ON A QUESTION OF SÁRKÖZY

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Abstract. Motivated by a question of Sárközy, we study the gaps in the product sequence $\mathcal{B} = \mathcal{A} \cdot \mathcal{A} = \{b_n = a_i a_j,\ a_i, a_j \in \mathcal{A}\}$ when $\mathcal{A}$ has upper Banach density $\alpha > 0$. We prove that there are infinitely many gaps $b_{n+1} - b_n \ll \alpha^{-3}$ and that for $t \geq 2$ there are infinitely many t-gaps $b_{n+t} - b_n \ll t^2\alpha^{-4}$. Furthermore we prove that these estimates are best possible.

We also discuss a related question about the cardinality of the quotient set $\mathcal{A}/\mathcal{A} = \{a_i/a_j,\ a_i, a_j \in \mathcal{A}\}$ when $\mathcal{A} \subset \{1, \ldots, N\}$ and $|\mathcal{A}| = \alpha N$.

1. Introduction

Let $\mathcal{A} = \{a_1 < a_2 < \ldots\}$ be an infinite sequence of positive integers. The lower and upper asymptotic densities of $\mathcal{A}$ are defined by

$$\underline{d}(\mathcal{A}) = \liminf_{N\to\infty} \frac{|\mathcal{A} \cap \{1, \ldots, N\}|}{N} \quad \text{and} \quad \overline{d}(\mathcal{A}) = \limsup_{N\to\infty} \frac{|\mathcal{A} \cap \{1, \ldots, N\}|}{N}.$$

The lower and upper Banach density of $\mathcal{A}$ are defined by

$$d_*(\mathcal{A}) = \liminf_{|I|\to\infty} \frac{|\mathcal{A} \cap I|}{|I|} \quad \text{and} \quad d^*(\mathcal{A}) = \limsup_{|I|\to\infty} \frac{|\mathcal{A} \cap I|}{|I|}$$

where I runs through all intervals. Clearly $d_*(\mathcal{A}) \leq \underline{d}(\mathcal{A}) \leq \overline{d}(\mathcal{A}) \leq d^*(\mathcal{A})$.

Sárközy considered the set

$$\mathcal{B} = \mathcal{A} \cdot \mathcal{A} = \{b_1 < b_2 < \ldots\}$$

of all products $a_i a_j$ with $a_i, a_j \in \mathcal{A}$ and asked the following question, stated as problem 22 in [4].

Question 1. *Is it true that for all $\alpha > 0$ there is a number $c = c(\alpha) > 0$ such that if $\mathcal{A} \subset \mathbb{N}$ is an infinite sequence with $\underline{d}(\mathcal{A}) > \alpha$, then $b_{n+1} - b_n \leq c$ holds for infinitely many n?*

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This question is not trivial, since for any $0<\alpha<1$ and $\epsilon>0$ there is a sequence $\mathcal{A}$ such that $\underline{d}(\mathcal{A})>\alpha>0$ but $\overline{d}(\mathcal{B})<\epsilon$, thus the gaps of $\mathcal{B}$ are greater than $\frac{1}{\epsilon}$ on average. See the construction in [1].

Bérczi [1] answered Sárközy's question in the affirmative by proving that $c(\alpha)\ll\alpha^{-4}$. Sándor [3] improved it to $c(\alpha)\ll\alpha^{-3}$ even assuming the weaker hypothesis $\overline{d}(\mathcal{A})>\alpha$.

In this work we consider Sárközy's question for the upper Banach density, that is to find a constant $c^*(\alpha)$ such that $b_{n+1}-b_n\le c^*(\alpha)$ infinitely often whenever $d^*(\mathcal{A})>\alpha$. In this setting we can find the best possible value for $c^*(\alpha)$ up to a multiplicative constant.

Theorem 1. *For every $0<\alpha<1$ and every sequence $\mathcal{A}$ with $d^*(\mathcal{A})>\alpha$, we have $b_{n+1}-b_n\le 2^4\alpha^{-3}$ infinitely often.*

Theorem 2. *For every $0<\alpha<1$, there exists a sequence $\mathcal{A}$ such that $d^*(\mathcal{A})>\alpha$ and $b_{n+1}-b_n\ge 2^{-12}\alpha^{-3}$ for every n.*

We observe that, since $d^*(\mathcal{A})\ge\overline{d}(\mathcal{A})$, Theorem 1 is stronger than Sándor's result.

We also extend this question and study the difference $b_{n+t}-b_n$ for a fixed t, namely to find a constant $c^*(\alpha,t)$ such that $b_{n+t}-b_n\le c^*(\alpha,t)$ infinitely often. The theorems above correspond to the case $t=1$. For greater t the answer is perhaps surprising, in that the exponent of α involved in $c^*(\alpha,t)$ is -4, not -3 like in the case $t=1$.

Theorem 3. *For every $0<\alpha<1$, every $t\ge 2$ and every sequence $\mathcal{A}$ with $d^*(\mathcal{A})>\alpha$, we have $b_{n+t}-b_n\le 2^5t^2\alpha^{-4}$ infinitely often.*

Theorem 4. *For every $0<\alpha<1$ and every $t\ge 2$, there is a sequence $\mathcal{A}$ such that $d^*(\mathcal{A})>\alpha$ and $b_{n+t}-b_n\ge 2^{-22}t^2\alpha^{-4}$ for every n.*

Notation. We will denote by $\lceil x\rceil$ the smallest integer greater or equal to x, $\lfloor x\rfloor$ the greatest integer small than or equal to x. For quantities A,B we write $A\ll B$ if there is an absolute constant $c>0$ such that $A\le cB$.

2. Proof of the results

Proof of Theorem 1. Let $L=\lceil 2\alpha^{-1}\rceil$. Since $d^*(\mathcal{A})\ge\alpha$, there are arbitrarily large intervals in which the density of $\mathcal{A}$ is $\ge\alpha$ and by the pigeonhole principle we can find infinitely many intervals I of length L^2 such that $|I\cap\mathcal{A}|\ge\alpha L^2$.

We divide each interval I into L subintervals of equal length L. For $i=1,\dots,L$, let A_i be the number of elements of $\mathcal{A}$ in the i-th interval. We count the number of differences $a-a'$ where $0<a'<a$ are in the same interval. On the one hand, it is

$$\begin{aligned}\sum_{1\le i\le L}\binom{A_i}{2} &= \frac{1}{2}\sum_{1\le i\le L}(A_i^2-A_i)\ge \frac{1}{2}\left(\frac{1}{L}\left(\sum_{1\le i\le L}A_i\right)^2-\sum_{1\le i\le L}A_i\right)\\ &= \frac{1}{2}\left(\frac{|\mathcal{A}\cap I|^2}{L}-|\mathcal{A}\cap I|\right)=\frac{|\mathcal{A}\cap I|}{2}\left(\frac{|\mathcal{A}\cap I|}{L}-1\right)\\ &\ge \frac{|\mathcal{A}\cap I|}{2}(\alpha L-1)=\frac{|\mathcal{A}\cap I|}{2}\left(\alpha\lceil 2\alpha^{-1}\rceil-1\right)\\ &\ge \frac{|\mathcal{A}\cap I|}{2}\ge\frac{\alpha L^2}{2}\ge L.\end{aligned}$$

On the other hand, the number of their possible values is at most $L-1$. Thus we can find 2 couples $(a,a'),(a'',a''')$ such that $0<a-a'=a''-a'''<L$. Then $0<|aa''-a'a'''|=|aa''-a'(a''+a'-a)|=|(a-a')(a''-a')|\le (L-1)(L^2-1)=(\lceil 2\alpha^{-1}\rceil-1)^2(\lceil 2\alpha^{-1}\rceil+1)\le 4\alpha^{-2}(4\alpha^{-1})=16\alpha^{-3}$. □

Proof of Theorem 3. Let $L=\lceil 4t/\alpha^2\rceil$. Again, since $d^*(\mathcal{A})>\alpha$, there exist infinitely many intervals $I=[x+1,x+L]$ which contains more than αL elements of $\mathcal{A}$. For each interval I, the number of sums $a+a',\ a\le a',\ a,a'\in I\cap\mathcal{A}$ is at least $(\alpha L)^2/2$ and they are all contained in an interval of length $2L$. Since $(\alpha L)^2/2\ge 2Lt$ then some sum must be obtained in t different ways, $a_i+a_i',\ a_i,a_i'\in I\cap\mathcal{A}$. If $i\ne j$ then $0<|a_ia_i'-a_ja_j'|=|a_i'-a_j||a_i-a_j|<L^2$, so the t products a_ia_i' lie in an interval of length $L^2<(4t/\alpha^2+1)^2\le 2^5t^2\alpha^{-4}$. □

For the proofs of Theorems 2 and 4 we will construct a special sequence $\mathcal{A}$ in the following way.

Definition 1. Given a positive value x_1 and an infinite sequence of finite sets of positive integers $\mathcal{A}_1,\mathcal{A}_2,\dots$ we define the associated sequence $\mathcal{A}$ to these inputs by

$$\mathcal{A}=\bigcup_{n=1}^{\infty}(x_n+\mathcal{A}_n). \tag{1}$$

where the sequence (x_n) is defined by

$$x_n=x_1+M_n^2+M_n(x_{n-1}+M_{n-1})+(x_{n-1}+M_{n-1})^2,\ n\ge 2$$

and M_n is the largest element of $\mathcal{A}_n$.

Lemma 1. *Let $\mathcal{A}$ be defined as in (1). Then, all the nonzero differences $d=c_1c_2-c_3c_4$, with $c_1,c_2,c_3,c_4\in\mathcal{A}$ but not all c_i in the same $x_n+\mathcal{A}_n$, satisfy $|d|\ge x_1$.*

Proof. Let n be the largest integer such that $c_i \in x_n + \mathcal{A}_n$ for some $i = 1,2,3,4$. We can assume that $c_1 \in \mathcal{A}_n$. Now we distinguish different cases.

- $c_2 \in x_n + \mathcal{A}_n$ and c_3 or $c_4 \notin x_n + \mathcal{A}_n$. In this case

$$\begin{aligned} |d| \geq x_n^2 - |c_3c_4| &\geq x_n^2 - (x_n + M_n)(x_{n-1} + M_{n-1}) \\ &= x_n(x_n - x_{n-1} - M_{n-1}) - M_n(x_{n-1} + M_{n-1}) \\ &\geq x_n - M_n(x_{n-1} + M_{n-1}) \geq x_1 \end{aligned}$$

- $c_2, c_3, c_4 \notin x_n + \mathcal{A}_n$. In this case

$$|d| \geq x_n - c_3c_4 \geq x_n - (x_{n-1} + M_{n-1})^2 \geq x_1.$$

- $c_3 \in x_n + \mathcal{A}_n$ and $c_2, c_4 \notin x_n + \mathcal{A}_n$. In this case we write $c_1 = x_n + a_1$ and $c_3 = x_n + a_3$. Then $|d| = |x_n(c_2 - c_4) + a_1c_2 - a_3c_4|$.
  If $c_2 = c_4$, then $|d| = c_2|a_1 - a_3| \geq x_1$. If $c_2 \neq c_4$, then

$$|d| \geq x_n - |a_1c_2 - a_3c_4| \geq x_n - M_n(x_{n-1} + M_{n-1}) \geq x_1.$$

□

In order to prove Theorems 2 and 4, we also need the following construction of Sidon sets due to Erdős and Turán [2]:

Lemma 2. *Let p be an odd prime number. Let $\mathcal{S} = \{s_i = 2pi + (i^2)_p : i = 0, \dots, p-1\}$, where $(x)_p \in [0, p-1]$ is the residue of x modulo p. Then $\mathcal{S}$ is a Sidon set in $[0, 2p^2)$ with p elements and $|s_i - s_j| \geq p$ for every $i \neq j$.*

Proof. It is clear that $|s_i - s_j| \geq 2p|i-j| - |(i^2)_p - (j^2)_p| \geq p$. Suppose we have an equation $s_i + s_j = s_k + s_l$ for some i, j, k, l. Then $2p(i + j - k - l) = (i^2)_p + (j^2)_p - (k^2)_p - (l^2)_p$. The left hand side is a multiple of $2p$ while the right hand side is strictly smaller than $2p$. Thus $i + j - j - l = 0$ and $(i^2)_p + (j^2)_p - (k^2)_p - (l^2)_p = 0$, i.e., $i^2 + j^2 \equiv j^2 + l^2 \pmod p$. Thus $i^2 + j^2 - k^2 + l^2 = (i-k)(i+k-j-l) \equiv 0 \pmod p$. Either $i = k$ and $j = l$, or $i + k - j - l \equiv 0 \pmod p$, in which case $k = l$ and $i = j$. □

Proof of Theorem 2. For $\alpha \geq 1/16$ we take $\mathcal{A} = \mathbb{N}$. Obviously $d^*(\mathcal{A}) = 1 \geq \alpha$ and all the gaps in $\mathcal{A} \cdot \mathcal{A}$ are $\geq 1 \geq 2^{-12}\alpha^{-3}$.

For $\alpha < 1/16$, let p be an odd prime such that $\frac{1}{4\alpha} \geq p > \frac{1}{8\alpha}$, $\mathcal{S}$ the Sidon set defined in Lemma 2 and $m = 2p^2$. We consider the sequence $\mathcal{A}$ defined in (1) with $x_1 = 4p^3$ and

$$\mathcal{A}_n = \bigcup_{k=1}^{n} (2km + \mathcal{S}). \tag{2}$$

First we observe that $\mathcal{A}_n$ is contained in the interval $I_n = [2m, 2mn+m)$ and then

$$d^*(\mathcal{A}) \geq \limsup_{n\to\infty} \frac{|\mathcal{A}_n|}{|I_n|} = \limsup_{n\to\infty} \frac{|np|}{|(2m-1)n|} > \frac{1}{4p} \geq \alpha.$$

Next we will prove that all the nonzero differences $d = c_1c_2 - c_3c_4$ with $c_1, c_2, c_3, c_4 \in \mathcal{A}$ satisfy $|d| \geq 4p^3$, and clearly $|d| \geq 2^{-7}\alpha^{-3}$.
By Lemma 1 it is true when not all c_i belong to the same $x_n + \mathcal{A}_n$.
Suppose then that $c_i = x_n + a_i,\ i = 1,2,3,4$. Then

$$\begin{aligned} d &= (x_n + a_1)(x_n + a_2) - (x_n + a_3)(x_n + a_4) \\ &= x_n(a_1 + a_2 - a_3 - a_4) + a_1a_2 - a_3a_4. \end{aligned}$$

If $a_1 + a_2 \neq a_3 + a_4$ then $|d| \geq x_n - |a_1a_2 - a_3a_4| \geq x_n - M_n^2 \geq 4p^3$.
If $a_1 + a_2 = a_3 + a_4$ then $|d| = |a_1 - a_2||a_1 - a_3|$. Now we write

$$a_i = 2k_im + s_i,\ 1 \leq k_i \leq n,\ s_i \in \mathcal{S}.$$

The condition $a_1+a_2 = a_3+a_4$ implies $2m(k_1+k_2-k_3-k_4) = s_3+s_4-s_1-s_2$. Since $|s_1+s_2-s_3-s_4| < 2m$, we have $k_1+k_2 = k_3+k_4$ and $s_1+s_2 = s_3+s_4$. Now we use the fact that $\mathcal{S}$ is a Sidon set to conclude that $\{s_1, s_2\} = \{s_3, s_4\}$. We can assume that $s_1 = s_3$ and $s_2 = s_4$, Then

$$|d| = |2m(k_1 - k_2) + (s_1 - s_2)||2m(k_1 - k_3)|.$$

By Lemma 2 we know that $p \leq |s_1 - s_2| < m$. If $k_1 \neq k_2$ then $|d| \geq |2m - m||2m| = 2m^2 = 8p^4$. If $k_1 = k_2$ then $|d| \geq p2m = 4p^3$. In any case $|d| \geq 4p^3$. □

Proof of Theorem 4. For $\alpha \geq 1/16$ we consider the sequence $\mathcal{A}$ defined in (1) with $x_1 = t^2$ and $\mathcal{A}_n = \{1, \dots, n\}$. Clearly $d^*(\mathcal{A}) = 1 \geq \alpha$.
Next, let $c_0c'_0, \dots, c_tc'_t$ distinct elements in $\mathcal{A} \cdot \mathcal{A}$. We will prove that $|c_ic'_i - c_jc'_j| \geq t^2/36$ for some $i, j,\ i \neq j$.

We have to study the case where all c_i, c'_i belong to the same $x_n + \mathcal{A}_n$. Otherwise we apply Lemma 1.

Clearly it is true for $2 \leq t \leq 6$. Suppose $t \geq 7$. We write

$$\begin{aligned} d_i = c_0c'_0 - c_ic'_i &= (x_n + a_0)(x_n + a'_0) - (x_n + a_i)(x_n + a'_i) \\ &= x_n(a_0 + a'_0 - a_i - a'_i) + a_0a'_0 - a_ia'_i. \end{aligned}$$

If the coefficient of x_n is non zero then $|d_i| \geq x_n - M_n^2 \geq x_1 = t^2$.
We suppose then that $a_0 + a'_0 - a_i - a'_i = 0$ for all i. It implies that $a_i \neq a_j$ if $i \neq j$. Otherwise $c_ic'_i = c_jc'_j$. Then we have $|c_0c'_0 - c_ic'_i| = |a'_0 - a_i||a_0 - a_i|$. Since there are at most $2(1+2(t/6)) < t$ values of i for which $|a_0 - a_i| \leq t/6$ or $|a'_0 - a_i| \leq t/6$ we obtain that $|a'_0 - a_i||a_0 - a_i| > (t/6)^2 \geq 2^{-22}t^2\alpha^{-4}$ for some i.

For $0<\alpha<1/16$ we take the same sequence $\mathcal{A}$ used in the proof of Theorem 2 but with $x_1=t^2p^4$. As we saw, this sequence has density $d^*(\mathcal{A})\ge\alpha$. As in that proof, we apply Lemma 1 to see that if c_i,c_i',c_j,c_j' not in the same $x_n+\mathcal{A}_n$ for some $i\ne j$ then $|c_ic_i'-c_jc_j'|\ge x_1=t^2p^4$ and it is done because $t^2p^4\ge 2^{-12}t^2\alpha^{-4}$.

Therefore, if $c_0c_0',\dots,c_tc_t'$ are different elements of $\mathcal{A}\cdot\mathcal{A}$, we can assume that all c_i,c_i' belong to the same $x_n+\mathcal{A}_n$ and we write them as $c_i=x_n+a_i,\ a_i\in\mathcal{A}_n$. Then

$$d_i=c_0c_0'-c_ic_i'=x_n(a_0+a_0'-a_i-a_i')+a_0a_0'-a_ia_i'$$

If $a_i+a_i'\ne a_0+a_0'$ for some $i\ne 0$ then $|d_i|\ge x_n-M_n^2\ge x_1=t^2p^4$. So we assume that $a_i+a_i'=a_0+a_0'$ for all $i=0,\dots,t$. We write $a_i=2mk_i+s_i$ and we can assume that $s_i\le s_i'$ for $i=0,\dots,t$. The condition $a_i+a_i'=a_0+a_0'$ for all $i=0,\dots,t$ implies that $2m(k_i+k_i'-k_0-k_0')=s_0+s_0'-s_i-s_i'$ and since $|s_0+s_0'-s_i-s_i'|<2m$ then $k_i+k_i'=k_0+k_0'$ and $s_i+s_i'=s_0+s_0'$.

Since S is a Sidon set and $s_i\le s_i'$ we have $s_i=s_0$ and $s_i'=s_0'$ for $i=0,\dots,t$. Then

$$c_ic_i'-c_0c_0'=2m(k_i-k_0)(2m(k_i-k_0')+s_0-s_0').$$

We observe that all k_i are distinct and $k_i\ne 0$. (Otherwise, if $k_i=k_j$ then $k_i'=k_j'$ and then $c_ic_i'=c_jc_j'$.)

If $k_i\ne k_0'$ then

$$\begin{aligned}|c_ic_i'-c_0c_0'| &= |2m(k_i-k_0)(2m(k_i-k_0')+s_0-s_0')|\\ &\ge 2m|k_i-k_0|(2m|k_i-k_0'|-m)\\ &> 2m^2|k_i-k_0||k_i-k_0'|\\ &= 8p^4|k_i-k_0||k_i-k_0'|.\end{aligned}$$

If $2\le t\le 6$ we consider k_1 and k_2. One of them (or both) is distinct from k_0'. For that k_i we have $|c_0c_0'-c_ic_i'|\ge 8p^4\ge 2^{-9}\alpha^{-4}\ge 2^{-14}t^2\alpha^{-4}$.

If $t\ge 7$ we observe that there are at most $2(1+2(t/6))<t$ values of i such that $|k_0-k_i|\le t/6$ or $|k_0'-k_i|\le t/6$. So there exists some i such that

$$|c_0c_0'-c_ic_i'|\ge 8p^4(t/6)^2\ge 2^{-14}t^2\alpha^{-4}.$$

□

3. A related question

We do not know if the exponent -3 in Theorem 1 can be improved when $\overline{d}(\mathcal{A})>\alpha$ or when $\underline{d}(\mathcal{A})>\alpha$, which is the original problem of Sárközy. Clearly nothing better than -2 is possible. We present an alternative approach to this question, which gives the bound of G. Bérczi quickly.

Let $\mathcal{A} \subset \{1, \dots, N\}$ a set with αN elements. We consider the set

$$\mathcal{A}/\mathcal{A} = \{a/a',\ a < a',\ a, a' \in A\}.$$

What can we say about the cardinality of $\mathcal{A}/\mathcal{A}$ when N is large? Clearly $|\mathcal{A}/\mathcal{A}| \ll \alpha^2 N^2$. Probably it is the true order of magnitude but we do not know how to improve the theorem below

Theorem 5. *If $\mathcal{A} \subset \{1, \dots, N\}$ with $|\mathcal{A}| = \alpha N$, then $|\mathcal{A}/\mathcal{A}| \gg \alpha^4 N^2$.*

Proof. Let $(\mathcal{A} \times \mathcal{A})_d = \{(a, a') \in \mathcal{A} \times \mathcal{A} : a < a', \gcd(a, a') = d\}$. Then for every d, all the quotients $\frac{a}{a'}, (a, a') \in (\mathcal{A} \times \mathcal{A})_d$ are distinct and contained in [0,1]. We first show that there exists d such that $|(\mathcal{A} \times \mathcal{A})_d| \geq \frac{1}{c(\alpha)} N^2$. Let T be an integer to be chosen later. Then

$$\begin{aligned}
(\alpha N)^2 \leq |\mathcal{A}|^2 &= \sum_d |(\mathcal{A} \times \mathcal{A})_d| \\
&= \sum_{d \leq T} |(\mathcal{A} \times \mathcal{A})_d| + \sum_{d > T} |(\mathcal{A} \times \mathcal{A})_d| \\
&\leq T \max_{d \leq T} |(\mathcal{A} \times \mathcal{A})_d| + \sum_{d > T} \left(\frac{N}{d}\right)^2 \\
&\leq T \max_{d \leq T} |(\mathcal{A} \times \mathcal{A})_d| + N^2/T
\end{aligned}$$

Thus there exists $d \leq T$ such that $|\mathcal{A} \times \mathcal{A})_d| \geq N^2(\frac{\alpha^2}{T} - \frac{1}{T^2})$. If we choose $T = \lceil \frac{2}{\alpha^2} \rceil$ then $\frac{\alpha^2}{T} - \frac{1}{T} \geq \frac{1}{T} \geq \frac{\alpha^2}{4}$. Thus for some d, $|(\mathcal{A} \times \mathcal{A})_d| \geq N^2 \alpha^4/4$. Finally we observe that $|\mathcal{A}/\mathcal{A}| \geq |(\mathcal{A} \times \mathcal{A})_d|$ for any d. □

We observe that if $\overline{d}(A) > \alpha$ there exist infinitely many intervals $[1, N]$ such that $|\mathcal{A} \cap [1, N]| > \alpha$. If we apply the theorem above we obtain $a/a', a''/a''' \in \mathcal{A}/\mathcal{A}$ such that $|\frac{a}{a'} - \frac{a''}{a'''}| \leq 4\alpha^{-4} N^{-2}$, so $|aa''' - a'a''| \leq 4\alpha^{-4}$.

Theorem 5 motivates the following questions:

Question 2. *Is it true that for some d, $|(A_N \times \mathcal{A}_N)_d| \gg \alpha^2 N^2$?*

Question 3. *Is it true that $|\mathcal{A}_N/\mathcal{A}_N| \gg \alpha^2 N^2$?*

Clearly an affirmative answer to Question 2 will answer Question 3 which in turn answers Question 1.

References

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