DENSE SETS OF INTEGERS WITH PRESCRIBED REPRESENTATION FUNCTIONS

JAVIER CILLERUELO AND MELVYN B. NATHANSON

Abstract. Let $\mathcal{A}$ be a set of integers and let $h \geq 2$. For every integer n, let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form $n = a_1 + \cdots + a_h$, where $a_i \in \mathcal{A}$ for $1 \leq i \leq h$, and $a_1 \leq \cdots \leq a_h$. The function $r_{\mathcal{A},h} : \mathbb{Z} \to \mathbf{N}$, where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$, is the *representation function of order h for $\mathcal{A}$*.

We prove that every function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g$ is the representation function of order h for a sequence $\mathcal{A}$ of integers, and that $\mathcal{A}$ can be constructed so that it increases "almost" as slowly as any given $B_h[g]$ sequence. In particular, given $h \geq 2$, for every $\varepsilon > 0$ and for any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g = g(h, \epsilon)$ there exists a sequence $\mathcal{A}$ satisfying $r_{\mathcal{A},h} = f$ and $\mathcal{A}(x) \gg x^{(1/h)-\varepsilon}$.

Roughly speaking we prove that the problem of finding a dense set of integers with prescribed representation function f of order h with $\liminf_{|n|\to\infty} f(n) \geq g$ is "equivalent" to the classical problem of finding a dense $B_h[g]$ sequences of positive integers.

1. Introduction

Let $\mathcal{A}$ be a set of integers and let $h \geq 2$. For every integer n, let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form

$$n = a_1 + \cdots + a_h$$

where $a_1 \leq \cdots \leq a_h$ and $a_i \in \mathcal{A}$ for $1 \leq i \leq h$. The function $r_{\mathcal{A},h} : \mathbb{Z} \to \mathbf{N}$ is the *representation function of order h for $\mathcal{A}$*, where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$.

Nathanson proved [7] that any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq 1$ is the *representation function of order h* of a set of integers $\mathcal{A}$ such that

$$\mathcal{A}(x) \gg x^{1/(2h-1)}, \tag{1}$$

where $\mathcal{A}(x)$ counts the number of positive elements $a \in \mathcal{A}$ no greater than x and $f(x) \gg g(x)$ means that there exists a constant $C > 0$ such that $f(x) \geq Cg(x)$ for x *large enough.*

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It is an open problem to determine how dense the sets $\mathcal{A}$ can be. In this paper we study the connection between this problem and the problem of finding dense $B_h[g]$ sequences. We recall that a set $\mathcal{B}$ of nonnegative integers is called a $B_h[g]$ sequence if

$$r_{\mathcal{B},h}(n) \le g$$

for every nonnegative integer n. It is usual to write B_h to denote $B_h[1]$ sequences.

Luczak and Schoen proved that any B_h sequence satisfying an additional kind of Sidon property (see [6] for the definition of this property, which they call the S_h property) can be enlarged to obtain a sequence with any prescribed representation function f satisfying $\liminf_{|x|\to\infty} f(x) \ge 1$. In particular, since they prove that there exists a B_h sequence $\mathcal{A}$ satisfying the S_h property with $\mathcal{A}(x) \gg x^{1/(2h-1)}$, they recover Nathanson's result.

In this paper we prove that any $B_h[g]$ sequence, without any additional property, can be modified slightly to have any prescribed representation function f of order h satisfying $\liminf_{|x|\to\infty} f(x) \ge g$. Our main theorem is the following.

Theorem 1.1. *Let $f : \mathbb{Z} \to \mathbf{N}$ be any function such that $\liminf_{|n|\to\infty} f(n) \ge g$ and let $\mathcal{B}$ be any $B_h[g]$ sequence. Then, for any decreasing function $\epsilon(x) \to 0$ as $x \to \infty$, there exists a sequence $\mathcal{A}$ of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \quad \mathcal{A}(x) \gg \mathcal{B}(x\epsilon(x)).$$

Roughly speaking, theorem above says that the problem of finding dense sets of integers with prescribed representation functions with $\liminf_{|n|\to\infty} f(n) \ge g$ is "equivalent" to the classical problem of finding dense $B_h[g]$ sequences of positive integers.

It is a difficult problem to construct dense $B_h[g]$ sequences. A trivial counting argument gives $\mathcal{B}(x) \ll x^{1/h}$ for these sequences. On the other hand, the greedy algorithm shows that there exists a B_h sequence $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{1/(2h-1)}. \tag{2}$$

For B_2 sequences, also called Sidon sets, Ruzsa proved [9] that there exists a Sidon set $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{\sqrt{2}-1+o(1)}. \tag{3}$$

This result and Theorem 1.1 give the following corollary.

Corollary 1. *Let $f : \mathbb{Z} \to \mathbf{N}$ any function such that $\liminf_{|n|\to\infty} f(n) \ge 1$. Then there exists a sequence of integers $\mathcal{A}$ such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg x^{\sqrt{2}-1+o(1)}.$$

This result gives an affirmative answer to the third open problem in [1], which was also posed previously in [8]. Unfortunately, nothing better than (2) is known for B_h sequences for $h \geq 3$.

Erdős and Renyi [3] proved however that, for any $\epsilon > 0$, there exists a positive integer g and a $B_2[g]$ sequence $\mathcal{B}$ such that $\mathcal{B}(x) \gg x^{1/2-\epsilon}$. They claimed that the same method could be extended to $B_h[g]$ sequences, but a serious problem with non-independent events appears when $h \geq 3$. As an application of a more general theory, Vu [11] overcame this problem. He proved that for any $\epsilon > 0$, there exist an integer $g = g(h, \epsilon)$ and a $B_h[g]$ sequence $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{1/h-\epsilon}.$$

This result and Theorem 1.1 imply the next corollary

Corollary 2. *Given $h \geq 2$, for any $\varepsilon > 0$, there exists $g = g(h, \varepsilon)$ such that, for any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g$, there exists a sequence $\mathcal{A}$ of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg x^{\frac{1}{h}-\varepsilon}.$$

The construction in [7] for the set $\mathcal{A}$ satisfying the growth condition (14) was based on the greedy algorithm. In this paper we construct the set $\mathcal{A}$ by adjoining a very sparse sequence $\mathcal{U} = \{u_k\}$ to a suitable $B_h[g]$ sequence $\mathcal{B}$. This idea was used in [2], but in a simpler way, to construct dense *perfect difference sets*, which are sets such that every nonzero integer has a unique representation as a difference of two elements of $\mathcal{A}$. The proof of the main theorem in [2] can be adapted easily to our problem in the simplest case $h = 2$.

Theorem 1.2. *Let $f : \mathbb{Z} \to \mathbf{N}$ be a function such that $\liminf_{|n|\to\infty} f(n) \geq g$, and let $\mathcal{B}$ be a $B_2[g]$ sequence. Then there exists a sequence of integers $\mathcal{A}$ such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg \mathcal{B}(x/3).$$

We omit the proof because it is very close to the proof of the main theorem in [2]. Unfortunately, that proof cannot be adapted to the case $h \geq 3$. We need another definition of a "suitable" $B_h[g]$ set. In section §2 we shall show how to modify a $B_h[g]$ sequence $\mathcal{B}$ so that it becomes "suitable." We do this by applying the "Inserting Zeros Transformation" to an arbitrary $B_h[g]$ set. This is the main ingredient in the proof of Theorem 1.1.

Chen [1] has proved that for any $\epsilon > 0$ there exists a unique representation basis $\mathcal{A}$ (that is, a set $\mathcal{A}$ with $r_{\mathcal{A},2}(k) = 1$ for all $k \in \mathbb{Z}$) such that $\limsup_{x\to\infty} \mathcal{A}(x)/x^{1/2-\epsilon} > 1$. J. Lee [5] has improved this result by proving that for any increasing function ω tending to infinity there exists a unique representation basis $\mathcal{A}$ such that $\limsup_{x\to\infty} \mathcal{A}(x)\omega(x)/\sqrt{x} > 0$.

Theorem 1.2 and the classical constructions of Erdős [10] and Krückeberg [4] of infinite Sidon sets $\mathcal{B}$ such that $\limsup_{x\to\infty} \mathcal{B}(x)/\sqrt{x} > 0$ provide a unique representation basis $\mathcal{A}$ such that $\limsup_{x\to\infty} \mathcal{A}(x)/\sqrt{x} > 0$. Indeed, we can easily adapt the proof of Theorem 1.3 in [2] to the case of the additive representation function $r(n)$ (instead of the subtractive representation function $d(n) = \#\{n = a - a',\ a, a' \in \mathcal{A}\}$).

Theorem 1.3. *There exists a unique representation basis $\mathcal{A}$ such that*

$$\limsup_{x\to\infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

Again we omit the proof because it is very close to the proof of Theorem 1.3 in [2].

Theorem above answers affirmatively the first open problem in [1]. Note also that if $\mathcal{A}$ is an infinite Sidon set of integers, then the set

$$\mathcal{A}' = \{4a : a \geq 0\} \cup \{-4a + 1 : a < 0\}$$

is also a Sidon set and, in this case, $\liminf |\mathcal{A} \cap (-x, x)|/\sqrt{x} = \liminf \mathcal{A}'(4x)/\sqrt{x}$. A well known result of Erdős states that $\liminf \mathcal{B}(x)/\sqrt{x} = 0$ for any Sidon set $\mathcal{B}$. Then the above limit is zero, so it answers negatively the second open problem in [1].

It is easy to prove that for any function ω tending to infinity there exists a B_h sequence such that $\limsup_{x\to\infty} \mathcal{B}(x)\omega(x)/x^{1/h} > 1$. We can construct the set $\mathcal{B}$ as follows: Let $x_1, \dots, x_k, \dots$ be a sequence of positive integers such that $\omega(x_k) > (hx_{k-1})^{1/h}$ and consider, for each k, a B_h sequence $\mathcal{B}_k \subset [1, x_k/(hx_{k-1})]$ with $|\mathcal{B}_k| \gg (x_k/(hx_{k-1}))^{1/h}$. The set $\mathcal{B} = \cup_k (hx_{k-1}) * \mathcal{B}_k$ satisfies the conditions, where we use the notation $t * \mathcal{A} = \{ta,\ a \in \mathcal{A}\}$.

The construction above and Theorem 1.1 yield the following Corollary, which extends the main theorem in [1] in several ways.

Corollary 3. *Let $f : \mathbb{Z} \to \mathbf{N}$ any function such that $\liminf_{|n|\to\infty} f(n) \geq 1$. For any increasing function ω tending to infinity there exists a set $\mathcal{A}$ such that $r_{\mathcal{A},h}(n) = f(n)$ for all integers n, and*

$$\limsup_{x\to\infty} \mathcal{A}(x)\omega(x)/x^{1/h} > 0.$$

2. The Inserting Zeros Transformation

Consider the binary expansion of the elements of a set $\mathcal{B}$ of positive integers. We will modify these integers by inserting strings of zeros at fixed places. We will see that this transformation of the set $\mathcal{B}$ preserves certain additive properties.

In this paper we denote by γ any strictly increasing function $\gamma : \mathbb{N}_0 \to \mathbb{N}_0$ with $\gamma(0) = 0$. For every positive integer r, we define the "Inserting Zeros Transformation" T_γ^r by

$$T_\gamma^r \left(\sum_{i\geq 0} \varepsilon_i 2^i \right) = \sum_{k\geq 0} 2^{2rk} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i. \tag{4}$$

In other words, if $\varepsilon_0\varepsilon_1\varepsilon_2\dots$ is the binary expansion of b, then

$$T_\gamma^r(b)=\varepsilon_0\cdots\varepsilon_{\gamma(1)-1}\underbrace{0\cdots0}_{2r}\varepsilon_{\gamma(1)}\cdots\varepsilon_{\gamma(2)-1}\underbrace{0\cdots0}_{2r}\varepsilon_{\gamma(2)}\cdots\varepsilon_{\gamma(k)-1}\underbrace{0\cdots0}_{2r}\varepsilon_{\gamma(k)}\cdots$$

Note that if $b<b'$, then $T_\gamma^r(b)<T_\gamma^r(b')$. We define the set

$$T_\gamma^r(\mathcal{B})=\{T_\gamma^r(b):b\in\mathcal{B}\}. \tag{5}$$

The next proposition proves that the function T_γ^r preserves some Sidon properties.

Proposition 2.1. Let $2r>\log_2 h$. If $b_1,\dots,b_h,b_1',\dots,b_h'$ are positive integers such that

$$T_\gamma^r(b_1)+\cdots+T_\gamma^r(b_h)=T_\gamma^r(b_1')+\cdots+T_\gamma^r(b_h'),$$

then

$$b_1+\cdots+b_h=b_1'+\cdots+b_h'.$$

In particular, if $\mathcal{B}$ is a $B_h[g]$ set and $2r>\log_2 h$, then $T_\gamma^r(\mathcal{B})$ is also a $B_h[g]$ set.

Proof. We write

$$t_k=\sum_{i=\gamma(k)}^{\gamma(k+1)-1}\varepsilon_i(b_1)2^i+\cdots+\sum_{i=\gamma(k)}^{\gamma(k+1)-1}\varepsilon_i(b_h)2^i. \tag{6}$$

For any $k\ge 1$ we define the integer

$$m_k=2^{2rk+\gamma(k)}. \tag{7}$$

It follows from (4), (6) and (7) that

$$T_\gamma^r(b_1)+\cdots+T_\gamma^r(b_h)\equiv\sum_{j=0}^{k-1}2^{2rj}t_j \pmod{m_k}.$$

Since $T_\gamma^r(b_1)+\cdots+T_\gamma^r(b_h)=T_\gamma^r(b_1')+\cdots+T_\gamma^r(b_h')$, we have

$$\sum_{j=0}^{k-1}2^{2rj}t_j\equiv\sum_{j=0}^{k-1}2^{2rj}t_j' \pmod{m_k}.$$

Notice that

$$0\le\sum_{j=0}^{k-1}2^{2rj}t_j\le 2^{2r(k-1)}\sum_{j=0}^{k-1}t_j\le 2^{2r(k-1)}h\sum_{i=0}^{\gamma(k)-1}2^i<2^{2r(k-1)}2^{2r}2^{\gamma(k)}=m_k,$$

and the same inequality works for $\sum_{j=0}^{k-1}2^{rj}t_j'$. Then

$$\sum_{j=0}^{k-1}2^{2rj}t_j=\sum_{j=0}^{k-1}2^{2rj}t_j'.$$

It follows that $t_k=t_k'$ for all $k\ge 0$, and so

$$b_1+\cdots+b_h=\sum_{k\ge0}t_k=\sum_{k\ge0}t_k'=b_1'+\cdots+b_h'.$$

This completes the proof. ∎

Definition 2.2. For all integers $m \geq 2$ and x, let

$$\|x\|_m = \min\{|y|,\ x \equiv y \pmod{m}\}.$$

Note that $\|x_1+x_2\|_m \leq \|x_1\|_m+\|x_2\|_m$ for all integers x_1 and x_2. Also, if $\|x\|_m \neq \|x'\|_m$ for some m, then $x \not\equiv x' \pmod{m}$ and so $x \neq x'$.

Proposition 2.3. For $k \geq 1$ and for any positive integer b

$$\|T_\gamma^r(b)\|_{m_k} < m_k 2^{-2r},$$

where m_k is defined in (7).

Proof. Let $b = \varepsilon_0\varepsilon_1\varepsilon_2\ldots$ be the binary expansion of b. Then

$$T_\gamma^r(b) \equiv \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \pmod{m_k}$$

and

$$0 \leq \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \leq \sum_{l=0}^{2r(k-1)+\gamma(k)-1} 2^l < m_k 2^{-2r}.$$

This completes the proof. □

3. Proof of Theorem 1.1

3.1. **Two auxiliary sequences.** Consider the sequence $\{z_j\}_{j=1}^\infty$ defined by

$$z_j = j - [\sqrt{j}]([\sqrt{j}]+1). \tag{8}$$

For every positive integer j there is a unique positive integer s such that $s^2 \leq j < (s+1)^2$. Then $j = s^2+s+i$ for some $i \in [-s,s]$ and $z_j = i$. It follows that for every integer i there are infinitely many positive integers j such that $z_j = i$. Moreover, $|z_j| \leq s \leq \sqrt{j}$ for all $j \geq 1$.

Let $f : \mathbb{Z} \to \mathbf{N}$ any function such that $\liminf_{|n|\to\infty} f(n) \geq g$. Let n_0 be the least positive integer such that $f(n) \geq g$ for all $|n| \geq n_0$. Choose an integer $r > 1+\log_2(h^2+n_0)$. Then

$$h^2 < 2^{r-1} \ \text{ and } \ n_0 < 2^{r-1}. \tag{9}$$

Let $\gamma : \mathbb{N}_0 \to \mathbb{N}_0$ be a strictly increasing function such that $\gamma(0) = 0$.

We consider the sequence $\mathcal{U} = \{u_i\}_{i=1}^\infty$ defined by

$$\begin{cases} u_{2k-1} &= -m_k 2^{-r}, \\ u_{2k} &= (h-1)m_k 2^{-r} + z_k \end{cases} \tag{10}$$

where $m_k = 2^{2rk+\gamma(k)}$. We write

$$\mathcal{U}_k = \{u_{2k-1}, u_{2k}\} \qquad \text{and} \qquad \mathcal{U}_{<k} = \bigcup_{s<k} \mathcal{U}_s. \tag{11}$$

Note that for all $j \leq k$ we have

$$|z_j| \leq \sqrt{k} < 2^k \leq 2^{\gamma(k)} < 2^{2r(k-1)+\gamma(k)} = m_k 2^{-2r}. \tag{12}$$

3.2. **The recursive construction.** For any $B_h[g]$-sequence $\mathcal{B}$ we consider the set $T_\gamma^r(\mathcal{B})$ defined in (5). Let $f : \mathbb{Z} \to \mathbb{N}$ be a function such that $f(n) \geq g$ for $|n| \geq n_0$. We construct an increasing sequence $\{\mathcal{A}_k\}_{k=0}^\infty$ of sets of integers as follows:

$$\mathcal{A}_0 = \{a \in T_\gamma^r(\mathcal{B}) : a \geq n_0\} \tag{13}$$

and, for $k \geq 1$,

$$\mathcal{A}_k = \begin{cases} \mathcal{A}_{k-1} \cup \mathcal{U}_k & \text{if } r_{\mathcal{A}_{k-1},h}(z_k) < f(z_k) \\ \mathcal{A}_{k-1} & \text{otherwise} \end{cases}$$

where z_k and $\mathcal{U}_k$ are defined in (8) and (11).

We shall prove that the set

$$\mathcal{A} = \bigcup_{k=0}^\infty \mathcal{A}_k \tag{14}$$

satisfies $r_{\mathcal{A},h}(n) = f(n)$ for all integers n as consequence of propositions 3.1 and 3.2.

Proposition 3.1. The sequence $\mathcal{A}$ defined in (14) satisfies $r_{\mathcal{A},h}(n) \geq f(n)$ for all integers n.

Proof. Since

$$\underbrace{u_{2k-1} + \cdots + u_{2k-1}}_{h-1} + u_{2k} = z_k$$

it follows that if $r_{\mathcal{A}_{k-1},h}(z_k) < f(z_k)$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k$ and

$$r_{\mathcal{A}_k,h}(z_k) \geq r_{\mathcal{A}_{k-1},h}(z_k) + 1.$$

Since the sequence (z_k) takes all the integers infinitely many times, then $r_{\mathcal{A}_k,h}(n) \geq f(n)$ for some k (if $f(n) < \infty$) or $\lim_{k\infty} r_{\mathcal{A}_k,h}(n) = \infty$ (if $f(n) = \infty$). □

Lemmas 3.1, 3.2 and 3.3 will allow us to give a clean proof of proposition 3.2.

Lemma 3.1. *Let $k \geq 1$. For nonnegative integers s and t with $s + t \leq h$, let*

$$\mathcal{A}_k^{(s,t)} = (h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}.$$

The sets $\mathcal{A}_k^{(s,t)}$ are pairwise disjoint, except possibly the sets $\mathcal{A}_k^{(0,0)}$ and $\mathcal{A}_k^{(h-1,1)}$.

Proof. If $n \in \mathcal{A}_k^{(s,t)}$ then

$$\begin{aligned} n &= a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} \\ &= a_1 + \cdots + a_{h-s-t} + (t(h-1) - s)m_k 2^{-r} + tz_k, \end{aligned}$$

with $a_1, \ldots, a_{h-s-t} \in \mathcal{A}_{k-1} \subset \mathcal{A}_0 \cup \mathcal{U}_{<k}$.

If $a_i \in \mathcal{A}_0$, then $\|a_i\|_{m_k} \leq m_k 2^{-2r}$ by Proposition 2.3. If $a_i \in \mathcal{U}_{<k}$ we use (10) and (12) to obtain

$$\|a_i\|_{m_k} \leq |a_i| \leq (h-1)m_{k-1}2^{-r} + m_{k-1}2^{-2r} < hm_k 2^{-2r}.$$

We use again (12) to obtain

$$\begin{aligned} \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} &\leq \|a_1\|_{m_k} + \cdots + \|a_{h-s-t}\|_{m_k} + \|tz_k\|_{m_k} \\ &\leq (h-s-t)m_k h 2^{-2r} + tm_k 2^{-2r} \\ &\leq h^2 m_k 2^{-2r}. \qquad (15) \end{aligned}$$

Now suppose that $n \in \mathcal{A}_k^{(s',t')}$ for some $(s',t') \neq (s,t)$.
If $\{(s,t),(s',t')\} \neq \{(0,0),(h-1,1)\}$, then $t(h-1)-s \neq t'(h-1)-s'$ and

$$\begin{aligned} m_k 2^{-r} &\leq \|\left((t(h-1)-s)-(t'(h-1)-s')\right) m_k 2^{-r}\|_{m_k} \\ &= \|(t(h-1)-s)m_k 2^{-r} - (t'(h-1)-s')m_k 2^{-r}\|_{m_k} \\ &= \|\left(n-(t(h-1)-s)m_k 2^{-r}\right) - \left(n-(t'(h-1)-s')m_k 2^{-r}\right)\|_{m_k} \\ &\leq \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} + \|a'_1 + \cdots + a'_{h-s'-t'} + t'z_k\|_{m_k} \\ &\leq 2h^2 m_k 2^{-2r}. \end{aligned}$$

It follows that $h^2 \geq 2^{r-1}$, which contradicts (9). This completes the proof. □

Lemma 3.2. *If $n \in \mathcal{A}_k^{(s,t)}$ for some $k \geq 1$ and $(s,t) \notin \{(0,0),(h-1,1)\}$, then $|n| > n_0$.*

Proof. If $n \in \mathcal{A}_k^{(s,t)}$, then

$$n = a_1 + \cdots + a_{h-s-t} + (t(h-1)-s)m_k 2^{-r} + tz_k$$

and

$$\begin{aligned} |n| &\geq \|n\|_{m_k} \\ &= \|a_1 + \cdots + a_{h-s-t} + tz_k + ((h-1)t-s)m_k 2^{-r}\|_{m_k} \\ &\geq \|((h-1)t-s)m_k 2^{-r}\|_{m_k} - \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} \\ &\geq |((h-1)t-s)m_k 2^{-r}| - h^2 m_k 2^{-2r} \\ &\geq m_k 2^{-r} - h^2 m_k 2^{-2r} \geq m_k 2^{-r-1} \geq 2^{2r} 2^{-r-1} \\ &\geq 2^{r-1} > n_0, \end{aligned}$$

We have used that if $|((h-1)t-s)m_k 2^{-r}| < m_k/2$, then

$$\|((h-1)t-s)m_k 2^{-r}\|_{m_k} = |((h-1)t-s)m_k 2^{-r}| \geq m_k 2^{-r}.$$

Also we have used $(h-1)t-s \neq 0$ and the inequalities (9) and (15) in the last inequalities.

□

Lemma 3.3. *For any $k \geq 0$, for any $h' < h$ and for any integer m we have that*

$$r_{\mathcal{A}_k,h'}(m) \leq g$$

Proof. By induction on k. Proposition 2.1 implies that $T^r_\gamma(\mathcal{B})$ and consequently $\mathcal{A}_0$ are $B_h[g]$-sequences. In particular, $\mathcal{A}_0$ is a $B_{h'}[g]$ sequence. Then $r_{\mathcal{A}_0,h'}(m) \le g$ for any integer m.

Suppose that it is true that for any $h' < h$, and for any integer m we have that $r_{\mathcal{A}_{k-1},h'}(m) \le g$.

Consider $m \in h'\mathcal{A}_k$.

- Suppose $m \notin (h' - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s,t) \neq (0,0)$. Then $r_{\mathcal{A}_k,h'}(m) = r_{\mathcal{A}_{k-1},h'}(m) \le g$ by the induction hypothesis.
- Suppose that $m \in (h' - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s,t) \neq (0,0)$. Consider an element $a \in \mathcal{A}_0$. Then

$$m + (h - h')a \in \mathcal{A}_k^{(s,t)} \in (h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}.$$

Since $(s,t) \neq (h-1,1)$ (because $h' < h$) we can apply lemma 3.1 and we have

$$r_{\mathcal{A}_k,h'}(m) \le r_{\mathcal{A}_k,h}(m + (h-h')a) = r_{\mathcal{A}_{k-1},h-s-t}(m + (h-h')a - su_{2k-1} - tu_{2k}).$$

We can the apply the induction hypothesis because $h - s - t < h$.

□

Proposition 3.2. The sequence $\mathcal{A}$ defined in (14) satisfies $r_{\mathcal{A},h}(n) \le f(n)$ for all integers n.

Proof. Next we show that, for every integer k, the sequence $\mathcal{A}_k$ satisfies $r_{\mathcal{A}_k,h}(n) \le f(n)$ for all n. The proof is by induction on k.

Let $k = 0$. Since $\mathcal{A}_0$ is a $B_h[g]$-sequences, we have $r_{\mathcal{A}_0,h}(n) \le g \le f(n)$ for $n \ge n_0$. If $n < n_0$, then $r_{\mathcal{A}_0,h}(n) = 0 \le f(n)$.

Now, suppose that it is true for $k-1$. In particular $r_{\mathcal{A}_{k-1},h}(z_k) \le f(z_k)$. If $r_{\mathcal{A}_{k-1},h}(z_k) = f(z_k)$ there is nothing to prove because in that case $\mathcal{A}_k = \mathcal{A}_{k-1}$. But if $r_{\mathcal{A}_{k-1},h}(z_k) \le f(z_k) - 1$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}\} \cup \{u_{2k}\}$. We will assume that until the end of the proof.

If $n \notin h\mathcal{A}_k$ then $r_{\mathcal{A}_k,h}(n) = 0 \le f(n)$.

If $n \in h\mathcal{A}_k$, since $\mathcal{A}_k = \mathcal{A}_{k-1} \cap \mathcal{U}_k$ we can write

$$h\mathcal{A}_k = \bigcup_{\substack{s,t=0 \\ s+t\le h}}^{h} \left((h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}\right).$$

Then

$$n = a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} \tag{16}$$

for some s,t, satisfying $0 \le s,t,\ s + t \le h$ and for some $a_1, \ldots, a_{h-s-t} \in \mathcal{A}_{k-1}$.

For short we write $r_{s,t}(n)$ for the number of solutions of (16).

- If $n \in (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s,t) \neq (0,0)$, $(s,t) \neq (h-1,1)$ then, due to lemma 3.1, we have that $r_{\mathcal{A}_k,h}(n) = r_{s,t}(n)$.
  - For $0 \leq n \leq n_0$ we have that $r_{s,t}(n) = 0 \leq f(n)$ (due to lemma 3.2).
  - For $n > n_0$ we apply lemma 3.3 in the first inequality below with $h' = h-s-t$ and $m = n - su_{2k-1} - tu_{2k}$,

$$r_{s,t}(n) = r_{\mathcal{A}_{k-1},h-s-t}(n - su_{2k-1} - tu_{2k}) \leq g \leq f(n)$$

- If $n \notin (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s,t) \neq (0,0)$, $(s,t) \neq (h-1,1)$, then $r_{\mathcal{A}_k,h}(n) = r_{0,0}(n) + r_{h-1,1}(n)$. Notice that $r_{0,0}(n) = r_{\mathcal{A}_{k-1},h}(n)$ and that $r_{h-1,1}(n) = 1$ if $n = z_k$ and $r_{h-1,1}(n) = 0$ otherwise.
  - If $n \neq z_k$, then $r_{\mathcal{A}_k,h}(n) = r_{A_{k-1},h}(n) \leq f(n)$ by the induction hypothesis.
  - If $n = z_k$, then $r_{\mathcal{A}_k,h}(n) = r_{\mathcal{A}_{k-1},h}(z_k) + r_{h-1,1}(z_k) \leq (f(z_k)-1)+1 = f(n)$.

□

3.3. **The density of $\mathcal{A}$.** Recall that $\gamma : \mathbb{N}_0 \to \mathbb{N}_0$ is a strictly increasing function with $\gamma(0) = 0$. Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. We extend γ to a strictly increasing function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. (For example, define $\gamma(x) = \gamma(k+1)(x-k) + \gamma(k)(k+1-x)$ for $k \leq x \leq k+1$.)

We have

$$\mathcal{A}(x) \geq \mathcal{A}_0(x) \geq T_\gamma^r(\mathcal{B})(x) - n_0.$$

Thus, to find a lower bound for $\mathcal{A}(x)$ it suffices to find a lower bound for the density of $T_\gamma^r(\mathcal{B})$.

Lemma 3.4. $T_\gamma^r(\mathcal{B})(x) > \mathcal{B}(x 2^{-2r\gamma^{-1}(\log_2 x)})$.

Proof. Let b be a positive integer such that

$$b \leq x 2^{-2r\gamma^{-1}(\log_2 x)}.$$

Let ℓ be such that $2^{\gamma(\ell)} \leq b < 2^{\gamma(\ell+1)}$. Then we can write

$$b = \sum_{k=0}^{\ell} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i. \tag{17}$$

It follows from the definition (4) of the Zeros Inserting Transformation that

$$\begin{aligned} T_\gamma^r(b) &= \textstyle\sum_{k=0}^{\ell} 2^{2rk} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i \\ &\leq 2^{2r\ell} b \\ &\leq 2^{2r\gamma^{-1}(\log_2 b)} b \\ &\leq 2^{2r(\gamma^{-1}(\log_2 b) - \gamma^{-1}(\log_2 x))} x \\ &\leq x. \end{aligned}$$

□

Recall that ϵ is a decreasing positive function defined on $[1, \infty)$ such that $\lim_{x\to\infty} \epsilon(x) = 0$. We complete the proof of Theorem 1 by choosing a function γ that satisfies the inequality

$$2^{-2r\gamma^{-1}(\log_2 x)} \geq \epsilon(x).$$

It suffices to take $\gamma(x) > \log_2(\epsilon^{-1}(2^{-2rx}))$.

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Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail address: franciscojavier.cilleruelo@uam.es

Department of Mathematics, Lehman College (CUNY), Bronx, New York 10468
E-mail address: melvyn.nathanson@lehman.cuny.edu