ARCS CONTAINING NO THREE LATTICE POINTS

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1. INTRODUCTION.

In [1], A.Cordoba and myself developed a method to study the location of lattice points on circles centered at the origin. There we proved the following theorem:

Theorem A.
On a circle of radius R centered at the origin, an arc whose length is not greater than

$$\sqrt{2}R^{\frac{1}{2}-\frac{1}{4[\frac{m}{2}]+2}}$$

contains, at most, m lattice points.

We could not prove wether the exponent $\frac{1}{2}-\frac{1}{4[\frac{m}{2}]+2}$ is sharp for each m. In particular, we do not know if the number of lattice points on arcs of length $R^{\frac{1}{2}}$ is bounded uniformly in R or not. Probably it is not.

Obviously, theorem A is sharp for $m=1$. The case $m=2$ was first proved by A.Schinzel and used by Zygmund [2] to prove a Cantor-Lebesgue theorem in two variables.

It is not too hard to prove that the exponent $\frac{1}{3}$ can not be improved.

In this paper we get the best constant C, such that an arc of length $CR^{\frac{1}{3}}$ can not contain three lattice points.

Theorem 1.

i) On a circle of radius R centered at the origin, an arc whose length is not greater than $2\sqrt[3]{2}R^{\frac{1}{3}}$, contains, at most, two lattice points.

ii) For every $\epsilon > 0$, there exist infinitely many circles $x^2 + y^2 = R_n^2$ with arcs of length $2\sqrt[3]{2}R_n^{\frac{1}{3}} + \epsilon$ containing three lattice points.

2 PRELIMINARY LEMMA AND NOTATION.

Let us denote by $r(n)$ the number of representations of the integer n as a sum of two squares, i.e. $r(n)$ is the number of lattice points on the circle $x^2 + y^2 = n$. Therefore we shall associate lattice points with Gaussian integers:

$a^2 + b^2 = n$ determines a Gaussian integer $a + bi = \sqrt{n}e^{2\pi i\Phi}$ for a suitable angle Φ.

If

$$n = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{\beta_k}$$

is the prime factorization of the integer n, then $r(n) = 0$ unless all the exponents β_k are even. In that case we have $r(n) = 4\prod(1 + \alpha_j)$.

A prime $p_j \equiv 1(4)$ can be represented as a sum of two squares, $p_j = a^2+b^2$, $0 < a < b$, in only one way. Then, for each p_j, the angle Φ_j,such that $a + bi = \sqrt{p_j}e^{2\pi i\Phi_j}$ is well defined.

With this notation we proved in [1] the following lemma:

Lemma.

If

$$n = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k}$$

then the Gaussian integers corresponding to the $4\prod(1+\alpha_j)$ lattice points on the circle $x^2 + y^2 = n$ are given by the formula

$$\sqrt{n}e^{2\pi i\{\Phi_0 + \sum_j \gamma_j\Phi_j + \frac{t}{4}\}}$$

where Φ_j *is the angle corresponding to* p_j,
γ_j *runs over the set* $\{\gamma \in Z; \quad |\gamma| \leq \alpha_j \quad \gamma \equiv \alpha_j(2)\}$
t *takes the values* $0, 1, 2, 3$ *and*

$$\Phi_0 = \begin{cases} 0 & \textit{if} \quad \nu \quad \textit{is even} \\ \frac{1}{8} & \textit{if} \quad \nu \quad \textit{is odd} \end{cases}$$

3. PROOF OF THEOREM 1.

i) Let us suppose that for the integer

$$n_0 = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k}$$

there is an arc, on the circle of radius $R_0 = \sqrt{n_0}$ centered at the origin, which contains three lattice points and whose length is $2\sqrt[3]{2}R_0^{\frac{1}{3}}$.

The previous lemma implies that the same must be true for the circle of radius $R = \sqrt{n}$ where $n = \prod_{p_j \equiv 1(4)} p_j^{\alpha_j}$.

Let ν_1, ν_2, ν_3 be three such lattice points. By the lemma, they have representations of the form

$$\sqrt{n} e^{2\pi i\{\sum_j \gamma_j^s \Phi_j + \frac{t^s}{4}\}} \qquad (s = 1, 2, 3)$$

$\gamma_j^s \in \{\gamma \in Z; \quad |\gamma| \leq \alpha_j \quad \gamma \equiv \alpha_j(2)\}, \quad t^s \in \{0, 1, 2, 3\}$
For each pair $\nu_s \neq \nu_{s'}$ of such points, let us consider the quantity

$$\Psi^{s,s'} = \sum_j \Phi_j\{\gamma_j^s - \gamma_j^{s'}\} + \frac{t^s - t^{s'}}{4} = 2\{\sum_j \Phi_j \frac{\gamma_j^s - \gamma_j^{s'}}{2} + \frac{t^s - t^{s'}}{8}\}$$

and observe that $\gamma_j^{s,s'} = \frac{\gamma_j^s - \gamma_j^{s'}}{2}$ takes always integer values.
We can write $\frac{t^s - t^{s'}}{8} = \frac{\delta(s,s')}{8} + \frac{t^{s,s'}}{4}$ where $t^{s,s'}$ is an integer and

$$\delta(s, s') = \begin{cases} 0 & \text{if} \quad t^s \equiv t^{s'}(2) \\ 1 & \text{if} \quad t^s \equiv t^{s'}(2) \end{cases}$$

Now, the angles $\frac{\Psi^{s,s'}}{2}$ correspond to a representation as a sum of two squares of

$$2^{\delta(s,s')}\prod_j p_j^{|\gamma_j^{s,s'}|} = n_{s,s'}^2 + m_{s,s'}^2 \qquad 1 \le n_{s,s'} \le m_{s,s'}$$

Then, $\frac{\Psi^{s,s'}}{2} = \frac{1}{2\pi} arctg \frac{n_{s,s'}}{m_{s,s'}}$ where

$$arctg\frac{n_{s,s'}}{m_{s,s'}} \ge arctg\frac{1}{m_{s,s'}} > \frac{1}{\sqrt{m_{s,s'}^2+1}} \ge \frac{1}{\sqrt{2^{\delta(s,s')}\prod p_j^{|\gamma_j^{s,s'}|}}}$$

And we have

$$\frac{\Psi^{1,2}}{2}\frac{\Psi^{1,3}}{2}\frac{\Psi^{2,3}}{2} > \frac{1}{(2\pi)^3\sqrt{2^{\delta(1,2)+\delta(1,3)+\delta(2,3)}\prod p_j^{|\gamma_j^{1,2}|+|\gamma_j^{1,3}|+|\gamma_j^{2,3}|}}}.$$

The maximun value of

$$|\gamma_j^{1,2}| + |\gamma_j^{1,3}| + |\gamma_j^{2,3}| = \frac{|\gamma_j^1 - \gamma_j^2|}{2} + \frac{|\gamma_j^1 - \gamma_j^3|}{2} + \frac{|\gamma_j^2 - \gamma_j^3|}{2}$$

is obtained when $\gamma_j^1 = \gamma_j^2 = \alpha_j$ and $\gamma_j^3 = -\alpha_j$.

Then $|\gamma_j^{1,2}| + |\gamma_j^{1,3}| + |\gamma_j^{2,3}| \le 2\alpha_j$.

Also we can observe that $\delta(1,2) + \delta(1,3) + \delta(2,3) \le 2$.

Then we get

$$\Psi^{1,2}\Psi^{1,3}\Psi^{2,3} > \frac{1}{2\pi^3 R^2}.$$

On the other hand, if P_1, P_2, P_3 are three points of the interval $[0,1]$, we have

$$|P_1 - P_2||P_1 - P_3||P_2 - P_3| \le \frac{1}{4}.$$

This implies that for three lattice points on an arc of length $2\sqrt[3]{2}R^{\frac{1}{3}}$, we have

$$\Psi^{1,2}\Psi^{1,3}\Psi^{2,3} \le \frac{1}{4}(\frac{2\sqrt[3]{2}R^{\frac{1}{3}}}{2\pi R})^3 = \frac{1}{2\pi^3 R^2}$$

and we get a contradiction.

ii) For each n we consider the circle $x^2+y^2=R_n^2$ where

$$R_n^2 = 16n^6 + 4n^4 + 4n^2 + 1.$$

We can see that

$$16n^6+4n^4+4n^2+1 = (4n^3-1)^2+(2n^2+2n)^2 = (4n^3)^2+(2n^2+1)^2 = (4n^3+1)^2+(2n^2-2n)^2.$$

The three lattice points $(4n^3-1, 2n^2+2n)$, $(4n^3, 2n^2+1)$, $(4n^3+1, 2n^2-2n)$ are on an arc of length

$$R_n\{arctg\frac{2n^2+2n}{4n^3-1} - arctg\frac{2n^2-2n}{4n^3+1}\} = R_n arctg\frac{16n^4+4n^2}{16n^6+4n^4-4n^2-1} = 2\sqrt[3]{2}R_n^{\frac{1}{3}} + o(1)$$

and the theorem follows.

REFERENCES.

[**1**] J.CILLERUELO and A.CORDOBA. *Trigonometric polynomials and lattice points,* Proceedings of the A.M.S., to appear.

[**2**] A.ZYGMUND. *A Cantor-Lebesgue theorem for double trigonometric series,* Studia Math. **64** (1972).