2.2.25. *Hallar una fórmula asintótica para* $\sum_{n\leq x}\sigma(n)$ *donde* $\sigma(n) = \sum_{d|n} d$.

Solución:

$$\sum_{n\leq x}\sigma(n) = \sum_{n\leq x}\sum_{d|n} d = \sum_{\substack{d,q\\ dq\leq x}} d = \sum_{q\leq x}\sum_{d\leq x/q} d$$

Ahora, hallamos $\sum_{n\leq x} n$. Usamos para ello la identidad de Abel, con $a(n) = 1$ ($A(x) = \lfloor x\rfloor$), y $f(x) = x$.

Entonces:

$$\sum_{n\leq x} n = \lfloor x\rfloor x - \int_1^x \lfloor t\rfloor \mathrm{d}t = x^2 + O(x) - \int_1^x t\mathrm{d}t + \int_1^x \{t\}\mathrm{d}t =$$

$$= x^2 - \frac{x^2}{2} + \frac{1}{2} + \int_1^x \{t\}\mathrm{d}t + O(x) = \frac{x^2}{2} + O(x)$$

Sustituyendo en lo anterior, obtenemos lo siguiente:

$$\sum_{q\leq x}\sum_{d\leq x/q} d = \sum_{q\leq x}\left[\left(\frac{x}{q}\right)^2 + O\left(\frac{x}{q}\right)\right] = \frac{x^2}{2}\sum_{q\leq x}\frac{1}{q^2} + O\left(x\sum_{q\leq x}\frac{1}{q}\right)$$

Sabemos que

$$\sum_{q\leq x}\frac{1}{q} = O(\log x)$$

por lo que:

$$\frac{x^2}{2}\sum_{q\leq x}\frac{1}{q^2} + O\left(x\sum_{q\leq x}\frac{1}{q}\right) = \frac{x^2}{2}\sum_{q\leq x}\frac{1}{q^2} + O(x\log x)$$

Ahora, hallamos $\sum_{n\leq x}\frac{1}{n^2}$. Usamos de nuevo la identidad de Abel, con $A(x) = \lfloor x\rfloor$ y $f(x) = \frac{1}{x^2}$.

$$\sum_{n\leq x}\frac{1}{x^2} = \frac{\lfloor x\rfloor}{x^2} + 2\int_1^x \frac{\lfloor t\rfloor}{t^3}\mathrm{d}t = \frac{1}{x} + O\left(\frac{1}{x^2}\right) + 2\int_1^x \frac{1}{t^2}\mathrm{d}t - 2\int_1^x \frac{\{t\}}{t^3}\mathrm{d}t =$$

$$= \frac{1}{x} - \frac{2}{x} + 2 - 2\int_1^x \frac{\{t\}}{t^3}\mathrm{d}t + O\left(\frac{1}{x^2}\right) =$$

$$= -\frac{1}{x} + 2 - 2\int_1^\infty \frac{\{t\}}{t^3}\mathrm{d}t + O\left(\frac{1}{x^2}\right) + 2\int_x^\infty \frac{\{t\}}{t^3}\mathrm{d}t = -\frac{1}{x} + 2 - 2\int_1^\infty + O\left(\frac{1}{x^2}\right)$$

Es decir:

$$\sum_{n\leq x} \frac{1}{n^2} = -\frac{1}{x} + C + O\left(\frac{1}{x^2}\right)$$

Podemos observar que:

$$\lim_{x\to\infty} \sum_{n\leq x} \frac{1}{n^2} = C = \zeta(2)$$

Por tanto,

$$\sum_{n\leq x} \frac{1}{n^2} = -\frac{1}{x} + \zeta(2) + O\left(\frac{1}{x^2}\right)$$

De aquí, obtenemos que:

$$\frac{x^2}{2}\sum_{q\leq x} \frac{1}{q^2} + O(x\log x) = \frac{x^2}{2}\left[-\frac{1}{x} + \zeta(2) + O\left(\frac{1}{x^2}\right)\right] + O(x\log x) =$$

$$= -\frac{x}{2} + \frac{x^2}{2}\zeta(2) + O(1) + O(x\log x) = \frac{x^2}{2}\zeta(2) + O(x + 1 + x\log x) =$$

$$= \frac{x^2}{2}\zeta(2) + O(x\log x)$$

Como $\zeta(2) = \frac{\pi^2}{6}$:

Conclusión:

$$\sum_{n\leq x} \sigma(n) = \frac{x^2\pi^2}{12} + O(x\log x)$$

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