Lattice points in the 3D torus

Introduction

• This talk is based on a joint work with my former student Dulcinea Raboso.

Lattice point problems are very classic. They can be traced back to the first half of the 19th century.

• C.F. Gauss (1837) circle problem, P.G.L. Dirichlet (1849) divisor problem.

The general situation is that we take a domain $\Omega \subset \mathbb{R}^d$, we scale it to $\Omega_R = R\Omega$ with a large constant R, and we approximate the number of integral points in Ω_R by the volume $|\Omega_R| = |\Omega| R^d$,

$$|\Omega_R \cap \mathbb{Z}^d| = |\Omega| R^d + O\big(R^\alpha\big).$$

Question: What is the best (minimal) value of α?

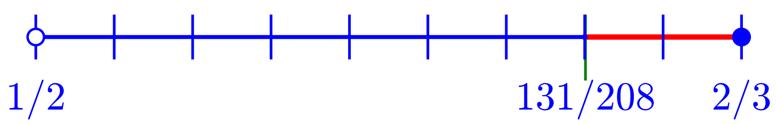
Example (circle problem)**.** Ω =unit circle, $d = 2$.

$$\#\big\{(n,m) \in \mathbb{Z}^2 \ : \ n^2 + m^2 \le R^2\big\} = \sum_{n \le R^2} r_2(n) = \pi R^2 + O\big(R^{\alpha_2}\big).$$

• W. Sierpinski 1906, proved that $\alpha_2 = 2/3$, the so-called “trivial” exponent, is valid. Actually, it is not trivial, let us say that it is easy once one knows a harmonic analytic approach.

• G.H. Hardy 1915, conjectured that any $\alpha_2 > 1/2$ is valid. Still open.

• M.N. Huxley 2003, proved that $\alpha_2 > 131/208$ is valid (for “general” regular convex plane regions).



In a century we have walked approximately 2 steps out of 9 towards the conjecture. And the improvements are really small. For instance, the last but one best known result was $46/73$ and $46/73 - 131/208 < 1/3000$. This is a classic and hard problem in analytic number theory.

Usual approach

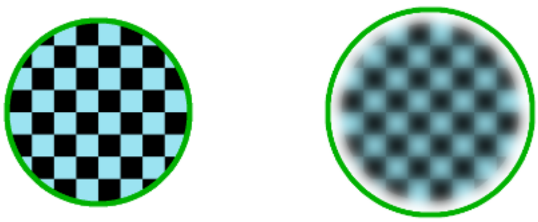
The basic idea is using Poisson summation formula

$$|\Omega_R \cap \mathbb{Z}^d| = \sum_{\vec{n}\in\mathbb{Z}^d} \chi_{\Omega_R}(\vec{n}) \underset{\text{formal}}{=} \sum_{\vec{n}\in\mathbb{Z}^d} \widehat{\chi}_{\Omega_R}(\vec{n})$$

The last series does not converges absolutely and even it is not expected to converge at all for large d. Here the hat indicates the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(\vec{x}) e^{-2\pi i \vec{\xi}\cdot\vec{x}}\, d\vec{x}.$$

To avoid the convergence problems one notes that details of size δ require frequencies at least comparable to δ^{-1}.



The effect in the formula is like losing the volume of a band of width δ

$$|\Omega_R \cap \mathbb{Z}^d| \underset{\text{approx.}}{=} \sum_{\|\vec{n}\|<\delta^{-1}} \widehat{\chi}_{\Omega_R}(\vec{n}) + O\big(R^{d-1}\delta\big).$$

It is obvious that $|\widehat{\chi}_{\Omega_R}(\vec{0})| = |\Omega| R^d$, then

$$\boxed{|\Omega_R \cap \mathbb{Z}^d| \underset{\text{approx.}}{=} |\Omega| R^d + \sum_{0\neq\|\vec{n}\|<\delta^{-1}} \widehat{\chi}_{\Omega_R}(\vec{n}) + O\big(R^{d-1}\delta\big)}.$$

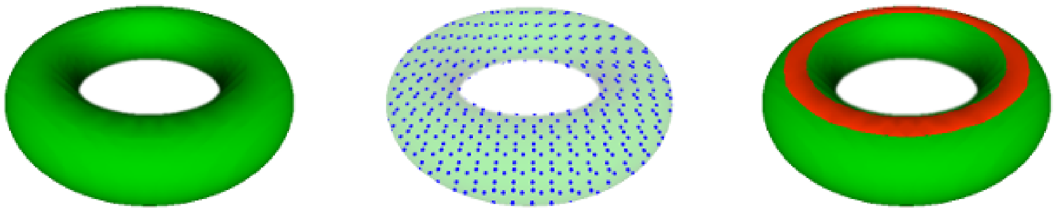
If we take δ small the last term is small but then we need good methods on exponential sums (perhpas too good) to deal with the middle term that oscillates.

Our result

The region that we consider is the 3D torus

$$\mathbb{T} = \Big\{ (x,y,z) \in \mathbb{R}^3 \ : \ \Big(\rho' - \sqrt{x^2+y^2}\Big)^2 + z^2 \le \rho^2 \Big\}$$

where $0 < \rho < \rho'$ are fixed constants. The torus has two circles of points with vanishing (Gaussian) curvature. It plays a role in the asymptotic.



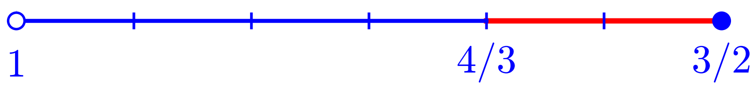
We have proved

$$\boxed{|\mathbb{T}_R \cap \mathbb{Z}^d| = |\mathbb{T}|R^3 + C(R)R^{3/2} + O\big(R^\alpha\big)} \qquad \text{for any } \alpha > 4/3,$$

where $C(R)$ is a periodic function (say a constant in arithmetic progressions).

$$|\mathbb{T}|R^3 = 2\pi^2\rho^2\rho' R^3 = \text{volume term}, \qquad C(R)R^{3/2} = \text{secondary main term}.$$

This problem is easier than the circle problem because we have walked approximately 2 steps out of 6 towards the conjecture that in this case claims the result for any $\alpha > 1$.



Our method

Following the scheme mentioned before, we apply

$$|\Omega_R \cap \mathbb{Z}^d| \underset{\text{approx.}}{=} |\Omega|R^d + \sum_{0 \ne \|\vec{n}\| < \delta^{-1}} \widehat{\chi}_{\Omega_R}(\vec{n}) + O\big(R^{d-1}\delta\big)$$

with $\Omega = \mathbb{T}$, $d = 3$ and $\delta = R^{-2/3}$, in this way $O\big(R^{d-1}\delta\big) = O\big(R^{4/3}\big)$. Then the problem boils down to estimate

$$\sum_{0 \neq \|\vec{n}\| < R^{2/3}} \widehat{\chi}_{\mathbb{T}_R}(\vec{n}).$$

A difference with respect to the classic scheme (for convex regions) is that when we apply the stationary phase approximation to study $\widehat{\chi}_{\mathbb{T}_R}(\vec{n})$, the method collapses when $\vec{n}$ is the normal at a point of vanishing curvature of $\mathbb{T}_R$. In our case, we have to separate the vertical vectors,

$$\sum_{0<n<R^{2/3}} \widehat{\chi}_{\mathbb{T}_R}(0,0,n) \quad \to \quad C(R)R^{3/2} + \text{negligible terms}$$

and we get the secondary main term.

For bounding the rest of the sum we elaborate a trick introduced in 1963 for the sphere problem by the celebrated exponential sum estimator И. М. Виноградов and independently and simultaneously by one of the most conspicuous Chinese number theorists

陈 景 润 (Chen Jingrun).

The rough idea is that the torus has cylindrical symmetry and it is preserved in the Fourier transform side, then we have to deal with oscillatory sums of the form

$$\sum_m \sum_n \sum_l f\Big(\sqrt{n^2+m^2}, l\Big)$$

When m is frozen, the derivative of $\sqrt{x^2+m^2}$ is like 1. If we glue the variables $k = n^2 + m^2$ we have something like $\sqrt{x}$ that has a smaller derivative

$$\sum_m \sum_n \sum_l f\Big(\sqrt{n^2+m^2}, l\Big) = \sum_k \sum_l r_2(k) f\Big(\sqrt{k}, l\Big).$$

In analytic terms this is, in principle, good because sums with less oscillation are easier to control. But we have to pay prize with a chaotic arithmetic coefficient $r_2(k)$ that escapes to analytic methods.

Although $r_2(k)$ is chaotic, it is close to be bounded, in fact its quadratic mean grows as a logarithm. Then it is not a fatal loss to use Cauchy-Schwarz inequality to get rid of it

$$\Big|\sum_{k\le K}\sum_l r_2(k) f\big(\sqrt{k}, l\big)\Big|^2 \underset{\text{approx.}}{\le} K \sum_{k\le K}\Big|\sum_l f\big(\sqrt{k}, l\big)\Big|^2.$$

Opening the square, and changing the order of summation we get

$$\sum_{l_1}\sum_{l_1}\sum_k f\big(\sqrt{k}, l_1\big)\bar{f}\big(\sqrt{k}, l_2\big)$$

and the inner sum is easier to study because of the control on the oscillation.

The preprint "Lattice points in the 3-dimensional torus" by F. Chamizo and D. Raboso will be posted on the arXiv soon.