THE ADDITIVE COMPLETION OF Kth-POWERS.

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ABSTRACT.

Let $k \geq 2$ be an integer. For fixed N, we consider a set A^N of non-negative integers such that for all integer $n \leq N$, n can be written as $n = a + b^k$, $a \in A^N$, b a positive integer.

We are interested in a lower bound for the number of elements of A^N.

Improving a result of Balasubramanian [1], we prove the following theorem:

Theorem 1.

$$|A^N| \geq N^{1-\frac{1}{k}}\Big\{\frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})} + o(1)\Big\}.$$

1. STATMENT OF RESULT AND PRELIMINARY LEMMAS.

Let $k \geq 2$ be an integer. For fixed N, we consider a set A^N of non-negative integers such that for all integer $n \leq N$, n can be written as $n = a + b^k$, $a \in A^N$, b a positive integer.

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Lemma 1. [1].

If $f(n) \geq 0$, then $\sum_{a+b^k \leq N} f(a+b^k) \geq \sum_{n=1}^{N} f(n)$.

Proof.

It is obvious. The equality doesn't hold in general because an integer n may have more than one representation as $n = a + b^k$.

Lemma 2.

If $f(x) = g(\frac{x}{N})$, g continuous and g differentiable except at a finite number of points, then

$$\sum_{n=1}^{N} f(n) = N \int_0^1 g(x)dx + O(1)$$

and

$$\sum_{b \leq (N-a)^{\frac{1}{k}}} f(a + b^k) = N^{\frac{1}{k}} h(\frac{a}{N}) + O(1)$$

where

$$h(x) = \int_0^{(1-x)^{\frac{1}{k}}} g(x + t^k)dt \tag{1}$$

and the constants appearing in the error terms are independent of N.

(The proof follows by a straightforward application of Euler's identity)

2. PROOF OF THE THEOREM.

Since the cardinal of A^N has the order of $N^{1-\frac{1}{k}}$, we have

$$\sum_{a \in A} h(\frac{a}{N}) \geq N^{1-\frac{1}{k}} \int_0^1 g(x)dx + O(N^{1-\frac{2}{k}}). \tag{2}$$

The function g which we will eventually choose will satisfy (as is proved later) the following conditions:

(i) $g(x) \geq 0$
(ii) g is continuous everywhere and differentiable except at a finite number of points.
(iii) $h(x)$ defined above has continuous derivative in $[0, 1)$
(iv) h reaches its maximum at some $0 < y_0 < 1$.
(v) $h'(x) > 0$, for all $x < y_0$.

Under these conditions let us write

$$A^N = B \bigcup (\bigcup_{m=1}^{M} A_m)$$

where

$$A_m = \{a \in A^N, \frac{(m-1)N}{M} y_0 \leq a < \frac{mN}{M} y_0\}$$

and

$$B = \{a \in A^N, a \geq N y_0\}.$$

Later we shall let M tend to infinity.

Then

$$\sum_{a \in A} h(\frac{a}{N}) = \sum_{m=1}^{M} \sum_{a \in A_m} h(\frac{a}{N}) + \sum_{a \in B} h(\frac{a}{N}) \leq$$

$$\leq \sum_{m=1}^{M} |A_m| h(\frac{m}{M} y_0) + |B| h(y_0) =$$

$$= \sum_{m=1}^{M} |A_m| \{h(\frac{m}{M} y_0) - h(y_0)\} + h(y_0)|A^N| \tag{3}$$

because $|B| = |A^N| - \sum_{m=1}^{M} |A_m|$.

Since $h(\frac{xy_0}{M}) - h(y_0)$ has a continuous derivative in $[1, M]$, due to (iii) and (iv), we can apply Abel's summation formula to get

$$\sum_{m=1}^{M} |A_m| \{h(\frac{m}{M} y_0) - h(y_0)\} = -\int_1^M \frac{y_0}{M} h'(\frac{xy_0}{M}) \sum_{m \leq x} |A_m| dx. \tag{4}$$

To estimate $\sum_{m \leq x} |A_m| = \#\{a \in A, a < \frac{[x]N}{M} y_0\}$ is precisely our initial problem but now for $\frac{[x]N}{M} y_0$.

Let us assume that we have proved

$$\liminf_{N \to \infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \geq c_0.$$

This is true for $c_0 = 1$ trivially. It means that for all $\epsilon > 0$, there exists $N(\epsilon)$, such that $|A^N| \geq (c_0 - \epsilon) N^{1-\frac{1}{k}}$ if $N > N(\epsilon)$.

Then, from (v) and for $N > N(\epsilon)/y_0$ we have

$$-\int_1^M \frac{y_0}{M} h'(\frac{xy_0}{M}) \sum_{m \le x} |A_m| dx \le -(c_0 - \epsilon) \int_{\frac{N(\epsilon)M}{Ny_0}}^M \frac{y_0}{M} h'(\frac{xy_0}{M})(\frac{[x]N}{M} y_0)^{1-\frac{1}{k}} dx =$$

$$-(c_0 - \epsilon) \int_{N(\epsilon)/N}^{y_0} h'(x)(\frac{[\frac{Mx}{y_0}]N}{M} y_0)^{1-\frac{1}{k}} dx.$$

Substituing in (3) and making $M \to \infty$, we get

$$\sum_{a \in A} h(\frac{a}{N}) \le -(c_0 - \epsilon) N^{1-\frac{1}{k}} \int_{N(\epsilon)/N}^{y_0} h'(x) x^{1-\frac{1}{k}} dx + h(y_0)|A^N|. \tag{5}$$

After, substituing (5) in (2) we have

$$|A^N| \ge N^{1-\frac{1}{k}} \frac{\int_0^1 g(x)dx + (c_0 - \epsilon) \int_{\frac{N(\epsilon)}{N}}^{y_0} h'(x) x^{1-\frac{1}{k}} dx}{h(y_0)} + O(N^{1-\frac{2}{k}}).$$

Then

$$\liminf_{N \to \infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \ge \frac{\int_0^1 g(x)dx + (c_0 - \epsilon) \int_0^{y_0} h'(x) x^{1-\frac{1}{k}} dx}{h(y_0)}$$

for every $\epsilon > 0$.

Therefore

$$\liminf_{N \to \infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \ge c_1 = \frac{\int_0^1 g(x)dx + c_0 \int_0^{y_0} h'(x) x^{1-\frac{1}{k}} dx}{h(y_0)}$$

and we apply the same process for c_1, to obtain

$$c_2 = \frac{\int_0^1 g(x)dx + c_1 \int_0^{y_0} h'(x) x^{1-\frac{1}{k}} dx}{h(y_0)}.$$

Reapeating the process indefinity we get

$$\liminf_{N \to \infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \ge \frac{\int_0^1 g(x)dx}{h(y_0) - \int_0^{y_0} h'(x) x^{1-\frac{1}{k}} dx}$$

for every g satisfying (i),(ii),(iii),(iv) and (v).

Let

$$g_\alpha(x) = \begin{cases} 0 & x < \alpha \\ x - \alpha & x \ge \alpha \end{cases} \qquad \alpha < 1$$

$h_\alpha(x), h'_\alpha(x)$ and $y_{0,\alpha}$ can be calculated explicitly.

$$h_\alpha(x) = \begin{cases} \frac{(1-x)^{\frac{1}{k}}}{k+1}(1-(k+1)\alpha+kx) + \frac{k(\alpha-x)^{\frac{k+1}{k}}}{k+1} & x < \alpha \\ \frac{(1-x)^{\frac{1}{k}}}{k+1}(1-(k+1)\alpha+kx) & x \geq \alpha \end{cases}$$

$$h'_\alpha(x) = \begin{cases} (1-x)^{\frac{1}{k}-1}(1-\frac{1-\alpha}{k}-x) - (\alpha-x)^{\frac{1}{k}} & x < \alpha \\ (1-x)^{\frac{1}{k}-1}(1-\frac{1-\alpha}{k}-x) & x \geq \alpha \end{cases}$$

$$y_{0,\alpha} = 1 - \frac{1-\alpha}{k}$$

One can verify directly that g_α satisfies all the conditions (i)-(v), so that

$$\liminf_{N\to\infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \geq c_\alpha = \frac{\int_0^1 g_\alpha(x)dx}{h_\alpha(y_{0,\alpha}) - \int_0^{y_{0,\alpha}} h'_\alpha(x)x^{1-\frac{1}{k}}dx} =$$

$$= \frac{\int_0^1 g_\alpha(x)dx}{\int_0^{y_{0,\alpha}} h'_\alpha(x)\{1-x^{1-\frac{1}{k}}\}dx + h_\alpha(0)}, \qquad \alpha < 1$$

Explicitly

$$c_\alpha = \frac{\frac{(1-\alpha)^2}{2}}{H_\alpha}$$

with

$$H_\alpha = \int_0^\alpha \{(1-x)^{\frac{1}{k}-1}(1-\frac{1-\alpha}{k}-x) - (\alpha-x)^{\frac{1}{k}}\}(1-x^{1-\frac{1}{k}})dx +$$

$$+ \int_\alpha^{1-\frac{1-\alpha}{k}} (1-x)^{\frac{1}{k}-1}(1-\frac{1-\alpha}{k}-x)(1-x^{1-\frac{1}{k}})dx + \frac{1-(k+1)\alpha}{k+1} + \frac{k\alpha^{\frac{k+1}{k}}}{k+1}$$

Then,

$$\liminf_{N\to\infty} \frac{|A^N|}{N^{1-\frac{1}{k}}} \geq \lim_{\alpha\to 1} c_\alpha = \frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})}$$

and we leave the straightforward details as an exercise to the reader.

3. OBSERVATIONS.

1) The following table shows our constants for the first values of k and compares them with preceeding results:

L.Moser	
Donagi and Herzog	$1+\frac{k-1}{2k^2}$
Abbott	
Balasubramanian	$(2-\frac{2}{k+1})^{\frac{1}{k}}$
Theorem 1	$1/\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})$

2)Theorem 1 is sharp in the following sense:

Let $r(n)=\#\{n=a+b^k, a\in A\}$ (in our case $r(n)\geq 1$ so that $\sum_1^N r(n)\geq N$); if we assume that for each N, there exists A^N such that $\sum_1^N r(n)=N+o(N)$, then theorem 1 is best possible.

Let us assume

$$\liminf_{N\to\infty}\frac{|A^N|}{N^{1-\frac{1}{k}}}=c \tag{6}$$

$$\sum_{a\in A}\sum_{a+b^k\leq N}1=\sum_{n=1}^{N}r(n)=N+o(N)$$

Then

$$\sum_{a\in A}(N-a)^{\frac{1}{k}}=N+o(N)$$

Let us write

$$A = \bigcup_{m=1}^{M} A_m$$

$$A_m = \{a \in A, \frac{m-1}{M}N < a \leq \frac{m}{M}N\}.$$

Now

$$\sum_{a\in A}(N-a)^{\frac{1}{k}} = \sum_{m=1}^{M}\sum_{a\in A_m}(N-a)^{\frac{1}{k}} \geq N^{\frac{1}{k}}\sum_{m=1}^{M}|A_m|(1-\frac{m-1}{M})^{\frac{1}{k}}.$$

Using Abel's summation formula we obtain

$$(\frac{N}{M})^{\frac{1}{k}}|A| + N^{\frac{1}{k}}\int_1^M \frac{1}{k}(1-\frac{t-1}{M})^{\frac{1}{k}}\sum_{m\leq t}|A_m|dt \leq N + o(N) \tag{7}$$

By (6), we have

$$\sum_{m\leq t}|A_m| \geq (c-\epsilon)(\frac{tN}{M})^{1-\frac{1}{k}} \tag{8}$$

for $t \geq \frac{N(\epsilon)M}{N}$.

Then

$$c\int_0^1 \frac{1}{k}(1-x)^{\frac{1}{k}}x^{1-\frac{1}{k}}dx \leq 1 + o(1)$$

after substituting (8) in (7) and making $M \to \infty$.

Therefore

$$c \leq \frac{1}{\Gamma(2-\frac{1}{k})\Gamma(1+\frac{1}{k})}.$$

Of course, we expect that $\sum_{n\leq N} r(n) = N + o(N)$ is false and that the correct conjecture is $\sum_{n\leq N} r(n) \geq kN + o(N)$.

Open problem: *for each k, find a constant $c_k > 1$ such that $\sum_{n\leq N} r(n) \geq c_k N + o(N)$*

3) In the proof of theorem 1 we didn't use the arithmetic properties of the kth-powers and the theorem can be generalized to sequences of the form

$b_n = \beta n^\gamma + o(n^\gamma)$, $\beta > 0$, $\gamma > 1$ getting

$$\liminf_{N\to\infty} \frac{|A|}{N^{1-\frac{1}{\gamma}}} \geq \frac{\beta^{\frac{1}{\gamma}}}{\Gamma(2-\frac{1}{\gamma})\Gamma(1+\frac{1}{\gamma})}.$$

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