Large Sieve

What is it?

Basic inequality: $x_1, x_2, \ldots \in \mathbb{T}$, $|x_\nu - x_\mu| > \delta$

$$\sum_\nu \left| \sum_{n \le N} a_n e(n x_\nu) \right|^2 \le (N + \delta^{-1}) \sum |a_n|^2.$$

<u>H. A. analog</u>: Sum of Sobolev's inequalities.

What for?

Take control of some Fourier series with rough coefficients appearing in Number Theory.

Example: $S(x) = \sum_{p \le N} e^{2\pi i p x}$ small for $x \notin \mathbb{Q}$ (p prime) $\Rightarrow$ Every large enough odd number is a sum of three primes (Vinogradov).

$$S(x) \longrightarrow \sum_{n \le N} \sum_{m \le N/n} \mu(n) e^{2\pi i m n x}.$$

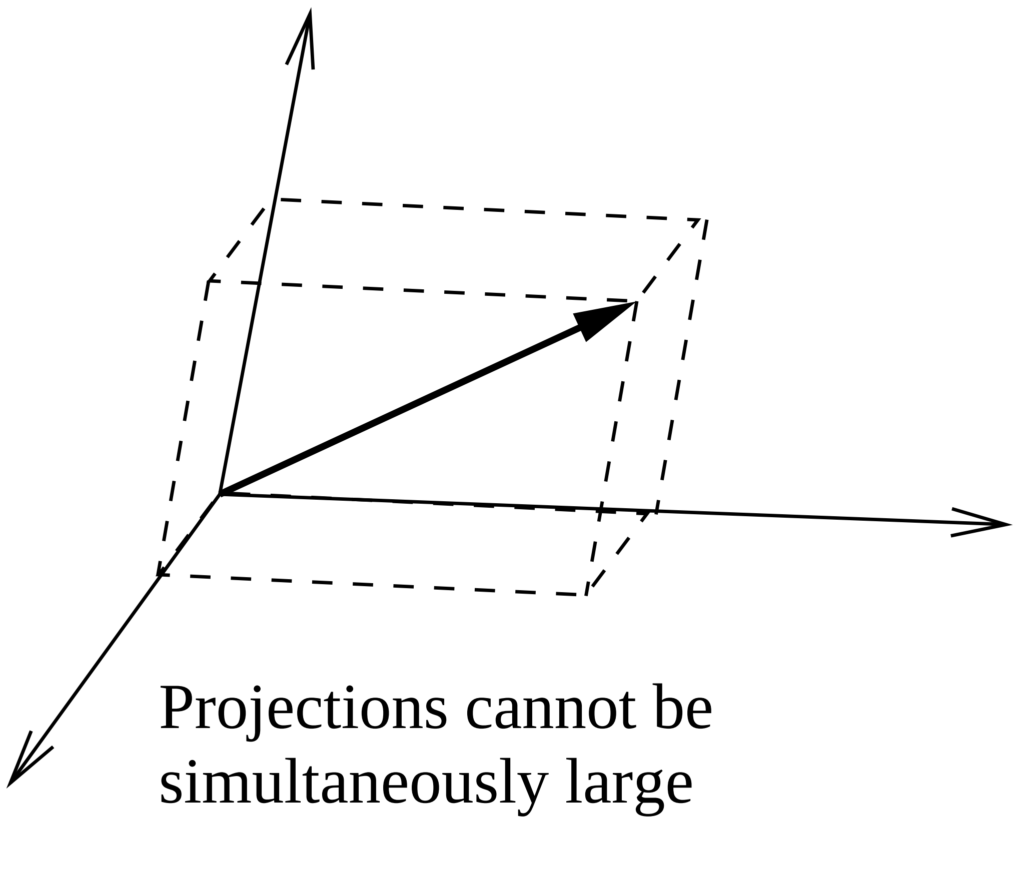
<u>H. A. analog</u>: Bilinear forms estimates.

Why does it work?

If the points x_ν are spaced the vectors $\vec{v}_\nu = \big(e(x_\nu), e(2x_\nu), \ldots, e(Nx_\nu)\big)$ are more or less orthogonal.

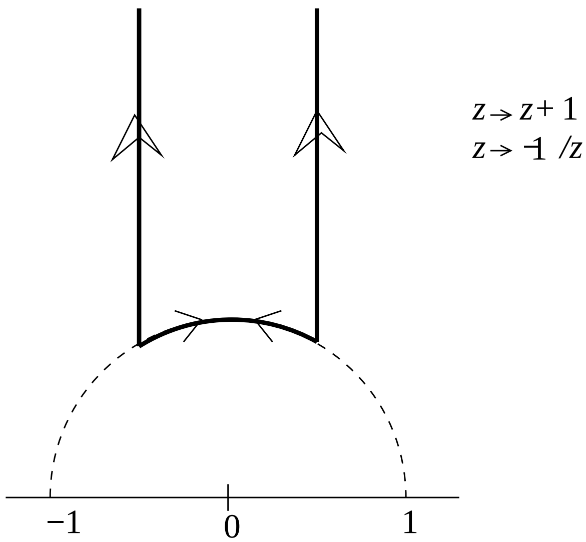
Quasi-orthogonality $\Rightarrow$ Cancellation

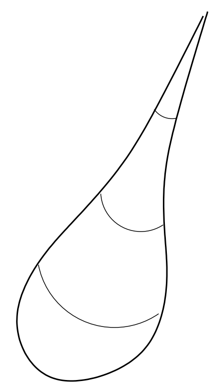
H. A. analog: Cotlar's lemma.



What is new?

Harmonic analysis on the upper half plane (spectral theory of automorphic forms).





$\mathbb{H}$ = upper half plane Γ = Fuchsian group

$d\mu = y^{-2} dx\, dy \qquad \Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$

Discrete Continuous

$-\Delta u_n = \lambda_n u_n,\ u_n \in L^2; \qquad -\Delta E = \lambda E,\ E \notin L^2$

Spectral theorem in $\Gamma\backslash\mathbb{H}$:

$f(z) = \sum a_n u_n(z)$+continuous spectrum.

Thm: *For* $\Gamma\backslash\mathbb{H}$, $d(z_\nu, z_\mu) > \delta$ *implies*

$$\sum_\nu \Big| \sum_{\sqrt{\lambda_n}\le\Lambda} a_n u_n(z_\nu) + \dots \Big|^2 \le K(\Lambda^2 + \delta^{-2})\|\mathbf{a}\|^2.$$

(Ext.) For a compact Riemannian D-manifold, $d(x_\nu, x_\mu) > \delta$ *implies*

$$\sum_\nu \Big| \sum_{\sqrt{\lambda_n}\le\Lambda} a_n \phi_n(x_\nu) \Big|^2 \le K(\Lambda^D + \delta^{-D}) \sum_{\sqrt{\lambda_n}\le\Lambda} |a_n|^2.$$

Proof→Study smoothed quasi-orthogonality

$$\sum_n e^{-\lambda_n/\Lambda^2} u_n(z_\nu) u_n(z_\mu) + \dots$$

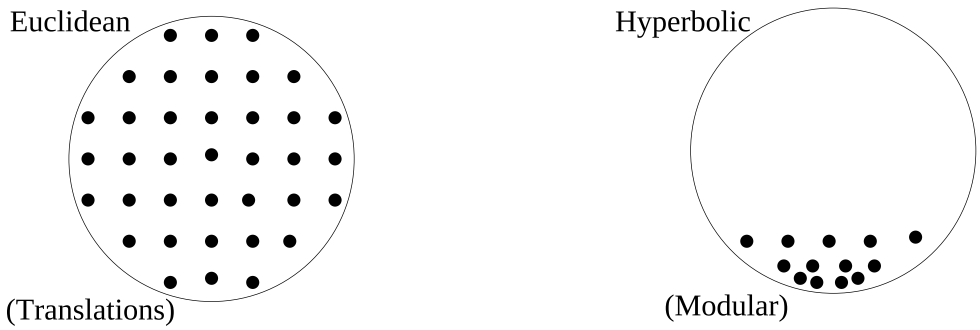
H. A. analog: Heat kernel estimates.

What follows?

$*$ For the most of the large circles in $\mathbb{H}$, it holds

$$\#\{\Gamma z \in \text{circle}\} = A + O\big(A^{1/2+\epsilon}\big)$$

where A is the area of the circle.



$*$ The number of integral solutions of $x^2 + y^2 - z^2 - t^2 = 1$ with $x^2 + y^2 \leq N$ is approximately $8N$ and the standard deviation of this circle is $O(N^{1/2+\epsilon})$.

$*$ Wave equation in a compact Riemannian manifold of dim $= D$ with frequencies cut-off up to δ^{-1}. If $\#\{\text{test particles}\}\delta^D > K$ then the energy average over test particles is bounded by total energy.