THE LEAST COMMON MULTIPLE OF A QUADRATIC SEQUENCE

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ABSTRACT. We obtain, for any irreducible quadratic polynomial $f(x) = ax^2 + bx + c$, the asymptotic estimate log l.c.m. $\{f(1), \dots, f(n)\} \sim n \log n$. When $f(x) = ax^2 + c$ we prove the more precise estimate, log l.c.m. $\{f(1), \dots, f(n)\} = n \log n + Bn + o(n)$ for a suitable constant $B = B(a, c)$.

1. Introduction

It is well known that $\log \text{l.c.m.}\{1, \dots, n\} \sim n$. Actually it is equivalent to the prime number theorem. The analogous for arithmetic progressions is also known [2],

$$\log \text{l.c.m.}\{a+b, \dots, an+b\} \sim n \frac{a}{\varphi(a)} \sum_{\substack{1 \le k \le a \\ (k,a)=1}} \frac{1}{k}.$$

We address here the problem of estimate $L_n(f) = \text{l.c.m.}\{f(1), \dots, f(n)\}$, where f is an irreducible quadratic polynomial in $\mathbb{Z}$. The case $f(x) = x^2 + 1$ has been considered in [1] where the estimate $\log L_n(f) = \log \text{l.c.m.}\{f(1), \dots, f(n)\} \ge An + B$ was obtained in this case for explicit constants $A, B > 0$.

In section §2 we give an asymptotic estimate for a general irreducible quadratic polynomial $f(x) = ax^2 + bx + c$.

Theorem 1.1. *For any irreducible quadratic polynomial we have*

$$\log \ \textit{l.c.m.} \ \{f(1), \dots, f(n)\} \sim n \log n.$$

In section §3, which is the main part of this work, we obtain a more precise estimate in some particular cases.

Theorem 1.2. *For any irreducible polynomial $f(x) = ax^2 + c$ we have*

$$\log \textit{l.c.m.} \ \{f(1), \dots, f(n)\} = n \log n + Bn + o(n),$$

for an explicit constant $B = B(a, c)$.

It includes the simplest case $f(x) = x^2 + 1$ with

$$B(1,1) = -1 - \frac{3 \log 2}{2} - \lim_{t \to \infty} \left(\sum_{\substack{p \le t \\ p \equiv 1 \pmod 4}} \frac{2 \log p}{p - 1} - \log t \right).$$

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To prove theorem 1.2 we need to use a deep result about the distribution of the solutions of the quadratic congruences $f(x) \equiv 0 \pmod p$ due to Duke, Friedlander and Iwaniec (for $D > 0$), and Toth (for $D < 0$), where $D = b^2 - 4ac$ is the discriminant of f.

Theorem 1.3 ([3], [4]). *For any irreducible quadratic polynomial f, the sequence $\{\nu/p,\ 0 \le \nu < p \le x,\ f(\nu) \equiv 0 \pmod p\}$ is well distributed in $[0,1)$ when $x \to \infty$.*

It would be interesting to extend these estimates to irreducible polynomials of higher degree, but we have found a serious obstruction in our argument. Some heuristic arguments allow us to conjecture that the asymptotic

$$\log \text{l.c.m. } \{f(1), \dots, f(n)\} \sim (\deg(f) - 1) n \log n$$

holds for any irreducible polynomial f.

2. Preliminaries and the general case

For any irreducible $f(x) = ax^2 + bx + c$ we write $P_n(f) = \prod_{i=1}^n f(i) = \prod_p p^{\alpha_p}$ and $L_n(f) = \text{l.c.m.}\{f(1), \dots, f(n)\} = \prod_p p^{\beta_p}$.

Lemma 2.1. *With the notation above we have that $\alpha_p = \beta_p$ for $p \ge 2an + b$.*

Proof. Notice that $\alpha_p \ne \beta_p$ if and only if there exist $i, j,\ i < j \le n$ such that $p|f(i)$ and $p|f(j)$. In that case we have $p|f(j) - f(i)$, so $p|(j-i)(a(j+i)+b)$ and then $p < 2an + b$. □

For short we write $L_n = L_n(f)$ and $P_n = P_n(f)$. The lemma above allow us to write

$$\log L_n = \log P_n + \sum_{p < 2an+b} (\beta_p - \alpha_p) \log p. \tag{2.1}$$

We have $\log P_n = \sum_{i \le n} \log(ai^2 + bi + c) = n \log a + \sum_{i \le n} 2 \log i + O(\sum_{i \le n} 1/i)$, so

$$\log P_n = 2n \log n + (\log a - 2)n + o(n). \tag{2.2}$$

Notice also that if $p^\beta | f(i)$ for some $i \le n$, then $\beta \le \log f(n)/\log p = O(\log n/\log p)$. Thus $\sum_{p < 2an+b} \beta_p \log p = O(\log n\ \pi(2an+b)) = O(n)$ and then

$$\log L_n = 2n \log n - \sum_{p \le 2an+b} \alpha_p \log p + O(n). \tag{2.3}$$

We observe that we can write $\alpha_p = \sum_{x \le n} \nu_p(f(x))$, where $\nu_p(m)$ denotes the maximum l such that $p^l | m$. As $\nu_p(m) = \sum_k \chi_{p^k}(m)$, with $\chi_{p^k}(m) = \begin{cases} 1, & p^k | m \\ 0, & \text{otherwise} \end{cases}$, we have

$$\alpha_p = \sum_k \sum_{x \le n} \chi_{p^k}(f(x)) = \sum_k \sum_{\substack{x \le n \\ p^k | f(x)}} 1 \tag{2.4}$$

and we obtain the trivial estimate

$$s(f;p^k)[n/p^k] \le \sum_{\substack{x \le n \\ p^k | f(x)}} 1 \le s(f;p^k)\left([n/p^k] + 1\right), \tag{2.5}$$

where $s(f;p^k)$ denotes the number of solutions of $f(x) \equiv 0 \pmod{p^k}$, $0 \le x < p^k$. Since $k \le \log f(n)/\log p$ we have

$$\alpha_p = n \sum_{k \le \log(f(n))/\log p} \frac{s(f,p^k)}{p^k} + O\left(\sum_{k \le \log f(n)/\log p} s(f;p^k) \right). \tag{2.6}$$

Lemma belove resumes all the casuistic for $s(f,p^k)$.

Lemma 2.2. *Let $f(x) = ax^2+bx+c$ be an irreducible polynomial and $D = b^2-4ac$.*

(1) *If $p \not| 2a$, $D = p^l D_p$, $(D_p,p) = 1$,then*

$$s(f,p^k) = \begin{cases} p^{k-\lceil k/2 \rceil}, & k \le l \\ 0, & k > l,\ l \text{ odd or } (D_p/p) = -1 \\ 2p^{l/2}, & k > l,\ l \text{ even } \ (D_p/p) = 1 \end{cases}.$$

(2) *If $p|a$, $p \ne 2$ then* $s(f,p^k) = \begin{cases} 0, & p|b \\ 1, & p \not| b \end{cases}$

(3) *If b is odd then $s(f,2^k) = s(f,2)$ for any $k \ge 2$.*

(4) *If b is even, let $D = 4^l D'$, $D' \not\equiv 0 \pmod 4$.*

(a) *If $D' \equiv 2,3 \pmod 4$,* $s(f;2^k) = \begin{cases} 2^{[k/2]}, & k \le 2l-1 \\ 0, & k \ge 2l \end{cases}$

(b) *If $D' \equiv 1 \pmod 8$,* $s(f;2^k) = \begin{cases} 2^{[k/2]}, & k \le 2l \\ 2^{l+1}, & k \ge 2l+1 \end{cases}$

(c) *If $D' \equiv 5 \pmod 8$,* $s(f;2^k) = \begin{cases} 2^{[k/2]}, & k \le 2l \\ 0, & k \ge 2l+1 \end{cases}$

Proof. The proof is a consequence of elementary manipulations and Hensel's lemma. □

Since for any prime $s(f;p^k)$ is bounded we have

$$\alpha_p = n \sum_{k \ge 1} \frac{s(f,p^k)}{p^k} + O(\log n/\log p) \tag{2.7}$$

which gives the trivial estimate $\alpha_p = O(n/p)$ for any prime p.

If $p \not| 2aD$ then $s(f;p^k) = 2$ or 0 according with $(D/p) = 1$ or -1. Then, for these primes we have

$$\alpha_p = \begin{cases} 0, & (D/p) = -1 \\ \frac{2n}{p-1} + O\left(\frac{\log n}{\log p}\right), & (D/p) = 1. \end{cases} \tag{2.8}$$

Now we put together the estimate $\alpha_p = O(n/p)$ for the primes $p|2aD$, the estimate $\pi(x) \ll x/\log x$ and the formulas (2.3) and (2.8) to obtain

$$\log L_n = 2n\log n - 2n \sum_{\substack{p \le 2an+b \\ (D/p)=1}} \frac{\log p}{p-1} + O(n). \tag{2.9}$$

The quadratic reciprocity law allow us to split the residues d, $(d,D) = 1$ in two sets D_1, D_{-1} of the same size, $\varphi(D)/2$, such that $(D/p) = 1 \iff p \equiv d \pmod D$

with $d \in D_1$. The prime number theorem for arithmetic progressions says that

$$\sum_{\substack{p \le t \\ p \equiv d \pmod D}} \log p = \frac{t}{\varphi(D)} + O(t/\log^2 t) \tag{2.10}$$

for $d \in D_1$. Then

$$\sum_{\substack{p \le t \\ (D/p)=1}} \log p = \frac{t}{2} + O(t/\log^2 t). \tag{2.11}$$

Actually a better error term is known in (2.10) and (2.11), but this one is enough to deduce, by partial summation, that

$$\sum_{\substack{p \le t \\ (D/p)=1}} \frac{\log p}{p-1} = \frac{\log t}{2} + A_D + o(1) \tag{2.12}$$

for a constant A_D. We finish the proof of theorem 1.1 by performing a substitution in (2.9) with the formula above.

3. A more precise estimate for $f(x) = ax^2 + c$

We can strength lemma 2.1 when $b = 0$.

Lemma 3.1. *If $f(x) = ax^2 + c$ we have that $\alpha_p = \beta_p$ for $p \ge 2n$.*

Proof. If $\alpha_p > \beta_p$ then there exists $i < j \le n$ such that $p|ai^2 + c$ and $p|aj^2 + c$. But it implies that $p|a(i-j)(i+j)$, so $p|i-j$, $p|a$ or $p|i+j$. In any case $p < 2n$. □

Then we can write

$$\log L_n = \log P_n + \sum_{p<2n} (\beta_p - \alpha_p) \log p. \tag{3.1}$$

Define $\mathcal{Q} = \{p,\ p|2c\}$ and $\mathcal{P} = \{p < 2n,\ p \not| 2ac,\ (-ac/p) = 1\}$. Lemma 2.2 implies that if $p \notin \mathcal{P} \cup \mathcal{Q}$ then $\beta_p = \alpha_p = 0$. We define also the bad and the good primes as

$$\mathcal{P}_{bad} = \{p \in \mathcal{P},\ p^2|ai^2 + c \text{ for some } i \le n\} \text{ and } \mathcal{P}_{good} = \mathcal{P} \setminus \mathcal{P}_{bad}. \tag{3.2}$$

Lemma 3.2. *Suppose that $p \in \mathcal{P}_{good}$. Then*

i) *$\alpha_p = \frac{2n}{p} + z_p$ where $z_p = 1 - \{\frac{n+x_p}{p}\} - \{\frac{n-x_p}{p}\}$, where x_p denotes the solution of $ax^2 + c \equiv 0 \pmod p$ such that $0 < x \le (p-1)/2$*

ii) *$\beta_p = 1$ for $p < 2n$.*

Proof. i) Since p is a good prime we have

$$\alpha_p = \#\{l,\ x_p + (l-1)p \le n\} + \#\{l,\ p - x_p + (l-1)p \le n\}.$$

So $\alpha_p = [\frac{n-x_p}{p}] + 1 + [\frac{n+x_p}{p}] = \frac{2n}{p} + z_p$ where $z_p = 1 - \{\frac{n+x_p}{p}\} - \{\frac{n-x_p}{p}\}$.
ii) Since p is good we have always that $\beta_p \le 1$. On the other hand, since $x_p \le (p-1)/2 < n$ and $p|ax_p^2 + c$ we have that $\beta_p \ge 1$. □

Lemma 3.3. *For any $J \ge 1$ we have $\#\{p,\ n/J < p \le 2n,\ p \in \mathcal{P}_{bad}\} \ll J^3$.*

Proof. If $p > n/J$ is bad, then there exists $i \leq n$ such that $ai^2 + c = p^2 r$ for some $2 \leq r \leq aJ^2$. For each r, $2 \leq r \leq aJ^2$, consider $P_r = \{p > n/J,\ ai_p^2 + c = p^2 r,\ \text{ for some } i_p \leq n\}$. If $p \in P_r$ we have that $|\frac{\sqrt{r}}{\sqrt{a}} - \frac{i_p}{p}| \leq \frac{|c|}{p^2\sqrt{ra}} \leq \frac{|c|J^2}{n^2\sqrt{ra}}$ and then all i_p/p lie on an interval of length $\frac{2|c|J^2}{n^2\sqrt{ra}}$. On the other hand, $|\frac{i_p}{p} - \frac{i'_p}{p'}| \geq \frac{1}{pp'} \geq \frac{1}{4n^2}$, so $|P_r| \leq \frac{8|c|J^2}{\sqrt{ra}} + 1$ and $\sum_{r \leq J^2} |P_r| \ll J^3$. □

We fix a large integer J and use the lemmas above to write

$$\sum_{p<2n} \beta_p \log p = \sum_{\substack{p<2n \\ p|2c}} \beta_p \log p + \sum_{\substack{p<2n \\ p\in\mathcal{P}}} \log p + \sum_{\substack{p\leq n/J \\ p\in\mathcal{P}}} (\beta_p - 1)\log p + \sum_{\substack{n/J<p<2n \\ p\in\mathcal{P}}} (\beta_p - 1)\log p =$$

$$O(\log n) + \sum_{\substack{p<2n \\ p\in\mathcal{P}}} \log p + \sum_{\substack{p\leq n/J \\ p\in\mathcal{P}}} O(\log n) + \sum_{\substack{n/J<p<2n \\ p \text{ bad}}} O(\log n).$$

By (2.11) and lemma 3.2 we have

$$\sum_{p<2n} \beta_p \log p = n + o(n) + O\left(\frac{n}{J\log(n/J)} \log n\right) + O(J^3 \log n), \tag{3.3}$$

when $n \to \infty$. Now we write

$$\sum_{p<2n} \alpha_p \log p = \sum_{p|2c} \alpha_p \log p + \sum_{\substack{p<n/J \\ p\in\mathcal{P}}} \left(2n\frac{\log p}{p-1} + O(\log n)\right) + \tag{3.4}$$

$$\sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} \left(\frac{2n}{p} + z_p\right)\log p + O\left(\sum_{\substack{n/J\leq p<2n \\ p \text{ bad}}} \frac{n}{p}\log p\right) = \tag{3.5}$$

$$nC(a,c) + O(\log n) + 2n \sum_{\substack{p<2n \\ p\in\mathcal{P}}} \frac{\log p}{p-1} + O\left(\frac{n}{J\log(n/J)}\log n\right) \tag{3.6}$$

$$-2n \sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} \frac{\log p}{p(p-1)} + \sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} z_p \log p + O(J^4 \log n), \tag{3.7}$$

where

$$C(a,c) = \sum_{p|2c} \log p \sum_{k\geq 1} \frac{s(f;p^k)}{p^k}. \tag{3.8}$$

We use (2.12) and the estimates

$$\sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} \frac{\log p}{p(p-1)} = O\left(\frac{\log(n/J)}{n/J}\right), \tag{3.9}$$

$$\sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} z_p \log p = \log n \sum_{\substack{n/J\leq p<2n \\ p\in\mathcal{P}}} z_p + O\left(\log J \frac{n}{\log n}\right) \tag{3.10}$$

to obtain

$$\sum_{p<2n} \alpha_p \log p = n\log n + (C(a,c) + \log 2 + 2A_D)n + \tag{3.11}$$

$$\log n \sum_{\substack{n/J\le p<2n\\ p\in\mathcal{P}}} z_p + o(n) + O(n/J) \tag{3.12}$$

when $n\to\infty$. Notice that most of the error terms have been included in $o(n)$.

To estimate $\sum_{\substack{n/J\le p<2n\\ p\in\mathcal{P}}} z_p$ we first split the primes $n/J < p < 2n$ in short intervals $[n/J, nH/J]$ and $I_j = (\frac{j-1}{J}n, \frac{j}{J}n]$, $H < j < 2J$ to write

$$\sum_{\substack{n/J\le p<2n\\ p\in\mathcal{P}}} z_p = \sum_{\substack{n/J\le p\le nH/J\\ p\in\mathcal{P}}} z_p + \sum_{H<j\le 2J}\sum_{\substack{p\in I_j\\ p\in\mathcal{P}}} z_p = \tag{3.13}$$

$$\sum_{H<j\le 2J}\sum_{\substack{p\in I_j\\ p\in\mathcal{P}}} z_p + O(\frac{nH}{J\log(nH/J)}) \tag{3.14}$$

where H is an integer which will be chosen later.

For $p\in I_j,\ j>H$ we can write $\frac{n}{p} = t_j - \epsilon_j(p)$ where $t_j = \frac{J}{j-1}$ and $\epsilon_j(p) = \frac{pJ-(j-1)n}{p(j-1)}$. Notice that $0\le \epsilon_j(p) \le \frac{J}{(j-1)^2} \le \frac{J}{H^2}$. Then we have

$$z_p = 1 - \{t_j + x_p/p + \epsilon_j(p)\} - \{t_j - x_p/p + \epsilon_j(p)\}.$$

We denote by E_j the set of the primes $p\in I_j$ such that

$$\{t_j\}\le x_p/p \le \{t_j\} + \frac{J}{H^2} \qquad \text{or} \qquad 1-\{t_j\}\le x_p/p\le 1-\{t_j\}+\frac{J}{H^2}. \tag{3.15}$$

If $p\in I_j\setminus E_j$ we have that $z_p = 1-\{t_j+x_p/p\}-\{t_j-x_p/p\}-2\epsilon_j(p)$, so

$$z_p = 1-\{t_j+x_p/p\}-\{t_j-x_p/p\}+O(J/H^2) \tag{3.16}$$

for these primes. For primes $p\in E_j$ it is useful to write

$$z_p = 1-\{t_j+x_p/p\}-\{t_j-x_p/p\}+O(1). \tag{3.17}$$

Theorem 1.3 implies that the sequences $\{t_j + x_p/p\}$ and $\{t_j - x_p/p\}$ are well distributed on $\mathcal{M}_j^+$ and $\mathcal{M}_j^-$ respectively, where $\mathcal{M}_j^\pm = t_j \pm [0,1/2) \pmod 1$. Observe also that $\mathcal{M}_j^+\cup\mathcal{M}_j^- = [0,1)$. Then we have that

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}}\{t_j+x_p/p\} = 2\int_{\mathcal{M}_j^+} sds\ \pi(\mathcal{P};y) + o(\pi(\mathcal{P};y)), \tag{3.18}$$

where $\pi(\mathcal{P};y) = \sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1$. For the same reason we have that

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}}\{t_j-x_p/p\} = 2\int_{\mathcal{M}_j^-} sds\ \pi(\mathcal{P};y) + o(\pi(\mathcal{P};y)) \tag{3.19}$$

and then

$$\sum_{p\le y,\ p\in\mathcal{P}} (1-\{t_j+x_p/p\}-\{t_j-x_p/p\}) = \tag{3.20}$$

$$\left(1-2\int_{[0,1)} sds\right)\pi(\mathcal{P};y)+o(\pi(\mathcal{P};y)) = o(y/\log y). \tag{3.21}$$

In particular we have that

$$\sum_{p\in I_j\cap\mathcal{P}} (1-\{t_j+x_p/p\}-\{t_j-x_p/p\}) = o(n/\log n). \tag{3.22}$$

So, if $j\le H$, by (3.16), (3.17) and (3.22) we obtain

$$\sum_{p\in I_j\cap\mathcal{P}} z_p = o(n/\log n)+O\left(\frac{J}{H^2}\sum_{p\in I_j}1\right)+O(|E_j|) \tag{3.23}$$

and then,

$$\sum_{H\le j\le 2J}\sum_{p\in I_j\cap\mathcal{P}} z_p = o(Jn/\log n)+O\left(\frac{J}{H^2}\frac{n}{\log n}\right)+O(\sum_{H\le j\le 2J}|E_j|). \tag{3.24}$$

Since x_p/p is well distributed we have that

$$|E_j| = \frac{2J}{H^2}\left(\pi(\mathcal{P},nj/J)-\pi(\mathcal{P},n(j-1)/J)+o(\pi(\mathcal{P},nj/J)\right).$$

Hence

$$\sum_{H<j\le 2J}|E_j| = \frac{2J}{H^2}\left(\pi(\mathcal{P};2n)-\pi(\mathcal{P};nH/J)\right)+o(Jn/\log n). \tag{3.25}$$

If we take $H=[J^{2/3}]$, formulas (3.25), (3.24), (3.13) and (3.14) give

$$\sum_{\substack{n/J\le p<2n\\ p\in\mathcal{P}}} z_p = O\left(\frac{n}{J^{1/3}\log n}\right)+o(n/\log n). \tag{3.26}$$

This and (3.11) yield

$$\sum_{p<2n}\alpha_p\log p = n\log n+(C(a,c)+\log 2+2A_D)n+O(n/J^{1/3})+o(n). \tag{3.27}$$

Putting (3.27), (3.3), (2.1) and (2.2), we have finally

$$\log L_n = n\log n+B(a,c)n+o(n)+O(n/J^{1/3})$$

where

$$B(a,c)=\log a-1-\log 2-\sum_{p|2c}\log p\sum_{k\ge1}\frac{s(ax^2+c;p^k)}{p^k}-\lim_{t\to\infty}\left(\sum_{p\in\mathcal{P}}\frac{2\log p}{p-1}-\log t\right)$$

and we finish the proof of theorem 1.2 observing that we can choose J arbitrary large.

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