Maximal coefficients of squares of Newman polynomials

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1 Introduction

Newman polynomials are those with all coefficients in $\{0,1\}$. We consider here the problem of finding Newman polynomials P such that all the coefficients of P^2 are so small as possible for $\deg P$ and $P(1)$ given.

A set $\mathcal{A} \subset [1,N]$ is called a $B_2[g]$ sequence if every integer n has at most g distinct representations as $n = a_1 + a_2$ with $a_1, a_2 \in \mathcal{A}$ and $a_1 \le a_2$. Gang Yu [4] introduced a new idea to obtain the upper bound $|\mathcal{A}| \le \sqrt{3.2\ gN}(1+o(1))$ for any $B_2[g]$ sequence $\mathcal{A} \subset [1,N]$ which improved all the previous ones. It has been conjectured by some authors that the constant 3.2 can be substituted by 2. Gang Yu observed that it would follow from conjecture 1 below.

Conjecture 1 (Gang Yu, [4]) *For any Newman polynomial P with $P(1) = o(\deg P)$, we have $\mathcal{M}(P^2) \gtrsim P^2(1)/\deg(P)$, where $\mathcal{M}(P^2)$ denotes the maximum coefficient of P^2.*

The notation $g(t) \gtrsim h(t)$ means that $\liminf_{t\to\infty} g(t)/h(t) \ge 1$. In the conjecture above the parameter tending to infinity is $\deg(P)$.

Berenhaut and Saidak [1] observed that the condition $P(1) = o(\deg P)$ in needed in Yu's conjecture by exhibiting an infinite sequence P_n of Newman polynomials such that $\mathcal{M}(P_n^2) \lesssim \frac{8}{9}P_n^2(1)/\deg P_n$. Dubikas [2] has improved 8/9 to 5/6. These sequences don't contradict Yu's conjecture, since they satisfy $P_n(1) \sim \frac{3}{4} \deg P_n$ and $P_n(1) \sim \frac{3}{5} \deg P_n$, respectively.

The aim of this note is to prove the next theorem, which in particular disproves Yu's conjecture.

Theorem 2 *There exists an infinite sequence of Newman polynomials P_n with $P_n(1) = o(\deg P_n)$ and such that*

$$\limsup_{n\to\infty} (\deg P_n)\mathcal{M}(P_n^2)/P_n^2(1) \le \pi/4.$$

We remark that the sequence P_n in theorem 2 satisfies not only $P_n(1) = o(\deg P_n)$ but $P_n(1) = O((\deg P_n)^{1/2}(\log(\deg P_n))^\beta)$ for any given $\beta > 1/2$. This growing is close to the best possible because it is easy to see that theorem 2 fails if we take $P_n(1) = o((\deg P_n)^{1/2})$, even if we allow to substitute $\pi/4$ for any greater constant. To see this, we observe that if $P_n(1) = o((\deg P_n)^{1/2})$ then $P_n^2(1)/\deg P_n = o(1)$ but clearly $\mathcal{M}(P_n^2) \ge 1$.

Our proof is based on the classic probabilistic method established by Erdős.

2 Proof of theorem 1.1

For any $\beta > 1/2$ we define an infinite sequence of positive integers $\mathcal{A}$ randomly by choosing each number i to be in $\mathcal{A}$ with probability

$$\mathbb{P}\left(i \in \mathcal{A}\right) = \frac{(\log i)^\beta}{\sqrt{i}}, \qquad i \in \mathbb{N}.$$

Let t_i the boolean random variable with values 1 or 0 according to $i \in \mathcal{A}$ or not. We consider the random formal polynomial $P(x) = \sum_i t_i x^i$. The coefficient of x^m in $P^2(x)$ is the random variable

$$Y_m = \sum_{i_1+i_2=m} t_{i_1} t_{i_2}. \tag{1}$$

Now, to each random polynomial $P(x)$ we associate the sequence of polynomials $\{P_n\}$ defined by

$$P_n(x) = \sum_{i \le n} t_i x^i. \tag{2}$$

The proof of theorem 2 will be accomplished by showing that this random sequence of polynomials satisfies the statement of theorem 2 with positive probability. The proof of this fact will follow from proposition 1 and proposition 2 below, whose proof, in turn, will be obtained by using a special case of Chernoff's inequality (see for example [3], Corollary 1.9).

Theorem 3 (Chernoff's inequality) *If X is a sum of independent boolean variables, then for any $0 < \epsilon < 1$,*

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \ge \epsilon \mathbb{E}(X)\right) \le 2e^{-\epsilon^2 \mathbb{E}(X)/4}.$$

Proposition 1 *Let $P_n(x)$ the random polynomials defined in (2). Then, for any $\epsilon > 0$ there exists N such that*

$$\mathbb{P}\left(P_n(1) > (1-\epsilon) 2n^{1/2} (\log n)^\beta \textit{ for all } n > N\right) \ge 0.9$$

Proof Fix $0 < \epsilon_0 < 1$. If we apply theorem 3 to $X = P_n(1)$ we obtain

$$\mathbb{P}\left(P_n(1) < (1-\epsilon_0)\mathbb{E}\left(P_n(1)\right)\right) \le 2e^{-\epsilon_0^2 \mathbb{E}(P_n(1))/4}. \tag{3}$$

Since $\mathbb{E}(P_n(1)) = \sum_{i \le n} \frac{(\log i)^\beta}{\sqrt{i}} \sim 2\sqrt{n}(\log n)^\beta$, there exists N_1 such that for any $n \ge N_1$

$$\mathbb{E}(P_n(1)) \ge (1-\epsilon_0) 2\sqrt{n}(\log n)^\beta. \tag{4}$$

By (3) and (4) we obtain, for $n \ge N_1$,

$$\begin{aligned}\mathbb{P}\left(P_n(1) \le (1-\epsilon_0)^2 2\sqrt{n}(\log n)^\beta\right) &\le \mathbb{P}\left(P_n(1) < (1-\epsilon_0)\mathbb{E}\left(P_n(1)\right)\right) \\ &\le 2e^{-\epsilon_0^2 \mathbb{E}(P_n(1))/4} \\ &\le 2e^{-\epsilon_0^2 (1-\epsilon_0)\sqrt{n}(\log n)^\beta/2}.\end{aligned}$$

Clearly
$$\sum_{n=1}^{\infty} \mathbb{P}\left(P_n(1) < (1-\epsilon_0)^2 2\sqrt{n}(\log n)^{\beta}\right) < \infty$$
and we can apply the Borel Cantelli lemma to deduce that there exists N such that
$$\mathbb{P}\left(P_n(1) > (1-\epsilon_0)^2 2\sqrt{n}(\log n)^{\beta} \text{ for all } n > N\right) \geq 0.9.$$
Finally we take $\epsilon_0 = 1 - \sqrt{1-\epsilon}$.

Proposition 2 *Let Y_m the random variable defined in (1). Then, for any $\epsilon > 0$ there exists M such that*

$$\mathbb{P}\left(Y_m < \pi(1+\epsilon)(\log m)^{2\beta} \textit{ for all } m > M\right) \geq 0.9$$

Proof Write
$$Z_m = \sum_{\substack{i_1+i_2=m \\ i_1 \leq i_2}} t_{i_1} t_{i_2},$$
so $Y_m = 2Z_m - t_{m/2}$. We observe that Z_m is a sum of independent boolean variables, because if $i_1+i_2 = i_1'+i_2'$, then $\{i_1, i_2\} = \{i_1', i_2'\}$ or $\{i_1, i_2\} \cap \{i_1', i_2'\} = \emptyset$. Write $\delta_m = 1$ if m is even and $\delta_m = 0$ if m is odd. We observe that
$$\begin{aligned}
\mathbb{E}(Z_m) &= \sum_{\substack{i+j=m \\ 1 \leq i < j}} \frac{(\log i)^{\beta}}{\sqrt{i}} \frac{(\log j)^{\beta}}{\sqrt{j}} + \delta_m \frac{(\log(m/2))^{\beta}}{\sqrt{m/2}} \\
&= \sum_{1 \leq i < m/2} \frac{(\log i)^{\beta}(\log(m-i))^{\beta}}{\sqrt{i(m-i)}} + \delta_m \frac{(\log(m/2))^{\beta}}{\sqrt{m/2}} \sim (\pi/2)(\log m)^{2\beta}
\end{aligned}$$
and so, for any $\epsilon_0 > 0$, there exists M_1 such that
$$(\pi/2)(1-\epsilon_0)(\log m)^{2\beta} \leq \mathbb{E}(Z_m) \leq (\pi/2)(1+\epsilon_0)(\log m)^{2\beta} \tag{5}$$
for any $m \geq M_1$. Then we can apply (5) and theorem 3 with $X = Z_m$ to deduce that
$$\begin{aligned}
\mathbb{P}\left(Z_m > (\pi/2)(1+\epsilon_0)^2(\log m)^{2\beta}\right) &\leq \mathbb{P}\big(Z_m > (1+\epsilon_0)\mathbb{E}(Z_m)\big) \\
&\leq 2e^{-\epsilon_0^2 \mathbb{E}(Z_m)/4} \\
&\leq 2e^{-C(\log m)^{2\beta}}
\end{aligned}$$

for $m > M_1$, where $C = \epsilon_0^2(1-\epsilon_0)\pi/8$. Since $2\beta > 1$ and $Y_m \le 2Z_m$ we have that

$$\sum_m \mathbb{P}\left(Y_m > \pi(1+\epsilon_0)^2(\log m)^{2\beta}\right) < \infty$$

and we can apply again the Borel-Cantelli lemma to deduce that there exists M such that

$$\mathbb{P}\left(Y_m < \pi(1+\epsilon_0)^2(\log m)^{2\beta_2} \text{ for all } m > M\right) > 0.9.$$

We complete the proof of proposition 2 by taking $\epsilon_0 = 1 - \sqrt{1-\epsilon}$.

To conclude the proof of theorem 2, it is clear that $\mathcal{M}(P_n^2) \le \max_{m\le 2n} Y_m$ and so, with probability > 0.9 we have that

$$\mathcal{M}(P_n^2) \le \max_{m\le M} Y_m + \max_{M<m<2n} Y_m \le M + \pi(1+\epsilon)(\log(2n))^{2\beta}.$$

Then, for any $\epsilon > 0$, for n large enough, say $n \ge N_2$, and with probability > 0.9, we have

$$\mathcal{M}(P_n^2) \le \pi(1+2\epsilon)(\log n)^{2\beta}. \tag{6}$$

On the other hand we know that with probability > 0.9, we have

$$P_n(1) > (1-\epsilon)2\sqrt{n}(\log n)^{\beta} \tag{7}$$

for all $n > N$. Then, with probability > 0.8, both (6) and (7) hold simultaneously and hence

$$\mathcal{M}(P_n^2) \le (\pi/4)\frac{1+2\epsilon}{(1-\epsilon)^2}P_n^2(1)/n \tag{8}$$

for all $n \ge \max\{N_2, N\}$. We finish the proof by observing that $n \ge \deg P_n$ and that we can take ϵ arbitrarily small.

References

[1] K.S. Berenhaut and F. Saidak, A note on the maximal coefficients of squares of Newmann polynomials, *Journal of Number Theory* **125** (2007), 285-288.

[2] A. Dubikas, Heights of powers of Newman and Littlewood polynomials, *Acta Arithmetica* **128** (2007), 167-176.

[3] T. Tao and V. Vu, Additive combinatorics, *Cambridge studies in advanced mathematics* **105** (2006)

[4] G. Yu, An upper bound for $B_2[g]$ sets, *Journal of Number Theory* **122** (2007), 211-220.