$B_2[g]$SEQUENCES WHOSE TERMS ARE SQUARES

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INTRODUCTION

Sixty years ago Sidon [7] asked, in the course of some investigations of Fourier series, for a sequence $a_1 < a_2 ...$ for which the sums $a_i + a_j$ are all distinct and for which a_k tends to infinity as slowly as possible. Sidon called these sequences, B_2 sequences. The greedly algorithm gives $a_k \ll k^3$ and this was the best result until Atjai, Komlos and Szemeredi [1] found a B_2 sequence satisfying $a_k = o(k^3)$. However, this result is far from the main conjecture about B_2 sequences.

Conjecture. *Corresponding to every $\epsilon > 0$, there exists a B_2 sequence $\mathcal{A}$ such that $a_j \ll j^{2+\epsilon}$.*

In general we say that a sequence $\mathcal{A}$ is a $B_2[g]$ sequence if $r_n(\mathcal{A}) \leq g$ for all integer n, where $r_n(\mathcal{A})$ is the number of representations of n in the form $n = a + b$, $a \leq b$ $(a, b \in \mathcal{A})$.

In 1960, P.Erdös and A.Renyi [5], using probabilistic methods, proved the following first steps towards the conjecture.

Theorem. *(Erdös-Renyi): Corresponding to every $\epsilon > 0$, there exists a natural number g and a $B_2[g]$ sequence $\mathcal{A}$ such that $a_j \ll j^{2+\epsilon}$.*

On the other hand it is easy to see that a $B_2[g]$ sequence has to satisfy $a_j \ll j^2$. Thus, it seems that the sequence of squares is on the border line for this kind of problem. For this reason and because of the conexion between the additive properties of the sequence of squares and the Fourier Series in the form $\sum a_k e^{ik^2x}$, we have been interested in the study of B_2 sequences whose terms are squares.

The entire sequence of squares cannot be a $B_2[g]$ sequence for any g because the function $r(n) = \#\{n; n = a^2 + b^2, a \leq b\}$ is not bounded uniformily in n.

However, in [3] we proved the existence of a B_2-sequence of squares $\{a_k^2\}$ such that $a_k^2 \ll k^4$. (The reader is referred also to [4] and to the discussion in [6].)

The main purpose of this note is to show that $\mathcal{A}$ in the Erdös-Renyi theorem can be taken to be a subsequence of the squares.

Theorem 1. *Corresponding to every $\epsilon > 0$, there exists a natural number g and a $B_2[g]$ sequence of squares $\{a_k^2\}$ such that $a_k \ll k^{1+\epsilon}$.*

We will use Erdös construction of a probability measure on the space of integer sequences such that (in the resulting probability space) almost all integer sequences have some prescribed rate of growth; thus the probable behaviour of the representation function in the addtion of sequences of precribed rates of growth may then be investigated without further reference to these rates of growth.

PROOF OF THE THEOREM 1.

We will prove the theorem showing, for every $\epsilon > 0$, the existence of a natural number g and a sequence $\{a_k\}$, with $a_k \ll k^{1+\frac{\epsilon}{2}}$ such that for every integer $n \geq n(\epsilon)$, the number of the representations of n in the form $n = a_k^2 + a_j^2$, $a_k \leq a_j$ is less or equal than g.

Let $\Omega = \{\omega\}$ be the space of all the sequences of integers. First of all we will construct, for every $\epsilon > 0$, a probability space such that with probability 1 the sequences in that space satisfy $a_j \sim c_\epsilon j^{1+\frac{\epsilon}{2}}$. We will need the following two theorems which can be found in [6], pg 142-144.

Theorem. *Let $\alpha_1, \alpha_2, \alpha_3, ...$ be real numbers satisfying $0 \leq \alpha_n \leq 1$. Then. there exists a probability space $(\Omega, \mathcal{S}, \mu)$ with the following two properties:*

i) For every natural number n, the event $B^{(n)} = \{\omega; n \in \omega\}$ is measurable and $\mu(B^{(n)}) = \alpha_n$.

ii) The events $B^{(1)}, B^{(2)}, ...$ are independents.

Theorem. *Let $\alpha_j = \dfrac{1}{j^c}$ for every integer $j \geq 1$, $0 < c < 1$. Then, with probability 1 in the space described above, the elements a_j of the sequences $\omega = \{a_j\}$ satisfy $a_j \sim (1-c) j^{\frac{1}{1-c}}$ as $j \to \infty$.*

For our purpose we choose $c = \dfrac{\epsilon}{2+\epsilon}$ in the latter theorem. Then, with probability 1, the elements a_j of the sequences $\omega = \{a_j\}$ satisfy $a_j \sim \frac{2}{2+\epsilon} j^{1+\frac{\epsilon}{2}}$.

We define $r_n(\omega) = \#\{n = a_j^2 + a_k^2; \quad a_j \leq a_k, \quad a_j, a_k \in \omega\}$. Next, we will prove that for every $g > \frac{1}{c} - 1 = \frac{2}{\epsilon}$, with probability 1, $r_n(\omega) \leq g$ for every integer $n \geq n(\epsilon)$. Erdös and Renyi also obtained the estimation $g > \frac{2}{\epsilon}$.

We appeal to the Borel-Cantelli lemma.

Theorem. *Let $\{E_n\}$ be a sequence of measurable events.*

If $\displaystyle\sum_{n=1}^{\infty} \mu(E_n) < +\infty$; then, with probability 1, at most a finite number of such events can occur.

For a natural number $g > \frac{1}{c} - 1$ we consider the events $E_n = \{\omega; r_n(\omega) > g\}$ and we will prove that $\sum_{n=1}^{\infty} \mu(E_n) < +\infty$. Then, with probability 1, at most a finity number of events can occur and the theorem will follow.

We have

$$\mu(E_n) = \sum_{d \geq g+1} \mu(E_{n,d}), \qquad \text{where} \qquad E_{n,d} = \{\omega; r_n(\omega) = d\}.$$

Let $r(n)$ be the function $r(n) = \#\{n = a^2 + b^2; \quad 0 < a \leq b\}$. Then, $n = a_1^2 + b_1 = \cdots = a_{r(n)}^2 + b_{r(n)}^2$, $a_i \leq b_i$.

If $r_n(\omega) = d$ then each component of exactly d pairs $(a_{j_1}, b_{j_1}), ..., (a_{j_d}, b_{j_d})$ among the $r(n)$ pairs $(a_1, b_1), ..., (a_{r(n)}, b_{r(n)})$ belongs to ω.

Let $E_{n,d}(j_1, ..., j_d)$ be the event

$$\{\omega; a_{j_1}, b_{j_1}, ..., a_{j_d}, b_{j_d} \in \omega, a_k \text{ or } b_k \notin \omega, k \neq j_i, i = 1, ..., d\}.$$

Then

$$\mu(E_{n,d}) = \sum_{1 \leq j_1 \cdots < j_d \leq r(n)} \mu(E_{n,d}(j_1, ..., j_d))$$

and

$$\mu(E_{n,d}(j_1, ..., j_d)) = \prod_{i=1}^{d} \mu\{\omega; a_{j_i}, b_{j_i} \in \omega\} \prod_{\substack{k \neq j_i \\ i=1,...,d}} (1 - \mu\{\omega; a_k, b_k \in \omega\});$$

here $\mu\{\omega; a, b \in \omega\} = \frac{1}{a^c}\frac{1}{b^c}$ except when $a = b$. In this case $\mu\{\omega; a, a \in \omega\} = \mu\{\omega; a \in \omega\} = \frac{1}{a^c}$.

<u>Estimation of $\mu(E_n)$ for $n \neq 2a^2$</u>.

We have

$$\mu(E_{n,d}(j_1, ..., j_d)) \leq \prod_{i=1}^{d} \frac{1}{(a_{j_i} b_{j_i})^c},$$

whence

$$\mu(E_{n,d}) = \sum_{1 \leq j_1 < \cdots < j_d \leq r(n)} \mu(E_{n,d}(j_1, ..., j_d)) \leq \sum_{1 \leq j_1 < \cdots < j_d \leq r(n)} \prod_{i=1}^{d} \frac{1}{(a_{j_i} b_{j_i})^c} \leq$$

$$\leq \frac{1}{d!}\left(\sum_{\substack{a^2+b^2=n\\ a\leq b}} \frac{1}{(ab)^c}\right)^d.$$

If $a^2+b^2=n$, $a\leq b$, then $b\geq\sqrt{n/2}$. We define $a_n=\min\limits_{a^2+b^2=n,a>0} a$ and we have

$$\mu(E_{n,d})\leq \frac{1}{d!}\left(\frac{r(n)}{(a_n\sqrt{n/2})^c}\right)^d.$$

It is a well known fact that, for every $\delta>0$, $\dfrac{r(n)}{n^\delta}\to 0$ as $n\to\infty$. Hence there exists $n_0=n_0(\epsilon)$ such that $\dfrac{r(n)}{(a_n\sqrt{n/2})^c}\leq\dfrac{1}{2}$ for every $n>n_0$, and we have

$$\mu(E_n)=\sum_{d\geq g+1}\mu(E_{n,d})\leq\frac{2}{(g+1)!}\left(\frac{r(n)}{(a_n\sqrt{n/2})^c}\right)^{g+1}$$

for $n>n_0$.

Estimation of $\mu(E_n)$ for $n=2a^2$.
Here

$$\mu\{\omega;r_n(\omega)>g\}=\mu\{\omega;r_n(\omega)>g,a\in\omega\}+\mu\{\omega;r_n(\omega)>g,a\notin\omega\}\leq$$

$$\leq\frac{1}{a^c}\frac{2}{g!}\left(\frac{r(n)-1}{(a_n\sqrt{n/2})^c}\right)^g+\left(1-\frac{1}{a^c}\right)\frac{2}{(g+1)!}\left(\frac{r(n)-1}{(a_n\sqrt{n/2})^c}\right)^{g+1}\leq$$

$$\leq\frac{4}{(g+1)!}\left(\frac{r(n)}{(\sqrt{n/2})^c}\right)^{g+1}$$

for $>n_0$.

Completion of proof.
Define $r_j=\max\limits_{2^j\leq n<2^{j+1}} r(n)$, and suppose $0<\delta<\frac{c}{4}$. From above, $r_j<2^{\delta j}$ for $j_0>j(\delta)$, and we may suppose also that $2^{j_0}\geq n_0$.

We write

$$\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n \le 2^{j_0}} \mu(E_n) + \sum_{n > 2^{j_0}} \mu(E_n).$$

The first sum is finite and the second sum can be written in the form

$$\sum_{j>j_0} \sum_{m=g+1}^{r_j} \left\{ \sum_{\substack{2^j \le 2^{j+1} \\ r(n)=m, n=2a^2}} \mu(E_n) + \sum_{\substack{2^j \le 2^{j+1} \\ r(n)=m, n \neq 2a^2}} \mu(E_n) \right\} = \Sigma_1 + \Sigma_2,$$

say; at once

$$\Sigma_1 \le \sum_{j>j_0} \sum_{m=g+1}^{r_j} \frac{4}{(g+1)!} \left(\frac{r_j}{2^{\frac{jc}{2}}} \right)^{g+1} \#\{n = 2a^2; 2^j \le n < 2^{j+1}\} \le$$

$$\le \frac{4}{(g+1)!} \sum_{j>j_0} \frac{2^{j\delta(g+2)}}{2^{\frac{jc(g+1)}{2}}} 2^{\frac{j}{2}}.$$

and this sum is finite if $g > \dfrac{1-c+4\delta}{c-2\delta}$.

Next,

$$\Sigma_2 \le \sum_{j>j_0} \sum_{m=g+1}^{r_j} \sum_{\substack{2^j \le n < 2^{j+1} \\ r(n)=m, n \neq 2a^2}} \frac{2}{(g+1)!} \left(\frac{r_j}{(a_n 2^{\frac{j}{2}})^c} \right)^{g+1} \le$$

$$\le \frac{2}{(g+1)!} \sum_{j>j_0} \sum_{m=g+1}^{r_j} \sum_{1 \le k \le 2^{\frac{j}{2}}} \left(\frac{r_j}{(k 2^{\frac{j}{2}})^c} \right)^{g+1} \#\{n; 2^j \le n < 2^{j+1}, a_n = k\} \le$$

$$\le \frac{2}{(g+1)!} \sum_{j>j_0} \frac{2^{j\delta(g+2)}}{2^{\frac{jc(g+1)}{2}}} 2^{\frac{j}{2}} \sum_{k \ge 1} \frac{1}{k^{c(g+1)}}.$$

The last serie is convergent because $g > \frac{1}{c} - 1$. Then Σ_2 is finite if g satisfies $g > \dfrac{1-c+4\delta}{c-2\delta}$.

Finally, it is clear that δ can be chosen small enough so that the natural number $g > \frac{1}{c} - 1$ satisfies even the last condition.

Acknownledgment. Thank Professor Erdös for his advise.

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