AN UPPER BOUND FOR $B_2[2]$ SEQUENCES

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Abstract. *We introduce a new counting method to deal with $B_2[2]$ sequences, getting a new upper bound for the size of these sequences, $F(N,2) \le \sqrt{6N}+1$.*

Introduction and Notation

For a sequence of positive integers A, the functions $r(n)$,$r'(n)$, $d(n)$ denote the number of solutions of

$$n = b + a, \quad a \le b, \quad a, b \in A.$$

$$n = b + a, \quad a < b, \quad a, b \in A.$$

$$n = b - a, \quad a < b, \quad a, b \in A.$$

We say that a sequence of integers A belongs to the class of $B_2[g]$ sequences if $r(n) \le g$ for any integer n.

To find $B_2[g]$ sequences as dense as possible is a standard problem in additive number theory ([2],[5],[6],[7]). We define $F(N,g) = \max\{|A|, A \subset [1,N], A \in B_2[g]\}$. One is interested in finding precise bounds of $F(N,g)$

For finite $B_2[g]$ sequences we have the following simple argument. Let $A \subset [1,N]$, be A a $B_2[g]$ sequence. Then

$$\binom{|A|+1}{2} = \sum_{n=1}^{2N} r(n) \le 2Ng,$$

which yields $F(N,g) \le 2\sqrt{gN}$.

$B_2[1]$ sequences are usually called Sidon sequences. It is easy to see that these sequences also satisfy that $d(n) \le 1$ for any integer $n \ge 1$. This special property of the differences makes it easier to get a better upper bound for the size of finite Sidon sequences. Erdős [3] used this fact to prove that

$$F(N,1) = F(N) = \sqrt{N}(1 + o(1)) \quad \text{as } N \to \infty.$$

Unfortunatly, the function $d(n)$ is not bounded for $B_2[g]$ sequences if $g > 1$ and Erdős argument does not apply. Recently, I.Ruzsa, C.Trujillo and the author [1], using a density argument, have proved that $F(N,g) \le 1.864\sqrt{gN}$.

In this paper we use a new combinatorial argument which allows us to take advantage of average behaviour of the differences for $B_2[2]$ sequences.

Theorem 1. $F(N,2) \le \sqrt{6N}+1$.

On the other hand, it is known [1] that $F(N,2) \ge (3/2 + o(1))\sqrt{N}$.

Theorem 1 improves the above result for $g = 2$ and the proof is completely different and easier.

Proof of theorem 1

Lemma 1. *For any finite sequence A of positive integers we have*
i) $\binom{|A|}{2} = \sum_{n\geq 1} r'(n)$
ii) $\binom{|A|}{2} = \sum_{n\geq 1} d(n)$

Proof. Obvious.

Let A a $B_2[2]$ sequence, $A \subset [1, N]$ and we define $R_j = \{n \notin 2\times A; r(n) = j\}$ and $R'_j = \{n \in 2\times A; r(n) = j\}$, for $j = 0, 1, 2.$ where $2 \times A$ denotes the set $\{2a; a \in A\}$.

Lemma 2. *If A is a $B_2[2]$ sequence then*

$$\sum_{n\geq 1} \binom{d(n)}{2} = 2|R_2| + |R'_2|$$

Proof. We associate, for each $m \in R_2$, the unique 4-tupla (a, b, c, d) such that

$$a < b < c < d \text{ and } a + d = b + c = m.$$

Conversely, each 4-tupla (a, b, c, d), $a < b < c < d$, $a + d = b + c$ corresponds to a unique integer $m \in R_2$.

Similary, for each $m \in R'_2$, we associate the 4-tupla (a, b, b, d), $a < b < d$ such that $a + d = b + b$. Again, each 4-tupla (a, b, b, d), $a < b < d$, $a + d = b + b$ corresponds to an unique integer $m \in R'_2$.

Now, for each positive integer n and for each pair $(b, a), (d, c), a < c$ of different solutions of $x - y = n$, we consider the 4-tupla (a, b, c, d) if $b < c$, the 4-tupla (a, c, b, d) if $c < b$ and the 4-upla (a, b, b, d) if $b = c$.

Then, each 4-tupla (a, b, c, d), $a < b < c < d$, $a + d = b + c$ comes from the pair of solutions $(b, a), (d, c)$ of $x - y = b - a$ and from the pair of solutions $(c, a), (d, b)$ of $x - y = c - a$.

Then, each $m \in R_2$ is counted exactly twice in the sum $\sum_{n\geq 1} \binom{d(n)}{2}$.

Similary, each 4-tupla (a, b, b, d), $a < b < d$, $a + d = b + b$ comes only from the unique pair of solutions $(b, a), (d, b)$ of $x - y = b - a$.

Then, each $m \in R'_2$ is counted once in the sum $\sum_{n\geq 1} \binom{d(n)}{2}$.

Lemma 3. $\sum_{n\geq 1} \binom{d(n)}{2} \leq \binom{|A|}{2}$

Proof. Observe that Lemma 1, i) can be written as $\binom{|A|}{2} = 2|R_2| + |R'_2| + |R_1|$

Proof of theorem 1.

From lemma 3 and lemma 1, ii) we have

$$\sum_{n\geq 1} d^2(n) = \sum_{n\geq 1} d(n) + 2\binom{|A|}{2} - 2|R_1| \leq 3\binom{|A|}{2}$$

On the other hand

$$\binom{|A|}{2}^2 = \left(\sum_{n\geq 1} d(n)\right)^2 \leq N \sum_{n\geq 1} d^2(n).$$

Then

$$\binom{|A|}{2} \leq 3N$$

and $|A| \leq \sqrt{6N} + 1$.

Note. *M.Helm [4] has proved independently and in a different way the upper bound* $F(N,2) \leq \sqrt{(6N)} + O(1)$.

References

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