and* $\mathcal{T}_k(s) = ks/|s|$ *if* $|s| \geq k$.

Proof. We take as test function $\mathcal{T}_k(u_n - u) \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\int_\Omega \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla \mathcal{T}_k(u_n - u)\rangle dx = \\ &-\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla \mathcal{T}_k(u_n - u)\rangle dx + \langle g_n, \mathcal{T}_k(u_n - u)\rangle + \langle f_n, \mathcal{T}_k(u_n - u)\rangle \end{aligned} \tag{3.7}$$

Fixed k, by 1) we have that

$$\langle f_n, \mathcal{T}_k(u_n - u)\rangle \to 0$$

and

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla \mathcal{T}_k(u_n - u)\rangle dx \to 0$$

as $n \to \infty$.

From 2) we have $|\langle g_n, \mathcal{T}_k(u_n - u)\rangle| \leq Ck$.

Then fixed k we get

$$\limsup_{n\to\infty} \int_\Omega \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla \mathcal{T}_k(u_n - u)\rangle dx \leq Ck.$$

Now the inequalities in Appendix A imply that the functions

$$e_n(x) = \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla \mathcal{T}_k(u_n - u)\rangle$$

are non negative and uniformly bounded in $L^1(\Omega)$. Take $0 < \theta < 1$ and split Ω in

$$S_n^k = \{x \in \Omega |\, |u_n - u| \leq k\},\;\; G_n^k = \{x \in \Omega |\, |u_n - u| > k\}$$

Hölder inequality provides the following estimate

$$\begin{array}{l}\int\limits_\Omega e_n^\theta dx = \int_{S_n^k} e_n^\theta dx + \int_{G_n^k} e_n^\theta dx \leq \\ \left(\int_{S_n^k} e_n dx\right)^\theta |S_n^k|^{1-\theta} + \left(\int_{G_n^k} e_n dx\right)^\theta |G_n^k|^{1-\theta}.\end{array}$$

Now, fixed k, $|G_n^k| \to 0$ as $n \to \infty$ and from the uniform boundedness in L^1 we get

$$\limsup_{n\to\infty} \int\limits_\Omega e_n^\theta dx \leq (Ck)^\theta |\Omega|^{1-\theta},$$

and letting $k \to 0$ we get that $e_n^\theta \to 0$ strongly in L^1. To finish the first part we use the inequalities for the p-laplacian obtained in appendix A.1 and we conclude.

If we assume now $3')$, the idea is to get the same type of estimate but with $\theta = 1$. We take as test function $\mathcal{T}_k(u_n - u)$ and proceed in a similar way. We refer to the reader to [25] for the details. ■

In our context the Lemma 3.3.2 will be read as follows.

Lemma 3.3.3 *Let $\{u_n\}_{n\in\mathbb{N}}$ verifying (i), (ii) and (iii) by the Variational Principle of Ekeland. Then for some subsequence,*

$$u_{n_j} \to u, \quad in \quad W_0^{1,q}(\Omega),\; q < p$$

and

$$\mathcal{T}_k(u_n) \to \mathcal{T}_k(u) \quad in \quad W_0^{1,p}(\Omega)$$

for all $k > 0$, where $\mathcal{T}_k(s) = s$ if $|s| \leq k$ and $\mathcal{T}_k(s) = ks/|s|$ if $|s| \geq k$.

According with the previous Lemma, and by a density argument, we can prove the required compactness property. We will call such a compactness property *singular Palais-Smale condition* in the sense of the following Lemma:

Lemma 3.3.4 *Consider $\{u_n\}_{n\in\mathbb{N}}$ the Palais-Smale sequence obtained above.*

Then $\{u_n\}_{n\in\mathbb{N}}$ satisfies the singular Palais-Smale condition, namely, there exists a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ such that

$$u_{n_j} \to u, \quad in \quad W_0^{1,q}(\Omega),\; q < p.$$

An inmediate consequence of Lemma (3.3.4) is that u is a solution of our problem in the sense of distributions. Moreover by density and taking into account that $u \in W_0^{1,p}(\Omega)$, we conclude that u is solution in the sense of $W_0^{1,p}(\Omega)$.

Finally the homogeneity of the problem implies that u is a minimum for J. Consider

$$J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle = (\frac{1}{p} - 1)\int\limits_\Omega f u_k,$$

where in the last term we can pass to the limit by weak $W_0^{1,p}(\Omega)$-convergence. Therefore

$$\begin{aligned} &\inf J = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} (J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle) = \\ &(\frac{1}{p} - 1)\int\limits_\Omega f u = J(u) - \frac{1}{p}\langle J'(u), u\rangle = J(u). \end{aligned}$$

We would like to point out that this approach solves the minimization problem, namely, the solution is obtained as a minimum of J. Also it is interesting to emphasize that this approach will be used to study problems with unbounded energy functionals in the following sections.

Remark 3.3.5
I) The uniqueness in the case $p = 2$ is obvious. Also it is easy to prove that in the linear case ($p = 2$) the sequence of minimizers converges strongly in the Sobolev space.
II) If $p > 2$ the uniqueness is in general not true as the following argument shows (see [42]). Assume $B \subset \Omega$ a ball and consider $u_0 \in \mathcal{C}_0^2(\Omega)$ such that $u = k > 0$ in B. Consider $f(x) = \mathcal{L}_\lambda u$ with $\lambda < \lambda_{N,p}$. In this way $f \in W^{-1,p'}(\Omega)$ and then we can find a solution v by minimization of J as above. Now it is clear that such $v \neq u_0$, because u_0 cannot be a minimum for J: in fact, taking into account that $p > 2$, we see that

$$\langle J''(u_0)z, z\rangle = (p-1)\left(\int\limits_\Omega |\nabla u_0|^{p-2}|\nabla z|^2 dx - \lambda \int\limits_\Omega \frac{|u_0|^{p-2}}{|x|^p} z^2 dx\right).$$

Hence, in particular if $z \in \mathcal{C}_0^\infty(\Omega)$ is such that supp $(z) \subset B$,

$$\langle J''(u_0)z, z\rangle = -\lambda(p-1)\int\limits_\Omega \frac{|u_0|^{p-2}}{|x|^p} z^2 dx < 0,$$

and then u_0 is not a minimum for J.
III) The uniqueness in the case $1 < p < 2$ seems to be an open problem.
IV) The strong convergence of the Palais-Smale sequence if $p \neq 2$, is unknown. The main difficulty is classical, we start with a weakly convergent sequence and the dependence in the gradient is nonlinear.

3.4 Critical potential and subcritical growth

In this section we will try to study the same kind of problems as in Chapter 2, but including the critical potential $\dfrac{\lambda}{|x|^p}$.

We will use again the variational approach to study the case of unbounded funcionals, more precisely the existence of solution via the *Mountain Pass Lemma.* (See the Appendix). For instance the following result holds.

Theorem 3.4.1 *Consider the Dirichlet problem*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u|x|^{-p} + |u|^{\alpha-2}u, \quad \lambda < \lambda_{N,p}\,,\ p < \alpha < Np/(N-p), \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then there exists at least a positive solution $u \in W_0^{1,p}(\Omega)$.

Proof. The proof of this theorem follows closely the previous variational approach; instead of minimizing and using the variational principle of Ekeland, the geometry of the energy functional allows us to use the Mountain Pass Lemma of Ambrosetti-Rabinowitz. In fact since $\lambda < \lambda_{N,p}$ then

$$\begin{aligned} &J(u) \equiv \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^p}dx - \frac{1}{\alpha}\int_\Omega |u|^\alpha dx \geq \\ &\gamma \int_\Omega |\nabla u|^p dx - C(p,\alpha)(\int_\Omega |\nabla u|^p dx)^{q/p}. \end{aligned}$$

Then we find the required Palais Smale sequence, which is easy to show has to be bounded in $W_0^{1,p}(\Omega)$. Using again Lemma 3.3.3, we get the compactness result, in the sense of the *singular Palais Smale condition* defined above. Then we can find a function $u \in W_0^{1,p}(\Omega)$ which is the strong limit in $W_0^{1,q}(\Omega)$, $q < p$, and therefore is a solution in the sense of distributions. Since $u \in W_0^{1,p}(\Omega)$, by density we get that u is a weak solution.

Finally, by homogeneity, and taking into account that we have strong convergence in L^α, we can see that $u \not\equiv 0$:

$$\begin{aligned} &c = \lim_{n\to\infty} J(u_n) = \\ &\lim_{n\to\infty} (J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle) = \\ &\lim_{n\to\infty} (\frac{1}{p} - \frac{1}{\alpha})\int_\Omega |u_n|^\alpha dx = (\frac{1}{p} - \frac{1}{\alpha})\int_\Omega |u|^\alpha dx, \end{aligned}$$

and we conclude. ■

3.5 Critical potential and critical growth

We will use the method developed by Pohozaev to show how our problem with critical Sobolev exponent is not regularized by the term in $|x|^{-p}u^{p-1}$. This is a deep difference with the autonomous case studied in the Chapter 2. In this sense, the following nonexistence result shows, from a different point of view, the critical character of the problem.

Lemma 3.5.1 *Consider the problem*

$$\begin{cases} -\Delta_p u &= \lambda \dfrac{u^{p-1}}{|x|^p} + \gamma f(u), \quad x \in \Omega,\ \lambda > 0, \\ u(x) &= 0, \qquad\qquad\qquad\quad on \quad \partial\Omega, \end{cases} \tag{3.8}$$

where $\Omega \subset I\!R^N$ is bounded, starshaped with respect to the origin, f is a continuous function and

$$\gamma(NF(u) - \frac{N-p}{p} u f(u)) \le 0, \quad F(u) = \int_0^u f(s)ds.$$

Then (3.8) has no positive solution $u \in W_0^{1,p}(\Omega)$.

Proof. We will use a Pohozaev type identity. The idea consists on multiply the equation by $\langle x, \nabla u\rangle$ and integrate by parts. (Observe that the regularity of u does not suffice to justify this calculus directly but as Pohozaev points out in [71] the results are valid also for weak solutions; an argument of approximation which justifies the previous observation can be seen for instance in [57]. See also [73].)

$$\begin{aligned} &(\frac{p-1}{p}) \int_{\partial\Omega} |\nabla u|^p \langle x, \nu\rangle d\sigma + (\frac{N-p}{p}) \int_\Omega |\nabla u|^p dx = \\ &\lambda(\frac{N-p}{p}) \int_\Omega \frac{u^p}{|x|^p} dx + \gamma N \int_\Omega F(u) dx, \end{aligned}$$

where ν is the outwards normal to $\partial\Omega$. On the other hand, multiplying the equation by u and integrating

$$\int_\Omega |\nabla u|^p dx = \lambda \int_\Omega \frac{u^p}{|x|^p} dx + \gamma \int_\Omega u f(u) dx. \tag{3.9}$$

Both identities give

$$(\frac{p-1}{p}) \int_{\partial\Omega} |\nabla u|^p \langle x, \nu\rangle d\sigma = \gamma \int_\Omega (NF(u) - \frac{N-p}{p} u f(u)) dx.$$

The conclusion is now obvious. ■

Notice the contrast of Lemma 3.5.1 with the results in Chapter 2 and in the papers [29] [68] in the case $p = 2$, and in [53] for $p \neq 2$, where the autonomous case is considered.

Remark 3.5.2 *In the case $\Omega = I\!R^N$, $p = 2$ and $f(u) = u^{(N+2)/(N-2)}$ an existence result can be seen in [64], Th.I.3, pg. 179. However the previous Lemma proves that this doubly critical problem has no positive solution in bounded starshaped domains. This means that the term with the potential cannot be seen as a lower order perturbation of the term with critical Sobolev exponent, although this is the case in terms of the growth in u.*

3.6 Subcritical potential and critical growth

We will prove that in fact the potential $\lambda|x|^{-p}$ is critical in the sense of the remark above by considering $\lambda|x|^{-q}$ with $0 < q < p$.

It is known that for the potential $\lambda|x|^{-q}$, which belongs to L^r for some $r > \dfrac{N}{p}$, there exists a first isolated and simple eigenvalue λ_1, for the corresponding Dirichlet problem. We will show that this subcritical potential produces a similar effect to the case $q = 0$, in the sense that the lack of compactness arises only from the highest power term. Moreover there are also *bad dimensions*, $p < N < p^2 - (p-1)q$ as in the autonomous case. This interval coincides with the known result in the case $q = 0$, and decreases when $q \to p$, dissappearing the solution in the limit case $q = p$, according to Lemma 3.5.1.

Theorem 3.6.1 *Consider the Dirichlet problem*

$$\begin{cases} -\Delta_p u &= \lambda \dfrac{|u|^{p-2}u}{|x|^q} + |u|^{p^*-2}u, \quad \lambda < \lambda_1 \\ u|_{\partial\Omega} &= 0,\ 0 < q < p,\ 1 < p < N,\ p^* = Np/(N-p). \end{cases} \tag{3.10}$$

If $N \geq p^2 - (p-1)q$, then there exists at least a positive solution $u \in W_0^{1,p}(\Omega)$.

Proof. The arguments are similar to those in [53] and then we will be sketchy. The geometry of the energy functional, J, satisfies the requirements of the *Mountain Pass Theorem* as can be checked following the same calculations as in Theorem 3.4.1. Then by using the *concentration-compactness method* by P.L. Lions, see [64], we get a local Palais-Smale condition. More precisely, let S the optimal constant in the Sobolev embedding and given a Palais-Smale sequence such that

$$J(u_k) \to c < \frac{S^{N/p}}{N},\ J'(u_k) \to 0, \quad \text{in } W^{-1,p'}(\Omega)$$

there exists a convergent subsequence. The only thing to check (and this is the main point), is the existence of a Palais-Smale sequence at this subcritical energy level. To get this particular sequence it is well known that it is sufficient to find a direction $v_\varepsilon \in W_0^{1,p}(\Omega)$ for which

$$\sup_{t>0} J(tv_\varepsilon) < c_0 < \frac{S^{N/p}}{N}, \tag{3.11}$$

because in this way the minimax critical value in the Mountain Pass Theorem verifies $c < c_0$. The natural election is

$$v_\varepsilon = \frac{u_\varepsilon}{||u_\varepsilon||_{p^*}}, \quad \text{where} \quad u_\varepsilon = \frac{\phi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}$$

are the minimizers of the Sobolev inclusion with a convenient truncation $\phi \in \mathcal{C}_0^\infty$ to adapt their support to Ω. As usual we compute:

1. $||\nabla v_\varepsilon||_p^p \approx S + c_1 \varepsilon^{\frac{N-p}{p}}$.
2. $\int\limits_\Omega \frac{v_\varepsilon^p}{|x|^q} dx \approx c_2 \varepsilon^{\frac{p^2-(p-1)q-p}{p}}$ if $N > p^2 - (p-1)q$.
3. $\int\limits_\Omega \frac{v_\varepsilon^p}{|x|^q} dx \approx c_2 \varepsilon^{\frac{N-p}{p}} |\log \varepsilon|$ if $N = p^2 - (p-1)q$.
4. $\int\limits_\Omega \frac{v_\varepsilon^p}{|x|^q} dx \approx c_2 \varepsilon^{\frac{N-p}{p}}$ if $N < p^2 - (p-1)q$.

Following the proofs in the previous Chapter, the decay as $\varepsilon \to 0$ of $||\nabla v_\varepsilon||_p^p$ must be faster than the decay of $\int\limits_\Omega \frac{v_\varepsilon^p}{|x|^q} dx$, to get the inequality (3.11) and this is true if $N \geq p^2 - (p-1)q$. So we conclude. (See also the proof of Theorem 3.3 in [53]). ■

Remark 3.6.2 *Something more can be said in the case $N < p^2 - (p-1)q$. Following the same argument as in the proof of Theorem 3 in [16] we find that there exists a positive solution to problem (3.10) if $\lambda \in (\lambda_1 - A, \lambda_1)$, where $A = S(\int\limits_\Omega |x|^{-Nq/p} dx)^{-p/N}$ (S is the optimal Sobolev constant).*

Remark 3.6.3 *A good exercise to the reader is to study different combinations of potentials and nonlinearities and describe the complete fauna.*

Appendix

Appendix A

The inverse operator of the p-Laplacian

We will study the *p-Laplacian* operator defined by

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad 1 < p < \infty$$

that appears in several situations. The p-Laplacian *is the paradigmatic example* of *degenerated/singular* quasilinear elliptic operator. Notice that if $p = 2$ it becomes the classical Laplace operator. In this section we deal mainly with the Dirichlet problem,

$$\begin{cases} -\Delta_p u = f(x) \quad \text{if} \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{if} \quad x \in \partial\Omega, \end{cases} \tag{A.1}$$

where $\Omega \in I\!R^N$, $f \in W^{-1,p'}(\Omega)$, $p' = p/(p-1)$. The boundary condition will be understood as $u \in W_0^{1,p}(\Omega)$.

We have the following elementary result, as application of the classical Calculus of Variations.

Theorem A.0.4 *Assume $\Omega \subset I\!R^N$ a bounded domain and $f \in W^{-1,p'}(\Omega)$, then problem A.1 has a solution $u \in W_0^{1,p}(\Omega)$ in the weak sense, namely*

$$\int_\Omega \{\langle |\nabla u|^{p-2}\nabla u, \nabla\phi\rangle - f\phi\} dx = 0, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{A.2}$$

Proof. The functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega f(x) u dx \quad u \in W_0^{1,p}(\Omega)$$

is Gateaux diferentiable and the Euler equation of J coincide with the partial differential equation under study. Then, we look for solution to problem A.1 as critical points of J. But, we have
$i)$ J is coercive since

$$\begin{aligned} J(u) &\geq \frac{1}{p}||\nabla u||_p^p - ||f||_{W^{-1,p'}(\Omega)}||\nabla u||_p \\ &\geq \frac{1}{2p}||\nabla u||_p^p - C\|f\|_{W^{-1,p'}(\Omega)}^{p'}. \end{aligned}$$

$ii)$ J is lower weakly semicontinuous because the first term is the norm in $W_0^{1,p}(\Omega)$ and the second one is continuous. Hence we can apply the abstract minimization result to J to find a minimum. ■

We are interested in the properties of the inverse operator

$$(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \longrightarrow W_0^{1,p}(\Omega)$$

and we need the following inequalities. (See [80]).

Lemma A.0.5 *Let $x, y \in I\!R^N$ and $\langle \cdot, \cdot \rangle$ the standard scalar product in $I\!R^N$. Then*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \begin{cases} c_p\, |x-y|^p, & \text{if} \quad p \geq 2 \\ c_p\, \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if} \quad 1 < p < 2. \end{cases} \tag{A.3}$$

Proof. By homogeneity we can assume that $|x| = 1$ and $|y| \leq 1$. Morover by choosing a convenient basis in $I\!R^N$ we can assume

$$x = (1, 0, ..., 0),\ y = (y_1, y_2, 0, ..., 0), \text{ and } \sqrt{y_1^2 + y_2^2} \leq 1.$$

$i)$ Case $1 < p < 2$. It is clear that the inequality is equivalent to the next one

$$\left\{ \left(1 - \frac{y_1}{(y_1^2 + y_2^2)^{\frac{2-p}{2}}} \right)(1 - y_1) + \frac{y_2^2}{(y_1^2 + y_2^2)^{\frac{2-p}{2}}} \right\} \frac{\left(1 - \sqrt{y_1^2 + y_2^2} \right)^{2-p}}{(1-y_1)^2 + y_2^2} \geq C. \tag{A.4}$$

But

$$1 - \frac{y_1}{\left(\sqrt{y_1^2 + y_2^2} \right)^{2-p}} \geq \begin{cases} 1 - \dfrac{y_1}{|y_1|^{2-p}} \geq (p-1)(1-y_1), & \text{if} \quad 0 \leq y_1 \leq 1 \\ 1 - y_1 \geq (p-1)(1-y_1), & \text{if} \quad y_1 \leq 0, \end{cases}$$

then

$$(p-1)\{(1-y_1)^2 + y_2^2\} \frac{(1 + y_1 + y_2)^{\frac{2-p}{2}}}{(1-y_1)^2 + y_2^2} \geq p - 1.$$

$ii)$ Case $p \geq 2$. The inequality is equivalent to prove that

$$\frac{\left[1 - y_1\left(y_1^2 + y_2^2\right)^{\frac{p-2}{2}}\right](1 - y_1) + y_2^2\left(y_1^2 + y_2^2\right)^{\frac{p-2}{2}}}{((1 - y_1)^2 + y_2^2)^{\frac{p}{2}}} \geq C.$$

Denote $t = \dfrac{|y|}{|x|}$ and $s = \dfrac{\langle x, y\rangle}{|x||y|}$ then, we must show that the function

$$f(t, s) = \frac{1 - (t^{p-1} + t)s + t^p}{(1 - 2ts + t^2)^{\frac{p}{2}}},$$

is bounded from below. Direct calculation shows that fixed t, $\dfrac{\partial f}{\partial s} = 0$ if

$$1 - (t^{p-1} + t)s + t^p = \frac{t^{p-2} + 1}{p}(1 - 2ts + t^2),$$

then for the critical s for f we have

$$f(t, s) = \frac{t^{p-2} + 1}{p}\frac{1}{(1 - 2ts + t^2)^{\frac{p-2}{2}}} \geq$$

$$\frac{1}{p}\frac{t^{p-2} + 1}{(t + 1)^{p-2}} \geq \frac{1}{p}\min_{0 \leq t \leq 1}\frac{t^{p-2} + 1}{(t + 1)^{p-2}} \geq \frac{1}{2p}.$$

■

The main properties of $-\Delta_p$ and $(-\Delta_p)^{-1}$ are summarized in the following theorem.

Theorem A.0.6 *Let $\Omega \subset \mathbb{R}^N$ a bounded domain.*
$A)$ $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is uniformly continuous on bounded sets.
$B)$ $(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$, and is continuous.
$C)$ The composed operator

$$(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact if $1 \leq q < \dfrac{pN}{N - p}$.

Proof.
$A)$ Consider $\mathcal{C} \subset W_0^{1,p}(\Omega)$ bounded set, i.e.,

$$\|u\|_{W_0^{1,p}(\Omega)} < M, \quad \text{if} \quad u \in \mathcal{C}.$$

Then for $u, v \in \mathcal{C}$ we get

$$\| -\Delta_p u - (-\Delta_p v)\|_{W^{-1,p'}(\Omega)} = \sup_{\|\phi\|_{W_0^{1,p}(\Omega)}=1} \int_\Omega \langle \Delta_p u - \Delta_p v, \nabla\phi\rangle dx \leq$$

$$\leq \begin{cases} c_p \sup_{\|\phi\|_{W_0^{1,p}(\Omega)}=1} \int_\Omega \left(|\nabla u|^{p-2} + |\nabla v|^{p-2}\right) |\nabla u - \nabla v||\nabla\phi| dx, & \text{if} \quad p \geq 2 \\ c_p \sup_{\|\phi\|_{W_0^{1,p}(\Omega)}=1} \int_\Omega |\nabla u - \nabla v|^{p-1} |\nabla\phi| dx, & \text{if} \quad 1 < p < 2 \end{cases}$$

then by Hölder inequality we conclude that

$$|| -\Delta_p u - (-\Delta_p v)||_{W^{-1,p'}(\Omega)} \leq \begin{cases} 2C_p M^{p-2} \|\nabla u - \nabla v\|_p, & \text{if} \quad p \geq 2, \\ C_p M \|\nabla u - \nabla v\|_p, & \text{if} \quad p \leq 2. \end{cases}$$

B) We need to prove uniqueness of solution to the Dirichlet problem A.1. Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ be solutions to the problems

$$-\Delta_p u_1 = f_1, \quad -\Delta_p u_2 = f_2,$$

then

$$\langle -\Delta_p u_1 - (-\Delta_p u_2), (u_1 - u_2)\rangle = \langle f_1 - f_2, u_1 - u_2\rangle ,$$

where the first term must be understood as the duality product. By the previous lemma

$$\langle -\Delta_p u_1 - (-\Delta_p u_2), (u_1 - u_2)\rangle =$$

$$\int_\Omega \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla(u_1 - u_2)\rangle dx \geq$$

$$\geq \begin{cases} C_p \int_\Omega |\nabla(u_1 - u_2)|^p dx, & \text{if} \quad p \geq 2 \\ C_p \int_\Omega \dfrac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx, & \text{if} \quad 1 < p < 2, \end{cases}$$

then if $p \geq 2$

$$\int_\Omega |\nabla(u_1 - u_2)|^p dx \leq D_p ||f_1 - f_2||_{W^{-1,p'}(\Omega)} ||\nabla(u_1 - u_2)||_p$$

$$||\nabla(u_1 - u_2)||_p \leq D_p^{\frac{1}{p-1}} ||f_1 - f_2||_{W^{-1,p'}(\Omega)}^{\frac{1}{p-1}},$$

in particular, if $f_1 \equiv f_2$ implies $u_1 \equiv u_2$.

If $1 < p < 2$

$$\int_\Omega \frac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx \leq C_p ||f_1 - f_2||_{W^{-1,p'}(\Omega)} ||u_1 - u_2||_{W_0^{1,p}(\Omega)},$$

and then by Hölder inequality

$$\begin{aligned}
&\int_\Omega |\nabla(u_1-u_2)|^p dx = \\
&\int_\Omega \frac{|\nabla(u_1-u_2)|^p}{(|\nabla u_1|+|\nabla u_2|)^{\frac{p(2-p)}{2}}} (|\nabla u_1|+|\nabla u_2|)^{\frac{p(2-p)}{2}} dx \leq \\
&\left(\int_\Omega \frac{|\nabla(u_1-u_2)|^2}{(|\nabla u_1|+|\nabla u_2|)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_1|+|\nabla u_2|)^p \, dx\right)^{\frac{2-p}{p}}.
\end{aligned}$$

Hence

$$\frac{\left(\int_\Omega |\nabla(u_1-u_2)|^p dx\right)^{\frac{1}{p}}}{\left(||u_1||_{W_0^{1,p}(\Omega)} + ||u_2||_{W_0^{1,p}(\Omega)}\right)^{2-p}} \leq C_p ||f_1-f_2||_{W^{-1,p'}(\Omega)}.$$

In this way we have obtained simultaneously the existence of the inverse operator (uniqueness) and its continuity.

C) Is a direct consequence of B) and the Sobolev embedding ■

Lemma A.0.7 *(Weak comparison Principle). Let Ω be a bounded domain in $I\!R^N$ with smooth boundary. Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} -\Delta_p u_1 \leq \Delta_p u_2 \quad \textit{in the weak sense in} \quad \Omega \\ u_1 \leq u_2 \quad \textit{on} \quad \partial\Omega \end{cases}$$

then $u_1 \leq u_2$ in Ω.

Proof. Take as test function $\phi = \max\{u_1 - u_2, 0\}$ that is nonnegative and belong to $W_0^{1,p}(\Omega)$ by hypothesis. So we get

$$\int_{\{u_1>u_2\}} \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, (\nabla u_1 - \nabla u_2)\rangle dx \leq 0,$$

and we conclude taking into account the Lemma A.0.5. ■

We will use the following extension of the Hopf Lemma, that can be found in [77].

Lemma A.0.8 *Let Ω be a bounded domain with smooth boundary. If $u \in \mathcal{C}^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ and verifies*

$$\begin{cases} -\Delta_p u \equiv -div(|\nabla u|^{p-2}\nabla u) & \geq & 0 & \textit{in} & \mathcal{D}' \\ u & > & 0 & \textit{in} & \Omega \\ u|_{\partial\Omega} & = & 0 & & \end{cases}$$

then $\dfrac{\partial u}{\partial n} < 0$ *on* $\partial\Omega$. *Here* n *is the outward normal to* $\partial\Omega$.

Proof. We will assume that the domain Ω verifies the interior sphere condition.

Consider $x_0 \in \partial\Omega$ and a interior ball tangent to $\partial\Omega$, namely,

$$B_r(y) \subset \Omega, \quad \bar{B}_r(y) \cap \partial\Omega = \{x_0\}.$$

Define

$$v(x) = \begin{cases} A|x-y|^{\frac{p-N}{p-1}} + B & N \neq p \\ A \log|x-y| + B, & N = p, \end{cases}$$

where $A = [2^{(N-p)/(p-1)} - 1]^{-1} r^{(N-p)/(p-1)}$, $B = [1 - 2^{(N-p)/(p-1)}]^{-1}$ if $p \neq N$, and $A = -(\log 2)^{-1}$, $B = \dfrac{\log r}{\log 2}$ if $p = N$. The function v verifies:

i) $v(x) \equiv 1$ in $\partial B_{\frac{r}{2}}(y)$ and $v(x) \equiv 0$ on $\partial B_r(y)$,

ii) $0 < v(x) < 1$ if $x \in B_r(y) - B_{\frac{r}{2}}(y)$ and

$$|\nabla v(x)| > c > 0, \quad \text{for some positive constant} \quad c.$$

But $u(x) > 0$ in Ω, then

$$\tau = \inf\{u(x) | \, x \in \partial B_{\frac{r}{2}}(y)\} > 0,$$

and putting $\tau v = w$ we find that w satisfies

$$\begin{cases} -\Delta_p w(x) = 0, & \text{if} \quad x \in B_r(y) - \bar{B}_{\frac{r}{2}}(y) \\ w(x) = \tau, & \text{if} \quad x \in \partial B_{\frac{r}{2}}(y) \quad \text{and} \\ w(x) = 0, & \text{if} \quad x \in \partial B_r(y). \end{cases}$$

Now $w \leq u$ on the boundary of the ring $B_r(y) - B_{\frac{r}{2}}(y)$ and $\Delta_p w \leq \Delta_p u$ in Ω in the weak sense. Hence, the weak comparison principle implies that $w \leq u$ in $B_r(y) - B_{\frac{r}{2}}(y)$. Taking into account that $u(x_0) = w(x_0) = 0$, then

$$\frac{\partial u}{\partial n}(x_0) = \lim_{t \to 0} \frac{u(x_0 - tn)}{t} \leq \lim_{t \to 0} \frac{w(x_0 - tn)}{t} = \frac{\partial w}{\partial n}(x_0) = \tau \frac{\partial v}{\partial n}(x_0) < 0.$$

■

Appendix B

Genus and its properties

We introduce in this section the classical idea of *genus* due to Krasnoselskii, as a tool to *measure in a convenient way* the size of the sets.

Given X a Banach space, we consider the class

$$\Sigma = \{A \subset X \mid A \text{ closed }, A = -A\}. \tag{B.1}$$

Definition B.0.9 *Then the genus, γ, is defined as follows.*

$$\begin{array}{rcl} \gamma : \Sigma & \longrightarrow & I\!N \cup \{\infty\} \\ A & \longrightarrow & \gamma(A) \end{array}$$

where

$$\gamma(A) = \min\{k \in I\!N \mid \exists \varphi \in \mathcal{C}(A, I\!R^k - \{0\}), \varphi(x) = -\varphi(-x)\}. \tag{B.2}$$

If the minimum does not exist, then we define $\gamma(A) = +\infty$.

Notice that if $C \subset X$ is such that $C \cap (-C) = \emptyset$ and we define A by $A = C \cup (-C)$ then $A \in \Sigma$ and moreover $\gamma(A) = 1$. It is sufficient define $\phi(x) = 1$ if $x \in C$ and $\phi(x) = -1$ if $x \in -C$. In this way $\phi \in \mathcal{C}(A, I\!R - \{0\})$.

As a consequence we have that if $A_1 \in \Sigma$ and $\gamma(A_1) > 1$ then A_1 contains infinitely many different points.

This elementary remark shows how to use the *genus*.

In general to calculate the exact genus of a set is a difficult task. Often it is sufficient to have some estimates which can be obtained by comparison with sets whose genus is known, for instance, with respect to spheres. In this way the following results are very useful.

Lemma B.0.10 *Let* $A, B \in \sum$. *Then:*

1. *If there exists* $f \in \mathcal{C}(A, B)$, *odd, then* $\gamma(A) \leq \gamma(B)$.
2. *If* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$.
3. *If there exists an odd homeomorphism between* A *and* B, *then* $\gamma(A) = \gamma(B)$.
4. *If* S^{N-1} *is the sphere in* $I\!R^N$, *then* $\gamma(S^{N-1}) = N$.
5. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
6. *If* $\gamma(B) < +\infty$, *then* $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.
7. *If* A *is compact, then* $\gamma(A) < +\infty$, *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(N_\delta(A))$ *where* $N_\delta(A) = \{x \in X : d(x, A) \leq \delta\}$.
8. *If* X_0 *is a subspace of* X *with codimension* K , *and* $\gamma(A) > K$, *then* $A \cap X_0 \neq \emptyset$.

For the proof and more details on this topic we refer to [76] and the references therein.

Appendix C

The variational principle of Ekeland

The idea of the ε*-variational principle* of Ekeland is the following: Assume f lower semicontinuous real valued function defined on the metric space $(\mathcal{M}, d)$ such that $f(x) \geq \beta$ for all $x \in \mathcal{M}$. The principle deals with the construction of minimizing sequences with some control, precisely, sequences verifying

$$\inf_{x \in \mathcal{M}} \{f(x)\} + \varepsilon > f(x_\varepsilon),$$

and

$$f(y) \geq f(x_\varepsilon) - \varepsilon d(x_\varepsilon, y).$$

We can read geometrically this conditions saying that for all $\varepsilon > 0$ the graph of f is above of this *cone*. See the original paper [45]; we follows here the proof in [67].

Theorem C.0.11 *Let $\mathcal{M}$ be a complete metric space and let*

$$\phi : \mathcal{M} \to (-\infty, \infty]$$

a proper function such that

i) $\phi(y) \geq \beta$,

ii) ϕ *lower semicontinuous.*

Given $\varepsilon > 0$ and $u \in \mathcal{M}$ such that

$$\phi(u) \leq \inf_{\mathcal{M}} \phi + \varepsilon,$$

then there exists $v \in \mathcal{M}$ such that:

1. $\phi(u) \geq \phi(v)$,

2. $d(u,v) \leq 1$,

3. If $v \neq w \in \mathcal{M}$ *then* $\phi(w) \geq \phi(v) - \varepsilon d(v,w)$.

Proof. Fixed $\varepsilon > 0$, we define the following order relation on $\mathcal{M}$, we say

$$w \leq v, \quad \text{if and only if} \quad \phi(w) + \varepsilon d(w,v) \leq \phi(v).$$

Consider $u_0 = u$ and by recurrence define the sequence $\{u_n\}$, as follows: for $n \in I\!N$ we take

$$S_n = \{w \in \mathcal{M} : w \leq u_n\},$$

choosing $u_{n+1} \in S_n$ such that

$$\phi(u_{n+1}) \leq \inf_{S_n}\{\phi\} + \frac{1}{n+1},$$

we get that $u_{n+1} \leq u_n$ and $S_{n+1} \subset S_n$. The lower semicontinuity of ϕ implies that S_n is a closed set. Now, if $w \in S_{n+1}$, we have $w \leq u_{n+1} \leq u_n$ and then,

$$\varepsilon d(w, u_{n+1}) \leq \phi(u_{n+1}) - \phi(w) \leq \inf_{S_n}\{\phi\} + \frac{1}{n+1} - \inf_{S_n}\{\phi\} = \frac{1}{n+1}.$$

Namely, if we call diameter$(S_{n+1}) = \delta_{n+1}$, $(\delta_{n+1}) \leq \dfrac{2}{\varepsilon(n+1)}$, so, $\lim_{n\to\infty} \delta_{n+1} = 0$. The completeness of $\mathcal{M}$ implies $\cap_{n=1}^{\infty} S_n = \{v\}$ for some $v \in \mathcal{M}$. But, in particular, $v \in S_0$, hence $v \leq u_0 = u$, i.e., $\phi(v) \leq \phi(u) + \varepsilon d(u,v) \leq \phi(u)$ and

$$d(u,v) \leq \frac{\phi(u) - \phi(v)}{\varepsilon} \leq \varepsilon^{-1}(\inf_{\mathcal{M}}\{\phi\} + \varepsilon - \inf_{\mathcal{M}}\{\phi\}) = 1.$$

Then, $d(u,v) \leq 1$.

To get 3), suppose that $w \leq v$. Then for all $n \in I\!N$, $w \leq u_n$, i.e.,

$$w \in \cap_{n=1}^{\infty} S_n$$

and then $w = v$.

So we conclude that if $w \neq v$ then $\phi(w) \geq \phi(v) - \varepsilon d(v,w)$. ■

The results that we explain below show how to use the variational principle to find *critical points* of functionals.

Corollary C.0.12 *Let* $\mathcal{X}$ *be a Banach space, and* $\varphi : \mathcal{X} \to I\!R$ *a differentiable lower bounded function in* $\mathcal{X}$. *Then for all* $\varepsilon > 0$ *and for all* $u \in \mathcal{X}$ *such that*

$$\varphi(u) \leq \inf_{\mathcal{X}} \varphi + \varepsilon$$

there exists $v \in \mathcal{X}$ *verifying*

$$\begin{aligned} &\varphi(v) \leq \varphi(u), \\ &\|u - v\|_{\mathcal{X}} \leq \varepsilon^{1/2}, \\ &\|\varphi'(v)\|_{\mathcal{X}'} \leq \varepsilon^{1/2}. \end{aligned}$$

Proof. In Theorem C.0.11 take $\mathcal{M} = \mathcal{X}$, $\phi = \varphi$, $\varepsilon > 0$, $\lambda = \dfrac{1}{\varepsilon^{1/2}}$, $d = \| \cdot \|$. We obtain $v \in \mathcal{X}$ such that

$$\varphi(v) \leq \varphi(u)$$

$$\|u - v\|_{\mathcal{X}} \leq \varepsilon^{1/2}$$

and for all $w \neq v$

$$\varphi(w) > \varphi(v) - \varepsilon^{1/2}\|(v - w)\|.$$

Taking in particular $w = v + th$ with $t > 0$ and $h \in \mathcal{X}$, $\|h\| = 1$, then

$$\varphi(v + th) - \varphi(v) > -\varepsilon^{1/2} t$$

that implies $-\varepsilon^{1/2} \leq \langle \varphi'(v), h \rangle$ for all $h \in \mathcal{X}$, $\|h\| = 1$, hence

$$\|\varphi'(v)\|_{\mathcal{X}'} \leq \varepsilon^{1/2}.$$

■

Corollary C.0.13 *If* $\mathcal{X}$ *and* φ *are as in Corollary C.0.12. Then, for all minimizing sequence of* φ, $\{u_k\} \subset \mathcal{X}$ *there exists a minimizing sequence* $\{v_k\} \subset \mathcal{X}$ *such that*

$$\begin{aligned} &\varphi(v_k) \leq \varphi(u_k) \\ &\|u_k - v_k\|_{\mathcal{X}} \to 0 \quad k \to \infty \\ &\|\varphi'(v_k)\|_{\mathcal{X}'} \to 0 \quad k \to \infty \end{aligned}$$

Proof. If $\varphi(u_k) \to c = \inf_{\mathcal{X}} \varphi$ consider $\varepsilon_k = \varphi(u_k) - c$ if is positive and $\varepsilon_k = \dfrac{1}{k}$ if $\varphi(u_k) = c$. Then for ε_k we take v_k that gives the Corollary C.0.12. ■

In the sense of the previous Corollary we find *almost minimum* for φ. The problem now is to get *almost critical points* of different types, for instance *mountain pass type* critical points in the sense of [13]. Here we point out how the variational principle of Ekeland separates the geometric aspect of the problem of finding critical points of functional from the analytical one, namely, we will need some compactness property to complete the search of critical points.

Appendix D

The Mountain Pass Theorem

Seeking by the completeness of the exposition we will give a proof of the classical *Mountain Pass Theorem* by Ambrosetti and Rabinowitz. (See [13]).

We will precise the notation as follows,

H1 $\mathcal{X}$ is a Banach space, $\mathcal{K} \subset \mathbb{R}^N$ a compact metric space, and $\mathcal{K}_0 \subset \mathcal{K}$ a closed set.

H2

$$\chi : \mathcal{K}_0 \longrightarrow \mathcal{X}$$

is a continuous function.

H3 Define

$$\mathcal{M} = \{g \in \mathcal{C}(\mathcal{K}, \mathcal{X}) : g(s) = \chi(s), s \in \mathcal{K}_0\}.$$

We recall the definition of *subdifferential* in the case of the norm in $\mathcal{X}$,

$$\partial\|x_0\| = \{p \in \mathcal{X}' : \|x_0\| - \|x\| \leq p(x_0 - x), \forall x \in \mathcal{X}\}$$

($\mathcal{X}'$ dual space of $\mathcal{X}$.

We will use the following abstract result.

Lemma D.0.14 *Let $\mathcal{X}$ be a Banach space, then for the norm*

$$\|\cdot\| : \mathcal{X} \to \mathbb{R}$$

we have

$$\partial\|x_0\| = \{p \in \mathcal{X}' : p(x_0) = \|x_0\|, \|p\|_{\mathcal{X}'} = 1\}$$

Proof. Given $x_0 \in \mathcal{X}$ by Hahn-Banach theorem there exists $p \in \mathcal{X}'$ such that $p(x_0) = \|x_0\|$ and $\|p\|_{\mathcal{X}'} = 1$. For such p we have

$$p(x - x_0) = p(x) - p(x_0) \leq \|p\|_{\mathcal{X}'}\|x\| - \|x_0\| \leq \|x\| - \|x_0\|,$$

hence $\|x_0\| - \|x\| \leq p(x_0 - x)$, namely, $p \in \partial\|x_0\|$. Reciprocally if $p \in \partial\|x_0\|$, we get

$$\|x_0\| - \|x\| \leq p(x_0 - x) \quad \forall x \in \mathcal{X}.$$

And in particular for $x = \lambda x_0$, $\lambda > 0$, we find $\|x_0\|(1-\lambda) \leq (1-\lambda)p(x_0)$, so $(p(x_0)-\|x_0\|)(1-\lambda) \geq 0$ for all $\lambda \in I\!R^+$. Then $p(x_0) = \|x_0\|$. Moreover $\|x_0\| - \|x\| \leq p(x_0) - p(x) \leq \|x_0\| - p(x)$ implies $p(x) < \|x\|$ and then $\|p\|_{\mathcal{X}'} \leq 1$ since $p(x_0) = \|x_0\|$ and $\|p\|_{\mathcal{X}'} = 1$. ■

Corollary D.0.15 *Let $\mathcal{C}(\mathcal{K}, I\!R)$ with the supremum norm. Denote by $\mathcal{M}(\mathcal{K})$ the Radon measures on $\mathcal{K}$. Then*

$$\partial\|f\|_\infty = \{\mu \in \mathcal{M}(\mathcal{K}) : \mu \geq 0, \int_{\mathcal{K}} d\mu = 1,\ supp(\mu) \subset \{t : f(t) = \|f\|_\infty\}\}.$$

Proof. It is clear that the Radon measures verifying the conditions above are the subdifferential of the norm. More precisely, by Lemma D.0.14 we have

i) If the conditions are satisfied then $\|\mu\| = 1$ and

$$\int_{\mathcal{K}} f(x) d\mu = \|f\|_\infty \int_{\text{supp}\mu} d\mu = \|f\|_\infty,$$

ii) If $\|\mu\| = 1$ and $\int_{\mathcal{K}} f(x) d\mu = \|f\|_\infty$ then $\mu > 0$, and

$$\text{supp}(\mu) \subset \{s |\, f(s) = ||f||_\infty\}.$$

■

Theorem D.0.16 *Consider $\mathcal{X}$, $\mathcal{K}$, $\mathcal{K}_0$ and $\mathcal{M}$ being as in* **H1 – H2 – H3**. *Let $u : \mathcal{X} \to I\!R$ a functional verifying:*

1) u is continuous,

2) u is Gateaux diferentiable and, moreover,

$$u' : \mathcal{X} \longrightarrow \mathcal{X}'$$

is continuous from the strong to weak-$*$ topologies.

consider

$$\begin{cases} \alpha & = \inf_{\varphi\in\mathcal{M}} \max_{s\in\mathcal{K}} u(\varphi(s)) \\ \alpha_1 & = \max_{\chi(\mathcal{K}_0)} u, \end{cases}$$

and assume that $\alpha > \alpha_1$. *Then for all* $\varepsilon > 0$ *and for all* $\varphi \in \mathcal{M}$ *such that* $\max_{s\in\mathcal{K}} u(\varphi(s)) \le \alpha+\varepsilon$ *there exists* $v_\varepsilon \in \mathcal{X}$ *such that*

$$\begin{array}{rcl} \alpha - \varepsilon & \le & u(v_\varepsilon) \le \max_{s\in\mathcal{K}} u(\varphi(s)) \\ d(v_\varepsilon, \varphi(\mathcal{K})) & \le & \varepsilon^{1/2} \\ \|u'(v_\varepsilon)\| & \le & \varepsilon^{1/2}. \end{array}$$

PROOF. We can assume that $0 < \varepsilon < \alpha - \alpha_1$. Let $\varphi \in \mathcal{M}$ verifiying

$$\max_{s\in\mathcal{K}} u(\varphi(s)) \le \alpha + \varepsilon.$$

Define

$$I : \mathcal{M} \to I\!R, \quad \text{by} \quad I(c) = \max_{s\in\mathcal{K}} u(c(s))$$

for $c \in \mathcal{M}$. By hypothesis $\alpha = \inf_{c\in\mathcal{M}} I(c) > \alpha_1$. Moreover I is lower semicontinuous, because if $\|c_k - c\|_\infty \to 0$ we have

$$I(c) = u(c(s_0)) = \lim_{k\to\infty} u(c_k(s_0)) \le \lim_{k\to\infty} I(c_k).$$

By the Ekeland variational principle applied to $I : \mathcal{M} \to I\!R$, given $\varepsilon > 0$ and $\varphi \in \mathcal{M}$ such that $I(\varphi) \le \alpha + \varepsilon$ there exists $c_\varepsilon \in \mathcal{M}$ such that

(i) $I(c_\varepsilon) \le I(\varphi) \le \alpha + \varepsilon$.

(ii) $I(c) \ge I(c_\varepsilon) - \varepsilon^{1/2}\|c - c_\varepsilon\|_\infty$ if $c \in \mathcal{M}$.

(iii) $\|c_\varepsilon - \varphi\|_\infty \le \varepsilon^{1/2}$.

Consider $\gamma : \mathcal{K} \to \mathcal{X}$ such that $\gamma(\mathcal{K}_0) = 0$. For $h \neq 0, h \in I\!R$ from (ii) we conclude that $I(c_\varepsilon + \gamma h) - I(c_\varepsilon) \ge -\varepsilon^{1/2}|h|\|\gamma\|_\infty$, namely

$$\frac{1}{|h|} I(c_\varepsilon + \gamma h) - I(c_\varepsilon) \ge -\varepsilon^{1/2}\|\gamma\|_\infty.$$

By definition of I,

$$\begin{array}{l} I(c_\varepsilon + \gamma h) - I(c_\varepsilon) = \max u(c_\varepsilon + \gamma h) - \max u(c_\varepsilon) \\ = \max_{s\in\mathcal{K}}\{u(c_\varepsilon(s)) + h\langle u'(c_\varepsilon(s)), \gamma(s)\rangle + o(h)\} - \max_{s\in\mathcal{K}} u(c_\varepsilon(s)). \end{array}$$

To simplify the printing we write

$$f(s) = u(c_\varepsilon(s)) \ \text{ and } \ g(s) = \langle u'(c_\varepsilon(s)), \gamma(s)\rangle.$$

We have by hypothesis that $f \in \mathcal{C}(\mathcal{K}, I\!R)$; also $g \in \mathcal{C}(\mathcal{K}, I\!R)$ because,

a) By Banach-Alaoglu theorem $\partial\|f\|_\infty$ is weak-$*$ compact.

b) By hypothesis if $v_n \to v$ en $\mathcal{X}$ then

$$\langle u'(v_n), y\rangle \to \langle u'(v), y\rangle$$

for all $y \in \mathcal{X}$.

c) If $u'(v_n) \rightharpoonup u'(v)$ weak-$*$ in $\mathcal{X}'$ and $y_n \to y$ in $\mathcal{X}$ then

$$\langle u'(v_n), y_n\rangle \to \langle u'(v), y\rangle, \quad \text{as} \quad n \to \infty.$$

Then the inequalities above can be written as

$$\begin{aligned}
&-\varepsilon^{1/2}\|\gamma\|_\infty \leq \lim_{h\to 0} \frac{1}{h}\{I(c_\varepsilon + h\gamma) - I(c_\varepsilon)\} \\
&= \lim_{h\to 0} \frac{1}{h}\{\|(f + hg)\|_\infty - \|f\|_\infty\} \\
&\leq \sup\{\textstyle\int_0^1 g d\mu : \mu \in \partial\|f\|_\infty\}.
\end{aligned}$$

Define $F : \partial\|f\|_\infty \to I\!R$ by $F(\mu) = \int_K g d\mu$ and consider a weak-$*$ convergent sequence $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\partial\|f\|_\infty$, then

$$F(\mu_n) = \int_{\mathcal{K}} g d\mu_n \to \int_{\mathcal{K}} g d\mu = F(\mu),$$

so, the supremum is attained. As a consequence we have

$$-e^{1/2}\|\gamma\|_\infty \leq \max\{\int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu : \mu \in \partial\|f\|_\infty\}.$$

Consider $\Gamma = \{\gamma \in \mathcal{C}(\mathcal{K}, \mathcal{X}) : \gamma(s) = 0 \text{ if } s \in \mathcal{K}_0, \|\gamma\|_\infty \leq 1\}$. Then,

$$-\varepsilon^{1/2} \leq -\varepsilon^{1/2}\|\gamma\|_\infty \leq \inf_{\gamma\in\Gamma}\{\max\{\int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu : \mu \in \partial\|f\|_\infty\}\}.$$

Take $\{\mu_n(\gamma)\} \subset \partial\|f\|_\infty$ such that fixed γ,

$$\int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu_n(\gamma) \longrightarrow M(\gamma),\ n \to \infty$$

where

$$M(\gamma) = \max_{u\in\partial\|f\|_\infty}\{\int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu\},$$

Then by choosing a convenient weak-$*$ convergent subsequence $\mu_n(\gamma) \rightharpoonup \mu(\gamma)$ we have

$$\int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu(\gamma) = M(\gamma).$$

Then by Lemma D.0.15,

$$-\varepsilon^{1/2} \leq \inf_{\gamma\in\Gamma} M(\gamma) = \inf_{\gamma\in\Gamma} \int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu(\gamma) \leq$$

$$\max_{\mu\in\partial\|f\|_\infty} \inf_{\gamma\in\Gamma} \int_{\mathcal{K}} \{u'(c_\varepsilon(s)), \gamma(s)\} d\mu = \max_{\mu\in\partial\|f\|_\infty} (-\int_{\mathcal{K}} \|u'(c_\varepsilon(s))\|_{\mathcal{X}'} d\mu) =$$

$$-\min\{\|u'(c_\varepsilon(s))\|_{\mathcal{X}'} : s \in \{t \in \mathcal{K} : u(c_\varepsilon(t)) = \|f\|_\infty\}\}$$

Namely

$$\min_{s\in\{t\in\mathcal{K}:u(c_\varepsilon(t))=\|f\|_\infty\}} \|u'(c_\varepsilon(s))\|_{\mathcal{X}'} \leq \varepsilon^{1/2}$$

In other words, there exists $s_\varepsilon \in \mathcal{K}$ such that, if $v_\varepsilon = c_\varepsilon(s_\varepsilon)$, then:

i) $u(v_\varepsilon) = \max_{s\in\mathcal{K}} u(c_\varepsilon(s)) = I(c_\varepsilon) \leq \inf I + \varepsilon^{1/2}$,

ii) $\|u'(v_\varepsilon)\| \leq \varepsilon^{1/2}$,

iii) $u(v_\varepsilon) \geq \alpha_1$,

iv) $d(v_\varepsilon, \varphi(\mathcal{K})) \leq \varepsilon^{1/2}$.

■

The classical condition of compactness is the condition of Palais-Smale that we formalize in the following definition.

Definition D.0.17 *Let $\mathcal{X}$ be a Banach space and $U : \mathcal{X} \longrightarrow I\!R$ a functional Gateaux diferentiable. U verifies the Palais-Smale condition to the level $c \in I\!R$ if and only if for all sequence $\{x_k\} \subset \mathcal{X}$ verifying*

i) $U(x_k) \to c$ as $k \to \infty$,

ii) $U'(x_k) \to 0$ in $\mathcal{X}'$ for $k \to \infty$

there exists a subsequence $x_{k_j} \to x$ in $\mathcal{X}$.

A sequence verifying i) and ii) is called a Palais-Smale sequence for U.

With this notions we can formulate the *Montain Pass Theorem.*

Theorem D.0.18 *Let $\mathcal{X}$ be a Banach space and $U : \mathcal{X} \to I\!R$ a continuous functional verifiying:*

i) U is Gateaux differentiable,

ii) U' is continuous from $\mathcal{X}$ to $\mathcal{X}'$ with the weak-$*$ topology.

Assume that there exists $u_0, u_1 \in \mathcal{X}$, and a ball B_r, centered in u_0 and with radius r such that if $u_1 \in \mathcal{X} - B_r$ and if $|x - u_0| = r$ then

$$\inf_{\partial B_r} U(x) > \max\{U(u_0), U(u_1)\}$$

Consider

$$\Gamma = \{g \in \mathcal{C}([0,1], \mathcal{X}) : g(0) = u_0, g(1) = u_1\}$$

and

$$c = \inf_{f \in \Gamma} \sup_{s \in [0,1]} U(f(s)).$$

If U satisfies the Palais-Smale condition at the level c, then c is a critical value for U and $c > \max\{U(u_0), U(u_1)\}$.

Proof. Take $\mathcal{K} = [0,1]$, $\mathcal{K}_0 = \{0,1\}$, $\chi(0) = u_0$, $\chi(1) = u_1$ and $\mathcal{M} = \Gamma$; since $c_0 = \inf_{\partial B_r} U(x) > \max\{U(u_0), U(u_1)\} = c_1$ and because all path $g \in \Gamma$ has nonempty intersection with ∂B_r, Theorem D.0.16 implies that for a given sequence $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$ we can find a sequence $\{x_k\} \subset \mathcal{X}$ such that

a) $U(x_k) \to c$,

b) $U'(x_k) \to 0$,

and then by the Palais-Smale condition for the level c, there exists a subsequence such that:

$$c = \lim_{k \to \infty} U(x_k) = U(x),$$

$$0 = \lim_{k \to \infty} U'(x_k) = U'(x).$$

Namely, x is a critical point at level c. ■

Appendix E

Boundedness of solutions

The boundedness of the solutions in the case $p \geq N$ is a consequence of Morrey's theorem and the classical bootstrapping.

One of the few general results for problems with critical growth is the following one on $\mathcal{C}^{1,\alpha}$ regularity. We will concentrate on the following problem

$$\begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u|_{\delta\Omega} = 0, \end{cases} \tag{E.1}$$

where $\Omega \subset I\!R^N$ is a smooth bounded domain, $1 < p < N$ and f satisfies

$$|f(x,u)| \leq C(1+|u|^r), \quad \text{with} \quad r+1 \leq p^* = \frac{Np}{N-p}, \tag{E.2}$$

that is the critical Sobolev exponent. The regularity of the solution of E.1 for $r < p^* - 1$ is a consequence of the results in the paper by Serrin [79] about the L^∞ estimates and the results by Di Benedetto, [44] and Tolksdorf, [87], for the $\mathcal{C}^{1,\alpha}$ regularity. The case $r = p^* - 1$ is much more delicate and will be obtained below, while for the supercritical case the result is not true: in general in the supercritical case a weak solution is not bounded.

The idea of the L^∞ estimate can be found in the argument used by Trudinger in [88] for Yamabe's Problem. The main point is to use some nonlinear test functions in the line of the classical Moser method.

If we have that the solutions of E.1 are bounded then the regularity $\mathcal{C}^{1,\alpha}$ is a consequence of the results in [44] or [87]. I would like to remark that a different (Schauder) aproach to the regularity of problem E.1, can be seen in Guedda-Véron, [58].

The first result in this section is the following theorem which comes from Stampacchia classical work [81].

Theorem E.0.19 *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution of*

$$\begin{cases} -\Delta_p u = g, & \text{in } \Omega,\ g \in W^{-1,q}(\Omega),\ q > \frac{N}{p-1} \\ u|_{\partial\Omega} = 0, \end{cases} \tag{E.3}$$

then $u \in L^\infty(\Omega)$.

Proof. We can write $g = \operatorname{div} F$, $F = (f_1, ..., f_N)$ where $f_i \in L^q(\Omega)$. Since u is the weak solution to (E.3) we can write

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle dx = \int_\Omega \langle F, \nabla v\rangle,\ \forall v \in W_0^{1,p}(\Omega) \tag{E.4}$$

For $k > 0$ we consider the test function

$$v = \operatorname{sign}(u)(|u| - k) = \begin{cases} u - k \ \text{ if } \ u > k \\ 0 \ \text{ if } \ u = k \\ u + k \ \text{ if } \ u < k \end{cases} \tag{E.5}$$

then $u = v + k\operatorname{sign}(u)$ and $u_{x_i} = v_{x_i}$ in $A(k) = \{x \in \Omega \,|\, |u(x)| > k\}$, $v = 0$ in $\Omega - A(k)$ and $v \in W_0^{1,p}(\Omega)$. For this election of test function (E.4) becomes

$$\int_{A(k)} |\nabla v|^p dx = \int_\Omega \langle F, \nabla v\rangle dx,$$

and by Hölder inequality

$$\int_\Omega \langle F, \nabla v\rangle dx \le \left(\int_{A(k)} |f|^q\right)^{1/q} \left(\int_{A(k)} |\nabla v|^p\right)^{1/p} |A(k)|^{1-(\frac{1}{p}+\frac{1}{q})},$$

here $|C|$ denotes the Lebesgue measure of the measurable set C. Then we have

$$\left(\int_{A(k)} |\nabla v|^p dx\right)^{1-1/p} \le \left(\int_{A(k)} |F|^q\right)^{1/q} |A(k)|^{1-(1/p+1/q)}$$

and by Sobolev inequality we obtain

$$S\left(\int_{A(k)} |v|^{p^*} dx\right)^{p/p^*} \le \left(\int_{A(k)} |F|^q\right)^{1/q} |A(k)|^{1-(1/p+1/q)}. \tag{E.6}$$

Notice that for $0 < k < h$, $A(h) \subset A(k)$ and then

$$\begin{aligned}|A(h)|^{1/p^*}(h-k) = \left(\int_{A(h)}(h-k)^{p^*}dx\right)^{1/p^*} \leq \\ \left(\int_{A(h)}|v|^{p^*}dx\right)^{1/p^*} \leq \left(\int_{A(k)}|v|^{p^*}dx\right)^{1/p^*}.\end{aligned} \tag{E.7}$$

From (E.6) and (E.7) we obtain

$$|A(h)| \leq \frac{||F||_q^{p^*/(p-1)}}{S^{p^*/p}}\frac{1}{(h-k)^{p^*}}|A(k)|^{p^*(\frac{1}{p}-\frac{1}{(p-1)q}}.$$

Because the hypothesis $q > N/(p-1)$,

$$p^*(\frac{1}{p}-\frac{1}{q(p-1)}) > 1.$$

Now we have the following

Claim: *Assume $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0$, $\beta > 1$,*

$$\phi(h) \leq \frac{c}{(h-k)^{\alpha}}[\phi(k)]^{\beta}.$$

Then $\phi(k_0+d) = 0$, where $d^{\alpha} = c2^{\frac{\alpha\beta}{\beta-1}}[\phi(k_0)]^{\beta-1}$.

In our case $\phi(h) = |A(h)|$, $\alpha = p^*$ and

$$\beta = p^*(\frac{1}{p}-\frac{1}{q(p-1)}) > 1.$$

We have $\phi(0) = |\Omega|$. By the Claim we get that $\phi(d) = 0$ for

$$d = c\frac{||F||_q^{1/(p-1)}}{S^{1/p}}|\Omega|^{(\frac{1}{p}-\frac{1}{q(p-1)})-\frac{1}{p^*}},$$

namely

$$||u||_{\infty} \leq c\frac{||F||_q^{1/(p-1)}}{S^{1/p}}|\Omega|^{(\frac{1}{p}-\frac{1}{q(p-1)})-\frac{1}{p^*}}.$$

To finish we need to prove the Claim.

Proof of the claim: Given d as above, define $d_n = d_0 + d - \frac{d}{2^n}$

By recurrence we have that

$$\phi(k_n) \leq \frac{\phi(k_0)}{2^{-n\mu}},\ \mu = \frac{\alpha}{1-\beta},$$

then

$$0 = \lim_{n\to\infty} \phi(k_k) \geq \phi(k_0 + d) \geq 0,$$

as we want to prove. ■

The second result in this section is the following adaptation of the announced result by Trudinger, [88].

Theorem E.0.20 *Let $u \in W_0^{1,p}(\Omega)$ be a solution of E.1. If f verifies E.2, then $u \in L^\infty(\Omega)$.*

Proof. We assume $r + 1 = p^*$. We can assume also that $u \geq 0$, in other case we can do the argument for the positive and negative parts of u by a minor change in the test function. We define for each $l > 0$ and $\beta > 1$,

$$F(u) = \begin{cases} u^\beta, \text{ if } u \leq l \\ \beta l^{\beta-1}(u - l) + l^\beta, \quad \text{if} \quad u > l \end{cases}$$

and

$$G(u) = \begin{cases} u^{(\beta-1)p+1}, \text{ if } u \leq l \\ \beta((\beta-1)p+1)l^{(\beta-1)p}(u-l) + l^{(\beta-1)p+1}, \quad \text{if} \quad u > l. \end{cases}$$

It is easy to show the following properties

i) $G(u) \leq uG'(u)$.

ii) For some constant independent of l, $c[F'(u)]^p \leq G'(u)$.

iii) $u^{p-1}G(u) \leq C[F(u)]^p$ with C independent of l.

iv) If $u \in W_0^{1,p}(\Omega)$, then $F(u)$, $G(u) \in W_0^{1,p}(\Omega)$ because F and G are Lipschitz functions.

We choose $\beta > 1$ such that $\beta p < p^*$. For this choice, consider the test function

$$\xi = \eta^p G(u),$$

where $\eta \in \mathcal{C}_0^\infty$ will be fixed below.

Then the equation gives us

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla(\eta^p G(u))\rangle dx = \int_\Omega f(x,u)\eta^p G(u) dx.$$

For $\varepsilon > 0$ and by direct computation on the left hand side, the previous identity becomes

$$\begin{aligned}&\int_\Omega |\nabla u|^p \eta^p G'(u) dx \leq \\ &\varepsilon \int_\Omega |\nabla u|^p \eta^p G'(u) dx + C\int_\Omega |\nabla \eta|^p (u^{p-1}G(u)) dx + \int_\Omega f(x,u)\eta^p G(u) dx,\end{aligned}$$

where we use the property i) above, Hölder and Young inequalities. Then, by choosing ε small enough,

$$\int_\Omega |\nabla u|^p \eta^p G'(u)dx \leq$$
$$\leq C_1 \int_\Omega |\nabla \eta|^p [F(u)]^p dx + C_2 \int_\Omega \eta^p u^{p^*-p}[F(u)]^p dx + C_3|\Omega|,$$

where we use that $G(u)f(x,u) \leq C_3(1 + u^{p^*-1}G(u))$ (for $l > 1$), motivated by the hypothesis E.2, and the property iii).

The left hand side can be estimated by using the property ii), i.e., $G' \geq c[F']^p$,

$$\int_\Omega |\nabla u|^p \eta^p G'(u)dx \geq c \int_\Omega |\eta F'(u)\nabla u|^p dx,$$

but then,

$$\int_\Omega |\nabla(\eta F(u))|^p dx \leq$$
$$\leq C_4 \int_\Omega |\nabla \eta|^p [F(u)]^p dx + C_5 \int_\Omega \eta^p u^{p^*-p}[F(u)]^p dx + C_6|\Omega|.$$

By the Sobolev inequality we get

$$\left(\int_\Omega [F(u)]^{p^*} \eta^{p^*} dx\right)^{\frac{p}{p^*}} \leq$$
$$C_4 \int_\Omega |\nabla \eta|^p [F(u)]^p dx + C_5 \int_\Omega \eta^p u^{p^*-p}[F(u)]^p dx + C_6|\Omega|.$$

Given $x_o \in \Omega$, we choose $\eta \in \mathcal{C}_0^\infty$ such that if $\operatorname{supp}(\eta) = B(x_o, R)$, then

$$||u||^{p^*-p}_{L^{p^*}(B(x_o,R))} \leq \frac{1}{2C_5}.$$

Then, by Hölder inequality,

$$C_5 \int_\Omega \eta^p u^{p^*-p}[F(u)]^p dx \leq \frac{1}{2}\left(\int_\Omega \eta^{p^*}[F(u)]^{p^*} dx\right)^{p/p^*}.$$

Finally we get the inequality

$$\int_\Omega \eta^{p^*}[F(u)]^{p^*} dx \leq C_5 \int_\Omega |\nabla \eta|^p [F(u)]^p dx + C_6|\Omega|.$$

Taking limits for $l \to \infty$ we conclude

$$\left(\int_\Omega \eta^{p^*} u^{\beta p^*} dx\right)^{p/p^*} \leq C(\eta) \int_\Omega u^{\beta p} dx + C_6|\Omega|$$

and, because $\beta p < p^*$, we have $u \in L^{\beta p^*}_{loc}$.

Then, u is a solution of the equation

$$-\Delta_p u = a(x)u^{p-1} + b(x),$$

where

$$a(x) = \begin{cases} 0 \quad \text{if} \quad u < 1 \\ \dfrac{f(x,u)}{u^{p-1}} \quad \text{if} \quad u \geq 1 \end{cases}$$

and

$$b(x) = \begin{cases} 0 \quad \text{if} \quad u > 1 \\ f(x,u) \quad \text{if} \quad u \leq 1. \end{cases}$$

Therefore, $a(x) \in L^r$, with $r > \dfrac{N}{p}$, and $b \in L^\infty$. We conclude, by Theorem E.0.19, that $u \in L^\infty$, and moreover, $u \in \mathcal{C}^\alpha$ by Theorem 1 in [79]. ■

Appendix F

Two Lemmas by P.L. Lions

By the sake of completness we will include in this appendix the proof of two lemmas by P.L. Lions. (See [64] and [65]). These results have been used in section 2.3. The first one is a real variable result that allows us to get precise representation of measures related by a *reverse Hölder inequality.*

Lemma F.0.21 *Let μ, ν be two non-negative and bounded measures on $\overline{\Omega}$, such that for $1 \leq p < r < \infty$ there exists some constant $C > 0$ such that,*

$$(\int_{\Omega} |\varphi|^r d\nu)^{\frac{1}{r}} \leq C(\int_{\Omega} |\varphi|^p d\mu)^{\frac{1}{p}} \quad \forall \varphi \in \mathcal{C}_0^\infty . \tag{F.1}$$

Then, there exist $\{x_j\}_{j \in J} \subset \overline{\Omega}$ and $\{\nu_j\}_{j \in I} \subset (0, \infty)$, where I is finite, such that:

$$\nu = \sum_{j \in I} \nu_j \delta_{x_j} \quad , \quad \mu \geq C^{-p} \sum_{j \in I} \nu_j^{\frac{p}{r}} \delta_{x_j}, \tag{F.2}$$

where δ_{x_j} is the Dirac mass at x_j.

Proof. By the reverse Hölder inequality (F.1), the measure ν is absolutely continuous with respect to μ. As a consequence there exists $f \in L^1(\Omega)$, $f \geq 0$, such that $\nu = f\mu$. Also by (F.1) we have,

$$\nu(A) \leq c_0^q (\mu(A))^{q/p},$$

for any borelian set $A \subset \Omega$. In particular, $f \in L^\infty(\Omega)$.

On the other hand the Lebesgue decomposition of μ with respect to ν gives us

$$\mu = g\nu + \sigma, \text{ where } g \in L^1(d\nu), \ g \geq 0$$

and σ is a bounded positive measure, singular respect to ν, namely, if K is the support of σ then $\nu(K) = 0$.

Now consider (F.1) applied to the test function

$$\phi = g^{1/(q-p)}\chi_{\{g\le n\}}\psi$$

we obtain

$$\begin{aligned}&(\int_\Omega |\varphi|^r d\nu)^{\frac{1}{r}} = (\int_\Omega g^{r/(r-p)}|\psi|^r\chi_{\{g\le n\}}d\nu)^{\frac{1}{r}}\\ &\le C(\int_\Omega g^{1+p/(r-p)}|\psi|^p\chi_{\{g\le n\}}d\nu)^{\frac{1}{p}} \le C(\int_\Omega g^{r/(r-p)}|\psi|^p\chi_{\{g\le n\}}d\nu)^{\frac{1}{p}}.\end{aligned} \tag{F.3}$$

Hence calling $d\nu_n = g^{r/(r-p)}\chi_{\{g\le n\}}d\nu$ the following reverse Hölder inequality holds

$$(\int_\Omega |\psi|^r d\nu_n)^{\frac{1}{r}} \le C(\int_\Omega |\psi|^p d\nu_n)^{\frac{1}{p}},$$

and in particular for each borelian $A \subset \Omega$ we have

$$(\nu_n(A))^{1/r} \le C(\nu_n(A))^{1/p}.$$

Taking into account that $p < r$ then either $\nu_n(A) = 0$ or

$$\nu_n(A) \ge c^{-(\frac{1}{p}-\frac{1}{r})} = \delta > 0.$$

As a consequence, given a point $x \in \Omega$, either $\nu_n(\{x\}) = 0$, or $\nu_n(\{x\}) \ge \delta > 0$ and this means that ν_n is a linear combination of Dirac masses, that necessarily must be finite because ν_n is bounded. Taking limits as $n \to \infty$ we conclude that

$$\nu = \sum_{j\in I}\nu_j\delta_{x_j}.$$

Moreover the inequality (F.1) applied in each x_j gives,

$$\mu \ge C^{-p}\sum_{j\in I}\nu_j^{\frac{p}{r}}\delta_{x_j}.$$

■

The second result is, roughly speaking, a description of how the lack of compactness happens in the Sobolev inclusion. More precisely we have.

Lemma F.0.22 *Let $\{u_j\}$ be a weakly convergent sequence in $W_0^{1,p}(\Omega)$ with weak limit u, and such that:*

i) $|\nabla u_j|^p \to \mu$ *weakly-$*$ in the sense of measures.*

ii) $|u_j|^{p^*} \to \nu$ *weakly-$*$ in the sense of the measures.*

Then, for some finite index set I we have:

$$\begin{cases} 1) \quad \nu = |u|^{p^*} + \sum_{j\in I} \nu_j \delta_{x_j} \quad , \quad \nu_j > 0 \\ 2) \quad \mu \geq |\nabla u|^p + \sum_{j\in I} \mu_j \delta_{x_j} \quad , \quad \mu_j > 0 \quad , \quad x_j \in \overline{\Omega} \\ 3) \quad \nu_j^{\frac{p}{p^*}} \leq \dfrac{\mu_j}{S}. \end{cases}$$

Proof. Given any $\phi \in \mathcal{C}_0^\infty(\Omega)$, by Sobolev inequality we have,

$$\left(\int_\Omega |\phi|^{p^*} |u_n|^{p^*} dx \right)^{\frac{1}{p^*}} S^{\frac{1}{p}} \leq \left(\int_\Omega |\nabla(\phi u_n)|^p dx \right)^{\frac{1}{p}}. \tag{F.4}$$

We write $v_n = u_n - u$ and by the convergence results in [28], we have,

$$\int_\Omega |\phi|^{p^*} |u_n|^{p^*} dx - \int_\Omega |\phi|^{p^*} |v_n|^{p^*} dx \to \int_\Omega |\phi|^{p^*} |u|^{p^*} dx, \ n \to \infty.$$

Moreover $v_n \rightharpoonup 0$ as $n \to \infty$ then applying the Sobolev inequality to ϕv_n and taking limits we arrive to a reverse Hölder inequality as in Lemma F.0.21, so we have the representation

$$\nu = |u|^{p^*} + \sum_{j\in I} \nu_j \delta_{x_j}.$$

Now by (F.4) and taking into account that $u_n \to u$ in L^p, we obtain,

$$\left(\int_\Omega |\phi|^{p^*} d\nu \right)^{\frac{1}{p^*}} S^{\frac{1}{p}} \leq \left(\int_\Omega |\phi|^p d\mu \right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla \phi|^p |u|^p dx \right)^{\frac{1}{p}}, \tag{F.5}$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega)$. Consider a $\phi \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and supported in the unit ball $B \subset I\!R^N$. Fixed $j \in I$ and $\varepsilon > 0$, we consider $\phi(\dfrac{x - x_j}{\varepsilon})$. For ε small enough we have,

$$\nu_j^{\frac{1}{p^*}} S^{\frac{1}{p}} \leq \mu(B(x_j, \varepsilon))^{\frac{1}{p}} + \left(\int_{B(x_j,\varepsilon)} \varepsilon^{-p} |\nabla \phi(\frac{x_j - x}{\varepsilon})|^p |u|^p dx \right)^{\frac{1}{p}}$$

and by Hölder inequality,

$$\left(\int_{B(x_j,\varepsilon)} \varepsilon^{-p}|\nabla\phi(\frac{x_j-x}{\varepsilon})|^p|u|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{B(x_j,\varepsilon)} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \varepsilon^{-1}\left(\int_{\Omega} |\nabla\phi(\frac{x_j-x}{\varepsilon})|^N dx\right)^{\frac{1}{N}},$$

then

$$\nu_j^{\frac{1}{p^*}} S^{\frac{1}{p}} \leq \mu(B(x_j,\varepsilon))^{\frac{1}{p}} + c\left(\int_{B(x_j,\varepsilon)} |u|^{p^*} dx\right)^{\frac{p}{p^*}}$$

and taking limit for $\varepsilon \to 0$ we obtain.

$$\mu_j \geq \nu_j^{\frac{p}{p^*}} S, \ j \in I,$$

that can be read in an equivalent way as

$$\mu \geq \sum_{j\in I} \nu_j^{\frac{p}{p^*}} S\delta_{x_j} \equiv \mu_1.$$

But because $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $\mu \geq |\nabla u|^p$ and $|\nabla u|^p$ is orthogonal to μ_1, we conclude. ■

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