JOURNAL OF NUMBER THEORY **43**, 198–202 (1993)

The Distribution of the Lattice Points on Circles

JAVIER CILLERUELO

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049-Madrid, España

Communicated by Hans Zassenhaus

Received May 7, 1991

1. INTRODUCTION AND STATEMENT OF RESULTS

Let us denote by $r(n)$ the number of the representations of the integer n as a sum of two squares, i.e., $r(n)$ is the number of lattice points on the circle $x^2 + y^2 = n$.

It is a well known fact that if

$$n = 2^{\nu} \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{\beta_k}$$

is the prime factorization of the integer n, then $r(n) = 0$ unless all the exponents β_k are even. In that case we have $r(n) = 4 \prod (1 + \alpha_j)$.

In [1], a new method was developed to study the location of lattice points on circles centered at the origin.

In the following we introduce the notation needed to understand the method:

We shall associate lattice points with Gaussian integers; $a^2 + b^2 = n$ determines a Gaussian integer $a + bi = \sqrt{n}\, e^{2\pi i \Phi}$ for a suitable phase Φ.

A prime $p_j \equiv 1 \pmod 4$ can be represented as a sum of two squares, $p_j = a^2 + b^2$, $0 < a < b$, in only one way. Then, for each p_j, then angle Φ_j such that $a + bi = \sqrt{p_j}\, e^{2\pi i \Phi_j}$ is well defined.

We proved in [1] the following lemma:

LEMMA 1. *If* $n = 2^{\nu} \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k}$ *then, the Gaussian integers corresponding to the* $r(n) = 4 \prod (1 + \alpha_j)$ *lattice points on the circle* $x^2 + y^2 = n$ *are given by the formula*

$$\sqrt{n}\, e^{2\pi i \{\Phi_0 + \sum_j \gamma_j \Phi_j + t/4\}},$$

0022-314X/93 \$5.00

where Φ_j *is the angle corresponding to* p_j, γ_j *runs over the set* $\{\gamma \in Z; |\gamma| \leqslant \alpha_j, \gamma \equiv \alpha_j \pmod 2\}$, t *takes the values* 0, 1, 2, 3, and

$$\Phi_0 = \begin{cases} 0 & \text{if } v \text{ is even} \\ \frac{1}{8} & \text{if } v \text{ is odd.} \end{cases}$$

One may think that the lattice points are well distributed around the circle when $r(n)$ is large enough, but also we could choose the Φ_j in such a way that all the lattice points are on small arcs. In the following we shall try to answer these questions.

First, we have to measure the distribution of the lattice points in some way. Let us consider the quanty $S(n)/\pi n$ for $r(n) \neq 0$, where $S(n)$ is the area of the polygon whose vertices are the lattice points on the circle $x^2 + y^2 = n$. These will be "better distributed" if $S(n)/\pi n$ is close enough to the number 1.

In this direction we have the following theorem.

THEOREM 1.

$$\left|\frac{S(n)}{\pi n} - 1\right| \ll \left(\frac{\log \log n}{\log n}\right)^2 \qquad \textit{for infinite many integers.}$$

We cannot hope for a much better result because it is easy to prove that $|S(n)/\pi n - 1| \gg 1/r^2(n)$ and $r(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$.

Obviously, Theorem 1 implies that

$$\limsup_{\substack{n \to \infty \\ r(n) \neq 0}} \frac{S(n)}{\pi n} = 1.$$

On the other hand, we have

$$\liminf_{\substack{n \to \infty \\ r(n) \neq 0}} \frac{S(n)}{\pi n} = \frac{2}{\pi}$$

because it is clear that $S(2^k)/\pi 2^k = 2/k$ for every integer k.

The following result also implies that $\liminf_{n \to \infty, r(n) \neq 0} S(n)/\pi n = 2/\pi$ but, in a certain sense, it is more unexpected.

THEOREM 2. *For each* $\varepsilon > 0$ *and for each integer* k, *there exists a circle* $x^2 + y^2 = n$ *such that all the lattice points are on the arcs* $\sqrt{n}\, e^{(\pi/2)(t+\theta)i}$, $|\theta| < \varepsilon$, $t = 0, 1, 2, 3$, and the number of them is greater than k.

We have also the more general result.

THEOREM 3. *The set* $\{S(n)/\pi n,\ r(n)\neq 0\}$ *is dense in the interval* $[2/\pi, 1]$.

Proof of Theorem 1. For each integer k let us consider $n_k = \prod_{m=1}^{50k}(m^2+1)$ and

$$\phi^l = \sum_{m=1}^{l} \arctan\frac{1}{m} - \sum_{m=l+1}^{50k} \arctan\frac{1}{m}, \qquad k < l < 50k$$

by Lemma 1, each angle $\phi^l/2\pi$ determines a lattice point (a_l, b_l) on the circle $x^2+y^2=n_k$.

We do not care if the numbers m^2+1 are primes or not, but, obviously we cannot expect that the lattice points described above are all the lattice points on the circle.

We observe that $\phi^l - \phi^{l-1} = 2\arctan(1/l) \leqslant 2\arctan(1/k)$ and $\sum_{k<l<50k} 2\arctan(1/l) > 2\pi$.

Then, the distance between two neighbour lattice points on the circle is smaller than $\sqrt{n_k}\, 2\arctan(1/k)$.

With an easy argument we can estimate the area $S_0(n_k)$ of the circle's region not included in the polygon whose vertices are $\sqrt{n_k}\, e^{i\phi^l}$, $k<l<50k$.

$$\begin{aligned}
0 &< \pi n_k - S(n_k) < S_0(n_k)\\
&< 50k\left(\frac{n_k}{2}\left(2\arctan\frac{1}{k}\right) - \frac{1}{2}\left(2\sqrt{n_k}\sin\arctan\frac{1}{k}\right)\left(\sqrt{n_k}\cos\arctan\frac{1}{k}\right)\right)\\
&= 50kn_k\left(\arctan\frac{1}{k} - \frac{1}{2}\sin\left(2\arctan\frac{1}{k}\right)\right)\\
&= 50kn_k\left(\arctan\frac{1}{k} - \frac{1}{2}\left(2\arctan\frac{1}{k} - \frac{1}{2}\left(2\arctan\frac{1}{k} + O\left(\frac{1}{k^3}\right)\right)\right)\right) = O\left(\frac{n_k}{k^2}\right).
\end{aligned}$$

It is easy to prove that $k \gg \log n_k/\log\log n_k$ and we conclude the proof of Theorem 1 by dividing by πn_k.

Proof of Theorem 2. We have to use a deep theorem about the distribution of the primes $\rho = a+bi \in Z(i)$, $ab\neq 0$ in the lattice.

THEOREM (I. V. Tchoulanovski). *Let* D *be a convex region contained inside the circle* $x^2+y^2 \leqslant R^2$ *with area* cR^2, $c>0$. *Then*

$$\sum_{\rho\in D} 1 \sim \frac{2cR^2}{\pi\log R}.$$

LEMMA 2. *For each* $\alpha\in[0, 2\pi)$ *and for every* $\varepsilon>0$, *there exists a prime* $p=a^2+b^2$, $a+bi=\sqrt{p}\,e^{i\Phi}$, *such that* $|\Phi-\alpha|<\varepsilon$.

Proof. Let us choose $D = \{re^{i(\alpha+\theta)}, 0 \leqslant r \leqslant R, -\varepsilon < \theta < \varepsilon\}$ and R large enough.

Taking $\alpha = 0$ in Lemma 2 we can find, for each $\varepsilon > 0$ and for each integer k, a prime $p_{\varepsilon,k} = a_{\varepsilon,k}^2 + b_{\varepsilon,k}^2$ such that

$$a_{\varepsilon,k} + ib_{\varepsilon,k} = \sqrt{p_{\varepsilon,k}}\, e^{i\Phi_{\varepsilon,k}}, \qquad |\Phi_{\varepsilon,k}| < \varepsilon/k.$$

Let $n_k = p_{\varepsilon,k}^k$. According to Lemma 1, all the lattice points on the circle $x^2 + y^2 = n_k$ are given by the formula

$$\sqrt{n_k}\, e^{i\{\gamma\Phi_{\varepsilon,k} + (\pi/2)t\}},$$

where γ runs over the set $\{\gamma \in Z;\, |\gamma| \leqslant k, \gamma \equiv k \pmod 2\}$ and t takes the values 0, 1, 2, 3.

To finish the proof of Theorem 2 we observe that $r(n) = 4(k+1) > k$ and $|\gamma\Phi_{\varepsilon,k}| < \varepsilon$ in all the cases.

Proof of Theorem 3. Let $\alpha \in (2/\pi, 1)$. Then, there exists $\beta \in (0, \pi/4)$ such that the area of the dotted region is $\pi\alpha n_k$. The idea is to look for circles such the polygons with vertices in the corresponding lattice points are close to the region described in Fig. 1.

Let us consider $\beta/2^2, \beta/2^3, \ldots, \beta/2^k$, and $\varepsilon = \beta/2^{2k}$.

According to Lemma 2, for each $j = 2, 3, \ldots, k$ we can find a prime $p_j = a_j^2 + b_j^2$, $a_j + ib_j = \sqrt{p_j}\, e^{2\pi i\Phi_j}$ such that $|2\pi\Phi_j - \beta/2^j| < \varepsilon$.

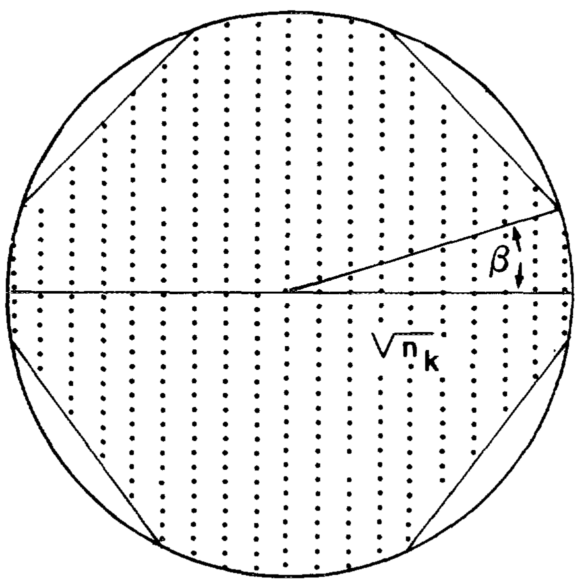

FIGURE 1

We choose $n_k = \prod_{j=2}^{k} p_j^2$. The lattice points on the circle $x^2 + y^2 = n_k$ are given by the formula

$$\sqrt{n_k}\, e^{2\pi i \{\sum_{j=2}^{k} \gamma_j \Phi_j + t/4\}},$$

where γ_j takes the values -2, 0, or 2, and $t = 0, 1, 2, 3$.

All the integers r, $0 \leqslant r < 2^{k-1}$ can be written in the form

$$r = a_0(r)\, 2^0 + a_1(r)\, 2^1 + \cdots + a_{k-2} 2^{k-2},$$

where the $a_j(r)$ takes values 0 or 1.

For every r we choose $\gamma_j^r = 2a_{k-j}(r)$ and we have

$$\sum_{j=1}^{k} \gamma_j^r \Phi_j = 2 \sum_{j=2}^{k} \frac{\beta a_{k-j}(r)\, 2^{k-j}}{2^k} + O\left(\frac{k\beta}{2^{2k}}\right) = \frac{\beta r}{2^{k-1}} + O\left(\frac{k\beta}{2^{2k}}\right).$$

Then, for each r, $0 \leqslant r < 2^{k-1}$ there exists a lattice point (a_r, b_r) on the circle $x^2 + y^2 = n_k$, $a_r + ib_r = \sqrt{n_k}\, e^{2\pi i \Phi_r}$, such that

$$\left|2\pi\Phi_r - \frac{\beta r}{2^{k-1}}\right| < \varepsilon', \qquad \varepsilon' = \frac{k\beta}{2^{2k}}.$$

Then

$$|2\pi\Phi_{r-1} - 2\pi\Phi_r| < \frac{\beta}{2^{k-1}} + 2\varepsilon', \qquad r = 1, ..., 2^{k-1} - 1$$

and

$$|2\pi\Phi_{2^{k-1}-1} - \beta| < \frac{\beta}{2^{k-1}} + \varepsilon'.$$

Furthermore there are no lattice points on the arcs

$$\sqrt{n_k}\, e^{i\theta + t/4}, \qquad \beta + \varepsilon < \theta < \pi/2 - \beta - \varepsilon, \qquad t = 0, 1, 2, 3.$$

Now, with the same geometric argument used in the proof of Theorem 1 and making $k \to \infty$ we obtain the theorem.

References

1. J. Cilleruelo and A. Córdoba, Trigonometrics polynomials and lattice points, *Amer. Math. Soc.* **115**, No. 4 (1992), 899–905.
2. J. Cilleruelo, Arcs containing no three lattice points, *Acta Arith.* **LIX**. 1, (1991), 87–90.
3. I. V. Tchoulanovsky, Démonstration élémentaire de la loi de distribution des nombres premiers de Gauss, *Vestnik L.G.U.* **13** (1956), 5–55.