Repunit Lehmer numbers

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Abstract

A Lehmer number is a composite positive integer n such that $\phi(n) \mid n-1$. In this paper, we show that given a positive integer $g>1$ there are at most finitely many Lehmer numbers which are repunits in base g. Our method is effective and we illustrate it by showing that there is no such Lehmer number when $g \in [2, 1000]$.

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1 Introduction

Let $\phi(n)$ be the Euler function of the positive integer n. Clearly, $\phi(n) = n-1$ if n is a prime. Lehmer [9] (see also B37 in [7]) conjectured that if $\phi(n) \mid n-1$, then n is prime. To this day, no counterexample to this conjecture has been found. A composite number m such that $\phi(m) \mid m-1$ is called a *Lehmer number*. Thus, Lehmer's conjecture is that Lehmer numbers don't exist but it is not even known that there should be at most finitely many of them.

Given a positive integer $g>1$ a base g repunit is a number of the form $m = (g^n-1)/(g-1)$ for some integer $n \geq 1$. We will refer to such numbers simply as repunits without mentioning the dependence on g. It is not known whether given g there are infinitely many repunit primes. When $g = 2$ such primes are better known as Mersenne primes. In [4], it was shown that there is no Lehmer number in the Fibonacci sequence. Here, we use some ideas from [4] together with finer arguments to prove the following results. In what follows, we write $u_n = (g^n - 1)/(g - 1)$.

Theorem 1. *For each fixed $g > 1$, there are only finitely many effectively computable positive integers n such that u_n is a Lehmer number.*

Theorem 2. *There is no Lehmer number of the form u_n when $2 \leq g \leq 1000$.*

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2 Prelimiaries

For a prime p and a nonzero integer m we write $\nu_q(m)$ for the exponent of q in the factorization of m. We start by collecting some elementary and well-known properties of the sequence of general term $u_n = (g^n - 1)/(g - 1)$ for $n \geq 1$.

Lemma 1. *i) $u_n = g^{n-1} + \cdots + g + 1$. In particular, u_n is coprime to g.*

ii) The sequence u_n satisfies the linear recurrence

$$u_1 = 1, \quad u_n = gu_{n-1} + 1, \quad n \geq 2. \tag{1}$$

iii) If $d \mid n$, then $u_d \mid u_n$.

iv) Let q be a prime. If $q \mid n$ and $q \nmid (g - 1)$, then $q \mid \phi(u_n)$.

v) Let q be a prime. If $q \mid n$, then $\nu_q(u_{n-1}) \leq \nu_q(u_f) \leq \nu_q(u_{q-1})$, where f is the order of $g \pmod q$.

vi) If u_n is a Lehmer number, then $(u_n, g - 1) = 1$.

Proof. i) and *ii)* are obvious. For *iii)*, we observe that

$$u_n = \frac{g^n - 1}{g - 1} = \frac{(g^d)^{n/d} - 1}{g^d - 1} \cdot \frac{g^d - 1}{g - 1} = \left((g^d)^{\frac{n}{d}-1} + \cdots + 1\right) u_d.$$

iv) Let p a prime which divides u_q. Then, $g^q \equiv 1 \pmod{p}$, so the order of g modulo p is 1 or q. If it is 1, then $p \mid g-1$. Since also $p \mid u_q$, we have

$$0 \equiv u_q \equiv \frac{g^q - 1}{g-1} = g^{q-1} + \cdots + g + 1 \equiv 1 + \cdots + 1 + 1 \equiv q,$$

where all congruences above are modulo p. Thus, $p \mid q$, therefore $p = q$, contradicting the fact that $q \nmid (g-1)$. So, the order of g modulo p is q, therefore $q \mid p-1$. On the other hand, by *iii)*, p divides u_n, so $p-1$ divides $\phi(u_n)$. Thus, $q \mid \phi(u_n)$.

v) Let f be the order of $g \pmod{q}$. We may assume that q does not divide g otherwise all three numbers are zero by *i)*. We may also assume that $q \mid u_{n-1}$, otherwise $\nu_q(u_{n-1}) = 0$ and the first inequality is clear. Now $g^{n-1} \equiv 1 \pmod{q}$, and so $f \mid n-1$. We now write

$$u_{n-1} = \left((g^f)^{\frac{n-1}{f}-1} + \cdots + 1\right) u_f.$$

The quantity in brackets above is not divisible by q since it is congruent to $(n-1)/f$ modulo q and $q \mid n$. Thus, $\nu_q(u_{n-1}) \le \nu_q(u_f) \le \nu_q(u_{q-1})$, where the last inequality follows because $f \mid q-1$ (so, $u_f \mid u_{q-1}$ by *iii)*).

vi) Suppose that q is a prime dividing both u_n and $g-1$. We then have that $g \equiv 1 \pmod{q}$ and $u_n = g^{n-1} + \cdots + 1 \equiv n \pmod{q}$. Thus, $q \mid n$ and $q \nmid g-1$. By *iv)*, we know that $q \mid \phi(u_n)$. Since u_n is a Lehmer number, we know that $\phi(u_n) \mid u_n - 1 = g u_{n-1}$. Thus, $q \mid u_{n-1}$ and $q \mid u_n - u_{n-1} = g^{n-1}$, which is not possible by *i)*. □

In the next lemma, we gather some known facts about Lehmer numbers.

Lemma 2. *i) Any Lehmer number must be odd and square-free.*

ii) If $m = p_1 \cdots p_K$ is a Lehmer number, then $K^{2^K} > m$.

iii) If $m = p_1 \cdots p_K$ is a Lehmer number, then $K \ge 14$.

Proof. i) If $m > 2$ then $\phi(m)$ is even, and since $\phi(m) \mid m-1$, we get that m must be odd. If $p^2 \mid m$, then $p \mid \phi(m)$, and since $\phi(m) \mid m-1$, we have $p \mid m-1$, which is not possible. Part *ii)* was proved by Pomerance in [5], while part *iii)* was proved by Cohen and Hagis in [2]. □

Lemma 3. *Theorems 1 and 2 hold when g is even.*

Proof. Note that

$$2^K \mid (p_1-1)\cdots(p_K-1) = \phi(u_n) \mid u_n - 1 = gu_{n-1}.$$

We observe that if g is even, then u_{n-1} is odd. In that case, we have

$$K \le \nu_2(\phi(u_n)) \le \nu_2(gu_{n-1}) = \nu_2(g), \tag{2}$$

implying, by Lemma 2-*ii)*, that

$$g^{n-1} < u_n < K^{2^K} \le (\nu_2(g))^{2^{\nu_2(g)}} \le (\nu_2(g))^g.$$

Thus,

$$n \le 1 + \left\lfloor \frac{g\log(\nu_2(g))}{\log g} \right\rfloor.$$

For Theorem 2, we observe that $\nu_2(g) \le 9$ for any $g \le 1000$, and we obtain a contradiction from (2) and Lemma 2-*iii)*. □

From Lemma 1-*i)*, we see that if g is odd and n is even, then u_n is even, so Lemma 2-*i)* shows that u_n cannot be a Lehmer number. From now on, we shall assume that both g and n are odd and ≥ 3 and that $u_n = (g^n-1)/(g-1)$ is a Lehmer number; i.e. $\phi(u_n) \mid u_n - 1 = gu_{n-1}$. We also keep the notation:

$$n = q_1^{\alpha_1}\cdots q_s^{\alpha_s}, \qquad \text{where } 2 < q_1 < \cdots < q_s \tag{3}$$

are primes and $\alpha_1, \ldots, \alpha_s$ are positive integers, and

$$u_n = p_1 \cdots p_K,\ 2 < p_1 < \cdots < p_K, \tag{4}$$

where $p_1, \ldots, p_K$ are also primes.

3 Proof of Theorem 1

3.1 Primitive divisors

Let $(A_n)_{n\ge1}$ denote a sequence with integer terms. We say that a prime p is a primitive divisor of A_n if $p \mid A_n$ and $\gcd(p, A_m) = 1$ for all non-zero terms A_m with $1 \le m < n$.

In 1886, Bang [1] showed that if $g > 1$ is any fixed integer, then the sequence $(A_n)_{n\ge1}$ of n-th term $A_n = g^n - 1$ has a primitive divisor for any index $n > 6$.

We will apply this important theorem to our sequence u_n.

Lemma 4. *If $d > 1$ is odd, then u_d has a prime divisor p_d such that $p_d \equiv 1 \pmod{2d}$ and $p_d \nmid u_{d'}$ for any $1 \le d' < d$.*

Proof. We write $v_n = g^n - 1$. We observe that $(v_n, v_m) = v_{(n,m)}$. Observe also that

$$\frac{v_d}{v_1} = u_d = g^{d-1} + \cdots + 1 \equiv d \pmod{g-1},$$

therefore if d is a prime not dividing $g-1$, then v_d has primitive divisors. If $d > 2$ is a prime dividing $g-1$, then the above argument shows that $\gcd(v_d, v_1)$ is a power of d. Writing $g - 1 = d\lambda$ and observing that

$$\begin{aligned}
\frac{v_d}{v_1} &= (1+d\lambda)^{d-1} + (1+d\lambda)^{d-2} + \cdots + 1 \\
&\equiv (1 + (d-1)d\lambda) + (1 + (d-2)d\lambda) + \cdots + 1 \\
&= d + d\lambda((d-1) + (d-2) + \cdots + 1) \pmod{d^2} \\
&\equiv d + \frac{d^2(d-1)}{2}\lambda \pmod{d^2} \equiv d \pmod{d^2}.
\end{aligned}$$

Thus, $d \,\|\, v_d/v_1$, and therefore

$$\frac{v_d}{dv_1} = \frac{1}{d}(g^{d-1} + \cdots + 1) > 1$$

is an integer coprime to v_1, so v_d again has primitive divisors. Thus, v_3 and v_5 (and, of course, v_1 if $g > 2$) have primitive divisors. The fact that v_d has primitive divisors for all odd $d \ge 7$ follows from Bang's result.

We now note that if p is a primitive prime divisor of v_d, then $g^d \equiv 1 \pmod{p}$, and d is the order of $g \pmod{p}$. Indeed, for if not, then $f < d$ and $p \mid v_f$, contradicting the fact that p is primitive for v_d. So, $d \mid p - 1$, and since d is odd, we get that $d \mid (p-1)/2$. Thus, $p \equiv 1 \pmod{2d}$.

Since a prime factor of $g - 1$ cannot be a primitive divisor for v_d except for $d = 1$, we deduce that if $d > 1$, then the primitive prime divisors for v_d are exactly those of $u_d = v_d/(g-1)$, and we get the first assertion of the lemma. $\square$

Lemma 5. *If u_n is square-free, n is odd and $(u_n, g-1) = 1$, then*

$$\begin{aligned}
\log\left(\frac{u_n}{\phi(u_n)}\right) &< \frac{\omega(n)}{2q}\left(1 + \log\left(\frac{q\log g}{\log(2q+1)}\right)\right) \\
&+ \frac{\tau(n)-1}{2q^2}\left(1 + \log\left(\frac{q^2\log g}{\log(2q^2+1)}\right)\right),
\end{aligned}$$

where q is the smallest prime dividing n.

Proof. We write $\mathcal{P}_d = \{p \text{ is primitive prime divisor for } u_d\}$. We shall first prove that

$$\prod := \prod_{1<d|n} \prod_{p \in P_d} p = u_n.$$

To see the above formula, we observe that if $p \mid u_d$ and $p \nmid g-1$, then $p \in \mathcal{P}_d$ for some $1 < d \mid n$. Since u_n is square-free, we have that $u_n \mid \prod$. On the other hand, the sets $\mathcal{P}_d$ are disjoint, and if $p \in \mathcal{P}_d$, then $p \mid u_d \mid u_n$. Thus, $\prod \mid u_n$.

Now, since u_n is square-free,

$$\phi(u_n) = \prod_{1<d|n} \prod_{p \in \mathcal{P}_d} (p-1),$$

and then

$$\log\left(\frac{u_n}{\phi(u_n)}\right) = \sum_{\substack{d|n \\ d>1}} \sum_{p \in \mathcal{P}_d} \frac{1}{p-1}.$$

Since all the primes $p \in \mathcal{P}_d$ are congruent to $1 \pmod{2d}$, we have

$$S_d := \sum_{p \in \mathcal{P}_d} \frac{1}{p-1} \leq \frac{1}{2d} \sum_{j=1}^{\#\mathcal{P}_d} \frac{1}{j} \leq \frac{1}{2d}(1 + \log \#\mathcal{P}_d).$$

To bound the cardinality of $\mathcal{P}_d$, we observe that $(2d+1)^{\#\mathcal{P}_d} \leq u_d < g^d$, so

$$\#\mathcal{P}_d < \frac{d \log g}{\log(2d+1)}.$$

We observe that $d \geq q$ and if d is not a prime, then $d \geq q^2$. Then

$$\sum_{1<d|n} S_d = \sum_{\substack{d|n \\ d \text{ prime}}} S_d + \sum_{\substack{d|n \\ d \text{ composite}}} S_d \leq \omega(n)\frac{1}{2q}\left(1 + \log\left(\frac{d \log g}{\log(2d+1)}\right)\right)$$
$$+(\tau(n)-1)\frac{1}{2q^2}\left(1 + \log\left(\frac{d^2 \log g}{\log(2d^2+1)}\right)\right).$$

□

3.2 A bound for q_1 and $\tau(n)$

Recall that we keep the notations from (3) and (4).

Lemma 6. *If u_n is a Lehmer number and n is odd, then*

$$\begin{aligned} \tau(n/q_i) &\leq \frac{\alpha_i(\alpha_i+1)}{2}\,\tau(n/q_i^{\alpha_i}) \leq \nu_{q_i}(\phi(n)) \leq \nu_{q_i}(gu_{n-1}) \\ &\leq \begin{cases} \nu_{q_i}(g), & \text{if } q_i | g; \\ \nu_{q_i}(u_{q_i-1}), & \text{if } q_i \nmid g \end{cases} \end{aligned} \tag{5}$$

for all $i = 1, \ldots, s$.

Proof. Lemma 4 implies that for each divisor of the form $q_i^{\alpha}d$ with $1 \leq \alpha \leq \alpha_i$ and $d \mid (n/q_i^{\alpha_i})$, the divisor $u_{q_i^{\alpha}d}$ of u_n has a primitive prime factor $p_{q_i^{\alpha}d} \equiv 1 \pmod{dq_i^{\alpha}}$. In particular, $q_i^{\alpha} \mid p_{dq_i^{\alpha}} - 1$, and the primes $p_{dq_i^{\alpha}}$ are distinct as d ranges over the divisors of $n/q_i^{\alpha_i}$. Thus,

$$\begin{aligned} q_i^{(1+\cdots+\alpha_i)\tau(n/q_i^{\alpha_i})} &\mid \prod_{1\leq\alpha\leq\alpha_i}\prod_{d|n/q_i^{\alpha_i}}(p_{dq_i^{\alpha}} - 1) \mid \prod_{p|u_n}(p-1) \\ &= \phi(u_n) \mid u_n - 1 \mid gu_{n-1}, \end{aligned}$$

which gives the two central inequalities. The first inequality is trivial and the equality holds when $\alpha_i = 1$. For the last inequality, if $q_i \mid g$, then $\nu_{q_i}(gu_{n-1}) = \nu_{q_i}(g(gu_{n-2}+1)) = \nu_{q_i}(g)$. If $q_i \nmid g$, then $\nu_{q_i}(gu_{n-1}) = \nu_{q_i}(u_{n-1})$, and we apply Lemma 1-*v)*. □

Lemma 7. *Let u_n be a Lehmer number with both n and g odd. If $q_i > \sqrt{g}$, then*

$$\tau(n/q_i) \leq q_i - 2.$$

Proof. If $q_i \mid g$ and $q_i > \sqrt{g}$, then $\nu_{q_i}(g) = 1$, and Lemma 6 above gives

$$\tau(n/q_i) \leq \nu_{q_i}(g) = 1 \leq q_i - 2. \tag{6}$$

If $q_i \nmid g$, then, again by Lemma 6 above, we have

$$\tau(n/q_i) \leq \nu_{q_i}(u_{q_i-1}).$$

Observe that

$$u_{q_i-1} \mid g^{q_i-1} - 1 = \left(g^{(q_i-1)/2} - 1\right)\left(g^{(q_i-1)/2} + 1\right).$$

Since q_i cannot divide both factors above, we have that

$$\tau(n/q_i) \leq \nu_{q_i}(g^{(q_i-1)/2} + \epsilon) \qquad \text{for some } \epsilon \in \{-1, +1\}.$$

If $\tau(n/q_i) \geq q_i - 1$, then

$$q_i^{q_i-1} \leq q_i^{\tau(n/q_i)} \leq g^{(q_i-1)/2} + 1 \leq (q_i^2 - 1)^{(q_i-1)/2} + 1,$$

and we get a contradiction for $q_i > 3$, because then

$$q_i^{q_i-1} = ((q_i^2 - 1) + 1)^{(q_i-1)/2},$$

and we see that the above expression on the right is larger that $(q_i^2 - 1)^{(q_i-1)/2} + 1$ except when $q_i = 3$.

Finally, if $q_i = 3$, the only odd $g < q_i^2$ with $q_i \nmid g$ are $g = 5$ and $g = 7$. But in both cases we have $\tau(n/3) \leq \nu_3(u_2) \leq 1 \leq q_i - 2$, which completes the proof of this lemma. □

Lemma 8. *Let u_n be a Lehmer number with both n and g odd. Then*

$$q_1 \leq \max\{\sqrt{g}, 19\}. \tag{7}$$

Proof. Assume that the above inequality does not hold. Then $q_1 \geq 23$, $g \leq q_1^2 - 1$, and since $q_1 > \sqrt{g}$, we can apply Lemma 7 to deduce that $\tau(n) \leq 2\tau(n/q_i) \leq 2q_i - 4$. We also observe that $\tau(n) \geq 2^{\omega(n)}$, so $\omega(n) \leq \log(2q_1 - 4)/\log 2$.

Since u_n is a Lehmer number, we have that $2 \leq u_n/\phi(u_n)$. Now Lemma 5 and the bounds above give

$$\begin{aligned} \log 2 &< \frac{\log\left((2q_1-4)/\log 2\right)}{2q_1}\left(1 + \log\left(\frac{q_1 \log(q_1^2-1)}{\log(2q_1+1)}\right)\right) \\ &+ \frac{2q_1-5}{2q_1^2}\left(1 + \log\left(\frac{q_1^2 \log(q_1^2-1)}{\log(2q_1^2+1)}\right)\right), \end{aligned}$$

which is false for $q_1 \geq 23$. □

3.3 The conclusion of the proof of Theorem 1

Since we have already proved that both $s = \omega(n)$ and $\tau(n)$ are bounded, in order to conclude the proof of Theorem 1 it is enough to prove that all the primes q_i with $i = 1, \ldots, s$ are also bounded. We shall prove this by induction on $i = 1, \ldots, s$ observing that this has already been achieved for $i = 1$. Let $i \leq s - 1$ and assume that q_i has been bounded. Put $Q_i = \prod_{j=1}^{j=i} q_j^{\alpha_j}$. There are only finitely many possibilities for this number. We put $g_i = g^{Q_i}$, $n_i = n/Q_i$ and rewrite the condition that u_n is Lehmer as

$$a\phi\left(\frac{g^{Q_i}-1}{g-1} \cdot \frac{g_i^{n_i}-1}{g_i-1}\right) = u_n - 1 = \frac{g^{Q_i}-1}{g-1} \cdot \frac{g_i^{n_i}-1}{g_i-1} - 1$$

with some integer $a \geq 2$. We put $w_m = (g_i^m - 1)/(g_i - 1)$ for the sequence of repunits in base g_i. Then, since u_n is square-free, we get that

$$a\phi(u_{Q_i})\phi(w_{n_i}) = u_{Q_i}w_{n_i} - 1,$$

therefore

$$a\frac{\phi(u_{Q_i})}{u_{Q_i}} = \frac{w_{n_i}}{\phi(w_{n_i})} - \frac{1}{u_{Q_i}\phi(w_{n_i})}. \tag{8}$$

The left hand side takes only finitely many values. Assume that it takes some value $\delta < 1$. If n_i is sufficiently large such that $\phi(w_{n_i}) > 1/u_{Q_i}(1-\delta)$, we then get that

$$\frac{w_{n_i}}{\phi(w_{n_i})} = \delta + \frac{1}{u_{Q_i}\phi(w_{n_i})} < 1,$$

which is obviously impossible. Thus, n_i (therefore n) is bounded in case $\delta < 1$. If on the other hand $\delta = 1$, then

$$w_{n_i} - \frac{1}{u_{Q_i}} = \phi(w_{n_i}),$$

which is impossible since $u_{Q_i} > 1$. Thus, it remains to study the case when the right hand side in (8) is > 1. Let $\delta_i > 1$ be the smallest possible value larger than 1 of the left hand side of (8). We then get

$$\delta_i < \frac{w_{n_i}}{\phi(w_{n_i})}.$$

We observe that w_{n_i} is a sequence "like" u_n but the new value of g is $g_i = g^{Q_i}$ and the new value of n is $n_i = n/Q_i$. Thus, the smallest prime factor of n_i is q_{i+1}. We also note that $\tau(n_i) = \tau(n/Q_i) \leq \tau(n/q_1) \leq 2q_1 - 4$, and that $\omega(n_i) \leq \log(2q_1 - 4)/\log 2$. Finally, we observe that $(w_{n_i}, g^{Q_i} - 1) = 1$, otherwise, since $(w_{n_i}, g - 1) = 1$, the number $u_n = (g^{Q_i} - 1)/(g - 1)w_{n_i}$ would not be square-free.

We now apply Lemma 5 to obtain that

$$\begin{aligned} \delta_i &< \frac{\omega(n_i)}{2q_{i+1}}\left(1 + \log\left(\frac{Q_i q_{i+1}\log g}{\log(2q_{i+1}+1)}\right)\right) \\ &+ \frac{\tau(n_i)-1}{2q_{i+1}^2}\left(1 + \log\left(\frac{Q_i q_{i+1}^2\log g}{\log(2q_{i+1}^2+1)}\right)\right). \end{aligned} \tag{9}$$

Hence, $\log \delta_i \ll \frac{\log q_{i+1}}{q_{i+1}}$, where the constant involved only depends on g implying that q_{i+1} must be bounded. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

We assume that $3 \le q_1 \le 31,\ 3 \le g \le 999$ and g is odd.

Claim 1: If $q_1 \nmid g$, then $\nu_{q_1}(\phi(u_n)) \le \nu_{q_1}(u_{q_1-1}) \le 5$.

This can be checked with Mathematica.

Claim 2: $\tau(n/q_1) \le \nu_{q_1}(\phi(u_n)) \le 6$, and $s \le 3$.

Suppose first that $q_1 \mid g$. Then, by Lemma 6,

$$\tau(n/q_1) \le \nu_{q_1}(\phi(n)) \le \nu_{q_1}(gu_{n-1}) = \nu_{q_1}(g) \le \left\lfloor \frac{\log g}{\log q_1} \right\rfloor \le \left\lfloor \frac{\log 1000}{\log 3} \right\rfloor \le 6.$$

Furthermore, the above inequality is achieved only when $(q_1, g) = (3, 729)$. Assume now that $q_1 \nmid g$. By Claim 1, either $q_1 = 3$ and $\tau(n/q_1) \le 6$, or $\tau(n/q_1) \le 5$. In particular, $\tau(n) \le 2\tau(n/q_1) \le 12$, which shows that $s \le 3$.

Claim 3: $s \ge 2$.

Let us see indeed that for our particular case we cannot have $s = 1$. If this were so, then $n = q_1^{\alpha_1}$. Then each prime factor p_j of u_n is primitive for some divisor $d > 1$ of n, which is a power of q_1 (again, this is because $\gcd(u_n, g-1) = 1$). Thus, $p_j \equiv 1 \pmod{q_1}$ for all $j = 1, \ldots, K$, showing that $\nu_{q_1}(\phi(u_n)) \ge K \ge 14$ (see Lemma 2-*iii)*), which contradicts the fact that $\nu_{q_1}(\phi(u_n)) \le 6$. Hence, $s \ge 2$.

Claim 4: $\alpha_1 = 1$ except when $(\alpha_1, q_1, g) = (2, 3, 729)$.

Put again, as in the proof of Theorem 1, $Q_1 = q_1^{\alpha_1}$. By Lemma 6 and the fact that $s \ge 2$, we have

$$\alpha_1(\alpha_1 + 1) \le \frac{\alpha_1(\alpha_1+1)}{2}\tau(n/q_1^{\alpha_1}) \le \nu_{q_1}(\phi_n).$$

By Claims 1 and 2 above, we know that $\nu_{q_1}(\phi(u_n)) \le 5$, except when $(\alpha_1, q_1, g) = (2, 3, 729)$. So, $\alpha_1 = 1$ except for this case.

Note that, at any rate, since $s \ge 2$, it follows that $2 \le \tau(n/q_1) \le \nu_{q_1}(gu_{q_1-1})$. A computation with Mathematica revealed 431 possibilities for the pairs (q_1, g) in our range satisfying $\nu_{q_1}(gu_{q_1-1}) \ge 2$.

Claim 5: $q_2 \le 23$.

The smallest left hand side in (8) computed over all the 432 possible pairs (Q_1, g) has $\delta_1 > 1.49$ (it was obtained for $g = 809,\ Q_1 = q_1 = 3$ and $a = 2$, for which the obtained value is > 1.495). Of course, we did not factor all the numbers of the form $(g^{Q_1}-1)/(g-1)$. If $q_1 = 31$, then the smallest prime $p_1 \equiv 1 \pmod{q_1}$ is 311. The number K of prime factors of u_{31} satisfies therefore

$$K < \frac{\log u_{q_1}}{\log p_1} < \frac{3 \cdot 31 \cdot \log 10}{\log 311} < 38;$$

hence,

$$a\frac{\phi(u_{q_1})}{u_{q_1}} \geq 2\left(1 - \frac{1}{311}\right)^{37} > 1.7.$$

Similarly, using the fact that when $q_1 = 29$ and 23 the first two primes congruent to 1 $\pmod{q_1}$ are 59 and 233, and 47 and 139 respectively, and

$$\frac{3 \cdot 29 \cdot \log 10}{\log 233} < 37 \qquad \text{and} \qquad \frac{3 \cdot 23 \cdot \log 10}{\log 139} < 33,$$

we have that

$$\begin{aligned} a\frac{\phi(u_{q_1})}{u_{q_1}} &\geq 2\min\left\{\left(1-\frac{1}{59}\right)\left(1-\frac{1}{233}\right)^{36}, \left(1-\frac{1}{47}\right)\left(1-\frac{1}{139}\right)^{32}\right\} \\ &> 1.55, \end{aligned}$$

whenever $q_1 \in \{23, 29\}$. Thus, we have factored only the numbers u_{Q_1} with $Q_1 \leq 19$. We now use inequality (9) for $i = 1$ to obtain

$$\begin{aligned} \log(1.49) &< \frac{\omega(n_1)}{2q_2}\left(1 + \log\left(\frac{Q_1 q_2 \log g}{\log(2q_2+1)}\right)\right) \\ &+ \frac{\tau(n_1)-1}{2q_2^2}\left(1 + \log\left(\frac{Q_1 q_2^2 \log g}{\log(2q_2^2+1)}\right)\right). \end{aligned}$$

If $q_1 > 3$, then $Q_1 = q_1 \leq 31$. If $q_1 = 3$, then $Q_1 = q_1^2 = 9$. Thus, $Q_1 \leq 31$ in both cases. We also saw in Claims 1 and 2 that $\tau(n_1) \leq \tau(n/q_1) \leq 6$, so also $\omega(n_1) \leq 2$. Hence,

$$\log(1.49) < \frac{1}{q_2}\left(1 + \log\left(\frac{31 q_2 \log 999}{\log(2q_2+1)}\right)\right) + \frac{5}{2q_2^2}\left(1 + \log\left(\frac{31 q_2^2 \log 999}{\log(2q_2^2+1)}\right)\right),$$

and this inequality does not hold when $q_2 \geq 29$.

4.1 The conclusion of the proof of Theorem 2

Thus, $3 \leq q_1 < q_2 \leq 23$. The argument showing that $\alpha_1 = 2$ except if $(q_1, g) = (3, 729)$ now shows that $\alpha_2 = 1$. We are now able to show that $s = 2$. Indeed, if it were not so, then we would have both $\tau(n/q_1) \geq 4$ and $\tau(n/q_2) \geq 4$. A quick computation with Mathematica shows that while there are pairs (q, g) such that $\nu_q(gu_{q-1}) \geq 4$ in our ranges, there is no odd g in $[3, 999]$ that has the above property with respect to two different primes $3 \leq q_1 < q_2 \leq 19$. Thus, either $n = q_1q_2$, or $n = 9q_2$ and $g = 729$. To test these last pairs, we proceeded as follows. First we have detected all pairs (n, g) with $n = q_1q_2$ with $3 \leq q_1 < q_2 \leq 19$ and odd $g \in [3, 999]$ such that $\nu_{q_i}(gu_{n-1}) \geq 2$ holds for both $i = 1, 2$. There are 2043 such pairs. For each one of these we checked that $\nu_2(u_{n-1}) < 14$. Similarly, when $Q_1 = 9$ and $g = 729$, the only possibility for q_2 in our range such that $\nu_{q_2}(u_{q_2-1}) \geq 2$ is $q_2 = 11$, but in this case $n = 99$ and $\nu_2(u_{n-1}) = 1 < 14$. This finishes the proof of Theorem 2.

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