Modular forms and lattice point counting problems

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Thesis defense

Universidad Autónoma de Madrid, 2018

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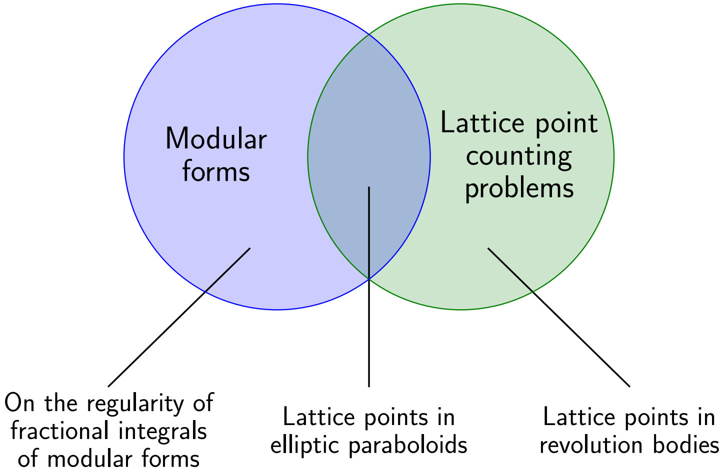
Outline of the talk

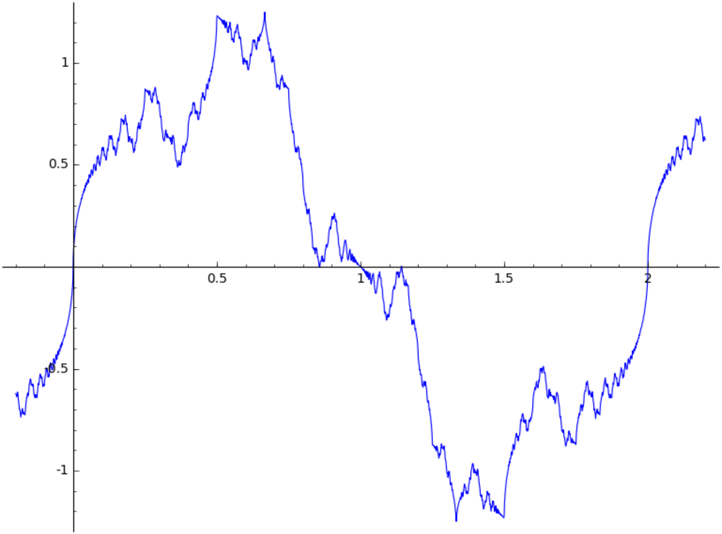


Outline of the talk

1. "Riemann's example".
2. Classical modular forms.
3. Fractional integrals of classical modular forms.
4. Lattice point counting problems.
5. Lattice points in bodies of revolution and elliptic paraboloids.

1. "Riemann's example"

$$\varphi(x) = \sum_{n \geq 1} \frac{\sin(n^2 \pi x)}{n^2}$$

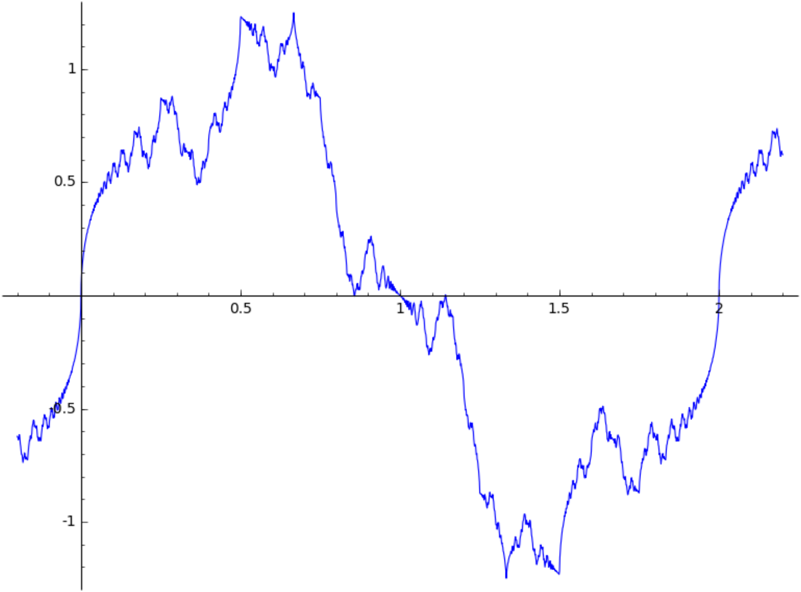


1. "Riemann's example"

$$\varphi(x) = \sum_{n \geq 1} \frac{\sin(n^2 \pi x)}{n^2}$$

- K. Weierstrass introduces this function in 1872 as an example given by Riemann of a continuous function which is nowhere differentiable.
- G. H. Hardy proves in 1916 that φ is not differentiable at any point except, perhaps, at the rational points of the form odd/odd or even$/(4n+3)$.
- J. Gerver completes this result in 1970 and 1971, showing that φ is indeed not differentiable in the rationals even$/(4n+3)$, but it *does have derivative* $-\pi/2$ *at the rationals odd/odd*.

1. “Riemann’s example”



"Riemann's example" and Jacobi's theta function,

$$\varphi(x) = \sum_{n\geq 1} \frac{\sin(n^2\pi x)}{n^2} \qquad \text{and} \qquad \theta(z) = \sum_{n\in\mathbb{Z}} e^{n^2\pi i z},$$

are *formally* related by

$$\varphi(x) = \frac{\pi}{2}\,\Re \int (\theta(x) - 1)\, dx.$$

The function θ is a classical modular form of weight $1/2$.

The group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; a, b, c, d \in \mathbb{Z},\;\; ad - bc = 1 \right\}$$

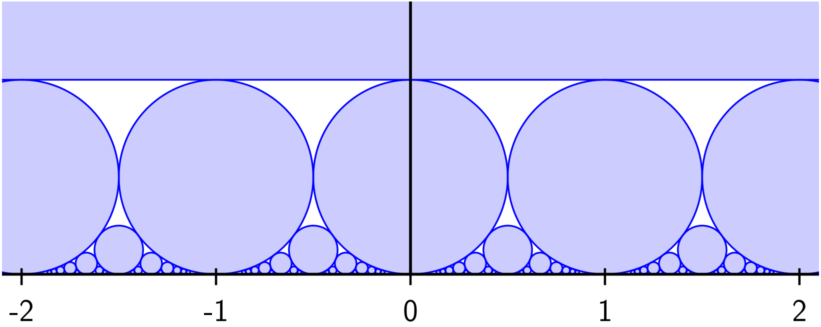
acts on the upper halfplane by Möbius transformations:

$$\gamma z := \frac{az+b}{cz+d} \qquad \text{given} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \quad \Im z > 0.$$

The action can be extended to the set $\mathbb{Q} \cup \{\infty\}$ (the **cusps**).

The **Ford circles** associated to the cusps are:

$$\left\{ \left| z - \frac{p}{q} - \frac{i}{2q^2} \right| \leq \frac{1}{2q^2} \right\} \qquad \text{and} \qquad \{ \Im z \geq 1 \}.$$



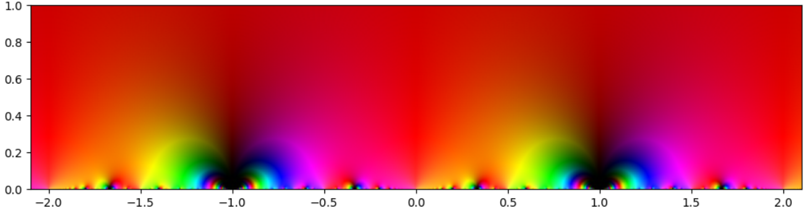
A **classical modular form** of weight $r > 0$ for a finite-index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is an analytic function defined on the upper halfplane, satisfying

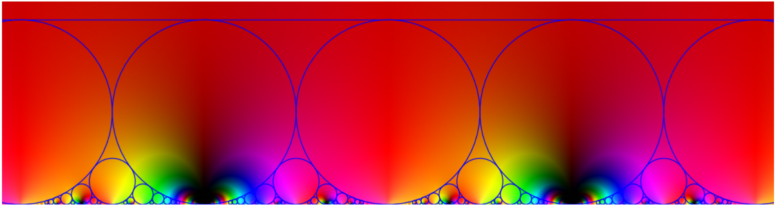
$$f(\gamma z) = \mu_\gamma (cz+d)^r f(z) \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

for some constants $|\mu_\gamma| = 1$, and which has at most polynomial growth as $\Im z \to 0^+$.

$$f^\gamma = \mu_\gamma f \qquad \text{where} \qquad f^\gamma(z) = \frac{f(\gamma z)}{(cz+d)^r}.$$

The graph of Jacobi's theta function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{n^2 \pi i z}$:





There exist $m \in \mathbb{N}$, $0 \leq \kappa < 1$, $a_n \in \mathbb{C}$ such that:

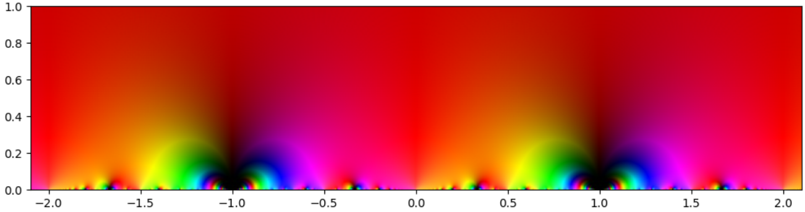
$$f(z) = \sum_{n \geq 0} a_n \, e^{2\pi i (n+\kappa) \frac{z}{m}} \qquad \text{(Fourier series at } \infty\text{).}$$

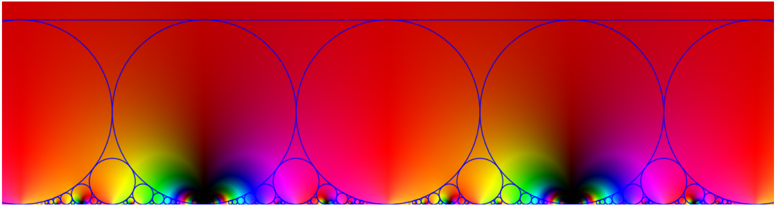
If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ then $f^\gamma(z) = f(\gamma z)/(cz+d)^r$ is a modular form of weight r for the group $\gamma^{-1}\Gamma\gamma \cap \mathrm{SL}_2(\mathbb{Z})$.

$$f^\gamma(z) = \sum_{n \geq 0} a_n^\gamma \, e^{2\pi i (n+\kappa_\gamma) \frac{z}{m_\gamma}} \qquad \text{(Fourier series at } \gamma(\infty) = a/c\text{).}$$

- There are essentially finitely-many functions f^γ.
- f is said to be **cuspidal at** a/c if either $a_0^\gamma = 0$ or $\kappa_\gamma > 0$.
- f is said to be a **cusp form** if it is cuspidal at every cusp.

θ is cuspidal at those rationals of the form odd/odd.





Fractional integrals of order α:

$$f(z) = \sum_{n+\kappa\geq 0} a_n e^{2\pi i(n+\kappa)\frac{z}{m}} \rightsquigarrow f_\alpha(x) = \sum_{n+\kappa>0} \frac{a_n}{(n+\kappa)^\alpha} e^{2\pi i(n+\kappa)\frac{x}{m}}.$$

Associated fractional integrals: $f^\gamma \rightsquigarrow f^\gamma_\alpha = \left(f^\gamma\right)_\alpha$.

For "Riemann's example" φ:

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{n^2\pi i z} \quad \rightsquigarrow \quad \theta_1(x) = 2\sum_{n\geq 1} \frac{1}{n^2} e^{n^2\pi i x},$$

$$\Im\theta_1(x) = 2\sum_{n\geq 1} \frac{\sin(n^2\pi x)}{n^2} = 2\varphi(x).$$

Let f be a nonzero classical modular form of weight $r > 0$.

$$f_\alpha(x) = \sum_{n+\kappa>0} \frac{a_n}{(n+\kappa)^\alpha} e^{2\pi i(n+\kappa)\frac{x}{m}} \qquad (*)$$

Theorem (F. Chamizo 2003)

Let $\alpha_0 = r/2$ if f is a cusp form and $\alpha_0 = r$ otherwise. Then:

- *For $\alpha \leq \alpha_0$ the series $(*)$ diverges in a dense subset of $\mathbb{R}$.*
- *For $\alpha > \alpha_0$ the series $(*)$ converges uniformly to a continuous periodic function in $\mathbb{R}$.*

From now on we will assume $\alpha > \alpha_0$.

A function f is said to lie in $\mathcal{C}^s(x_0)$ if for some polynomial P,

$$|f(x) - P(x - x_0)| \ll |x - x_0|^s \qquad (x \to x_0).$$

The **pointwise Hölder exponent** β of f is defined as

$$\beta(x) = \sup \{s \, : \, f \in \mathcal{C}^s(x)\}.$$

- For $0 < s < 1$, we have: $f \in \mathcal{C}^s(x_0) \iff f$ is s-Hölder at x_0.
- If f is smooth enough, P must equal the Taylor polynomial.

3. Fractional integrals of classical modular forms

Let f be a nonzero classical modular form of weight $r > 0$.

$$f_\alpha(x) = \sum_{n+\kappa>0} \frac{a_n}{(n+\kappa)^\alpha} e^{2\pi i(n+\kappa)\frac{x}{m}}$$

Theorem (Regularity at the rational numbers, P. 2016, ...*)

Let $x \in \mathbb{Q}$, and β the pointwise Hölder exponent of either f_α, $\Re f_\alpha$ or $\Im f_\alpha$. Then:

$$\beta(x) = \begin{cases} 2\alpha - r & \text{if } f \text{ is cuspidal at } x, \\ \alpha - r & \text{if } f \text{ is not cuspidal at } x. \end{cases}$$

*Chamizo, Petrykiewicz, Ruiz-Cabello (2015); Chamizo (2003); Duistermaat (1991); Gerver (1970, 1971); Hardy (1916).

An irrational x is said to be τ-**approximable** if

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^\tau} \text{ for infinitely many rationals } \frac{p}{q}.$$

We say that x is τ-**approximable with respect to** f if

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^\tau} \text{ for infinitely many rationals } \frac{p}{q} \text{ where } f \text{ is not cuspidal.}$$

We also define

$$\tau_x = \sup\left\{\tau \,:\, x \text{ is } \tau\text{-approximable with respect to } f\right\} \geq 2.$$

3. Fractional integrals of classical modular forms

Let f be a nonzero classical modular form of weight $r > 0$.

$$f_\alpha(x) = \sum_{n+\kappa>0} \frac{a_n}{(n+\kappa)^\alpha} e^{2\pi i(n+\kappa)\frac{x}{m}}$$

$$\tau_x = \sup\{\tau \,:\, x \text{ is } \tau\text{-approximable with respect to } f\}$$

Theorem (Regularity at the irrational numbers, P. 2016, ...*)

Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and β the pointwise Hölder exponent of either f_α, $\Re f_\alpha$ or $\Im f_\alpha$. If f is a cusp form, $\beta(x) = \alpha - r/2$. Otherwise,

$$\beta(x) = \alpha - \left(1 - \frac{1}{\tau_x}\right) r.$$

*Chamizo, Petrykiewicz, Ruiz-Cabello (2015); Ruiz-Cabello (2014); Chamizo (2003); Jaffard (1994); Hardy (1916).

In the case of $\varphi = \frac{1}{2}\Im\theta_1$:

$$\beta(x) = \begin{cases} 3/2 & \text{if } x \in \mathbb{Q} \text{ and } x = \text{odd/odd}, \\ 1/2 & \text{if } x \in \mathbb{Q} \text{ but } x \neq \text{odd/odd}, \\ \frac{1}{2}\left(1 + \frac{1}{\tau_x}\right) & \text{if } x \text{ is an irrational number}, \end{cases}$$

where

$$\tau_x = \sup\left\{\tau \;:\; \left|x - \frac{p}{q}\right| \leq \frac{1}{q^\tau} \text{ for infinitely many } \frac{p}{q} \neq \text{odd/odd}\right\}.$$

φ is differentiable only in those rationals of the form odd/odd.

Theorem (Approximate functional equation, P. 2016, ...*)

Given $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *with* $\gamma(\infty) = x_0 \in \mathbb{Q}$,

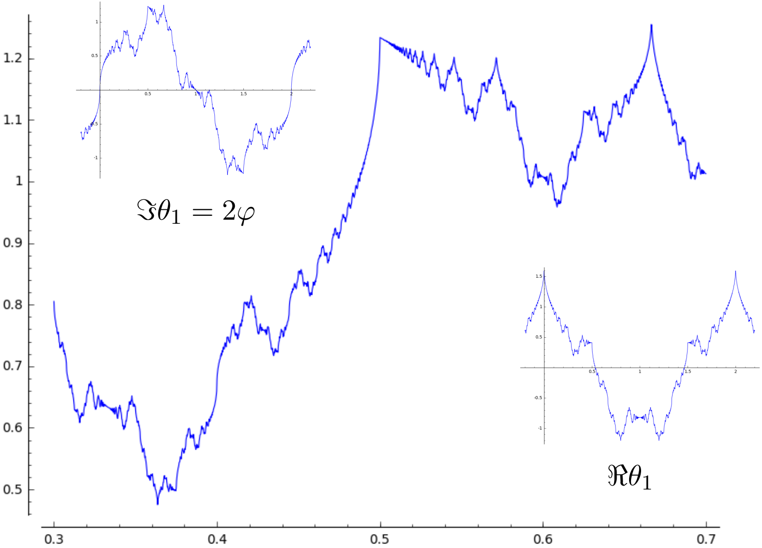
$$f_\alpha(x) = A\eta(x - x_0) + B|x - x_0|^{2\alpha}(x - x_0)^{-r} f_\alpha^\gamma(\gamma^{-1}x) + E(x)$$

- $A = 0$ *if and only if* f *is cuspidal at* x_0.
- $\eta(x) = \begin{cases} x^{\alpha-r} & \text{if } \alpha - r \notin \mathbb{Z}, \\ x^{\alpha-r}\log x & \text{if } \alpha - r \in \mathbb{Z}. \end{cases}$
- B *is a real constant,* $B > 0$.
- E *is differentiable in* $\mathbb{R} \setminus \{x_0\}$ *and lies in* $\mathcal{C}^{2\alpha-r+1}(x_0)$.

Recall $\gamma \in \Gamma$ implies $f^\gamma = \mu_\gamma f$ for some constant $\mu_\gamma \in \mathbb{C}$.

*Chamizo, Petrykiewicz, Ruiz-Cabello (2015); Miller, Schmid (2004); Duistermaat (1991).

$$\varphi(x) = \Im\left[A\sqrt{x - x_0}\right] + \Im\left[B(x - x_0)^{3/2}\theta_1^{\gamma}(\gamma^{-1}x)\right] + E(x).$$



3. Fractional integrals of classical modular forms

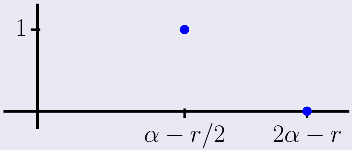
The **spectrum of singularities** is the map:

$$d(\delta) = \begin{cases} \dim_{\mathrm{H}} \{x : \beta(x) = \delta\} & \text{if } \beta(x) = \delta \text{ for some } x, \\ -\infty & \text{otherwise.} \end{cases}$$

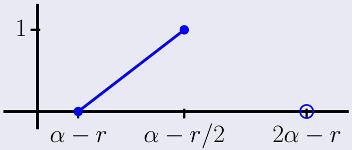
Theorem (Spectrum of singularities, P. 2016, ...[*])

Let d be the spectrum of singularities of f_α, $\Re f_\alpha$ or $\Im f_\alpha$. Then:

If f is a cusp form:



If f is not a cusp form:



[*] Chamizo, Petrykiewicz, Ruiz-Cabello (2015); Ruiz-Cabello (2014); Jaffard (1996).

Techniques involved:

- Wavelet transform.
- Approximate functional equation.
- Jarník-Besicovitch theorem for approximation by cusps.

Lemma (Regularity and growth)

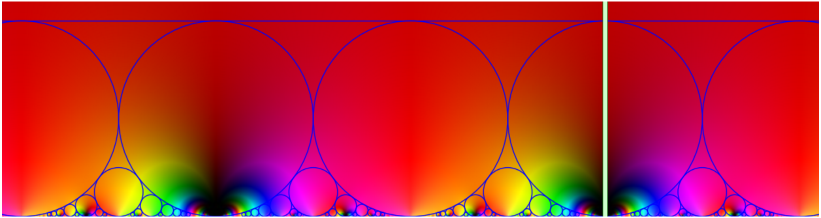
Fix $0 < \delta' < \delta < \alpha$ *and* $x_0 \in \mathbb{R}$*. If for every* $\epsilon > 0$ *small enough, one has when* $(x, y) \to (x_0, 0^+)$*,*

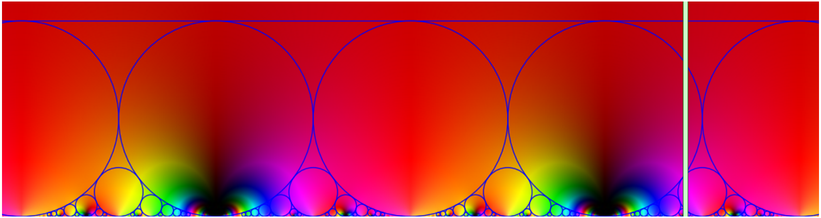
$$f(x+iy) = \Omega(y^{\delta-\alpha+\epsilon}) \quad \text{and}$$

$$f(x+iy) \ll y^{\delta-\alpha-\epsilon}\left(1 + \frac{|x-x_0|}{y}\right)^{\delta'-\epsilon},$$

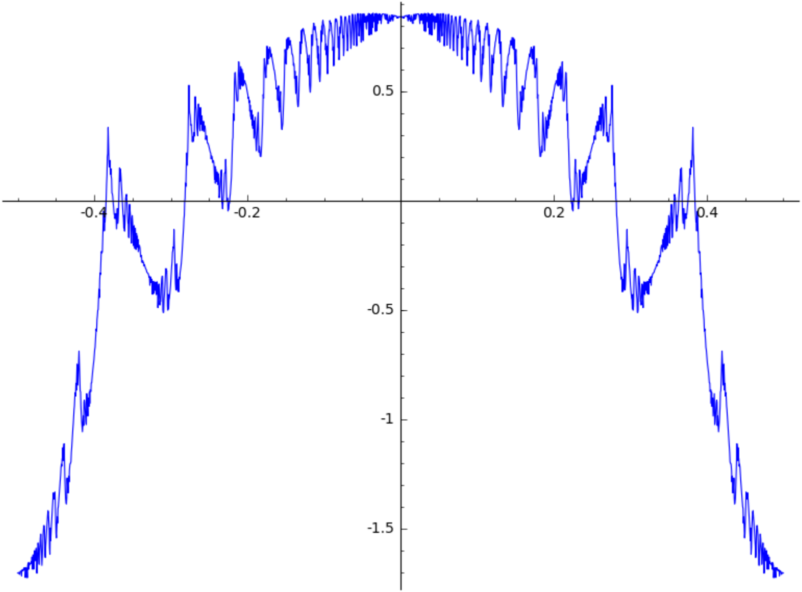
then $\beta(x_0) = \delta$*.*

Wavelet transform and Diophantine analysis:

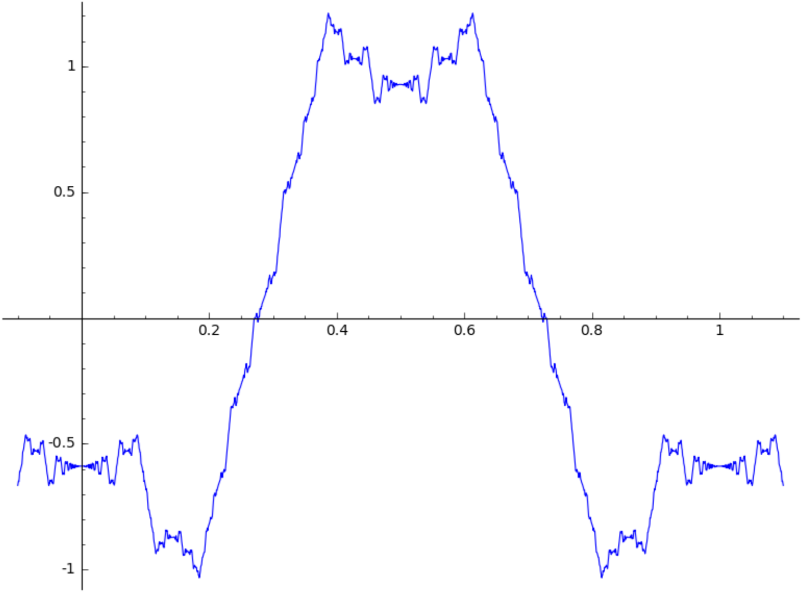




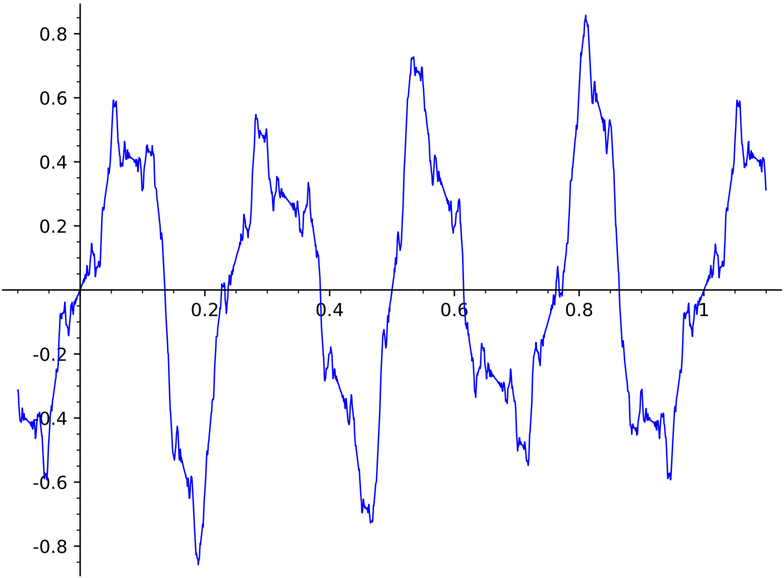
The function $\Re\Delta_{13/2}$, where Δ is the discriminant form: $(\alpha_0 = 6)$



$-\Re f_{9/5}$, where f is the newform of weight 2 and level 14: $(\alpha_0 = 1)$



$\Im f_{\frac{12}{5}}$, where f is a cusp form for a noncongruence group: $(\alpha_0 = \frac{3}{2})$



Given $K \subset \mathbb{R}^d$ and $R > 1$ we define

$$\mathcal{N}(R) = \#\{\, \vec{n} \in \mathbb{Z}^d \;:\; \vec{n} \in RK \,\}$$

and the error exponent

$$\alpha_K = \inf \{\, \alpha > 0 \;:\; \mathcal{N}(R) = \mathrm{vol}(K) R^d + O(R^\alpha) \,\}.$$

K is said to be a **smooth convex body** if it is convex, compact, and its boundary is a differentiable $(d-1)$-dimensional manifold with strictly positive Gaussian curvature.

K smooth convex body $\implies \alpha_K \le d - 1$ (Gauss 1837).

4. Lattice point counting problems

Lattice points and exponential sums:

$$\begin{aligned}\mathcal{N}(R) &= \sum_{\vec{n}\in\mathbb{Z}^d} \chi_{RK}(\vec{n}) \\ \text{“}{=}\text{”} &\sum_{\vec{n}\in\mathbb{Z}^d} \hat{\chi}_{RK}(\vec{n}) \\ &= \mathrm{vol}(K)R^d + \sum_{\vec{0}\neq\vec{n}\in\mathbb{Z}^d} \hat{\chi}_{RK}(\vec{n}).\end{aligned}$$

Bounds for α_K $\iff$ Bounds for $\displaystyle\sum_{0<\|\vec{n}\|\leq R^{-1}} \hat{\chi}_{RK}(\vec{n})$.

State of the art:

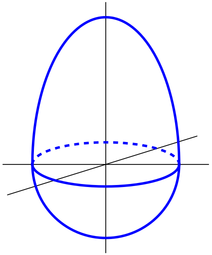
d	smooth convex body	d-dimensional ball	conjecture
2	$\alpha_K \leq \frac{131}{208} = 0.6298\ldots$ [1]	$\alpha_K \leq \frac{517}{824} = 0.6274\ldots$ [2]	$\alpha_K = 1/2$ [3]
3	$\alpha_K \leq \frac{231}{158} = 1.4620\ldots$ [4]	$\alpha_K \leq \frac{21}{16} = 1.3125$ [5]	$\alpha_K = 1$
≥ 4	$\alpha_K \leq d - 2 + r(d)$ [4]	$\alpha_K = d - 2$	$\alpha_K = d - 2$

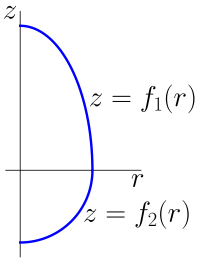
In the bottom-left entry, $r(d) = \frac{d^3+3d+8}{d^3+d^2+5d+4}$.

[1] Huxley (2003).
[2] Bourgain, Watt (2017).
[3] Hardy (1915).
[4] Guo (2010).
[5] Heath-Brown (1997); Iwaniec, Chamizo (1995).

5. Lattice points in bodies of revolution

Let K be a three-dimensional smooth convex body, that is also invariant by rotations around the z-axis:





Theorem (Chamizo 1998)

If the functions $\frac{1}{r} f_i'''(r)$ (extended by continuity to $r = 0$) do not vanish, then $\alpha_K \leq 11/8 = 1.375$.

Ball: $\alpha_K \leq 1.3125$ Smooth convex bodies: $\alpha_K \leq 1.4620\ldots$.

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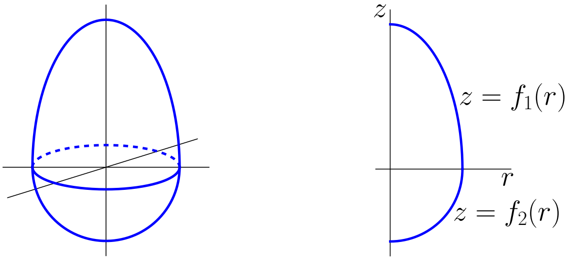
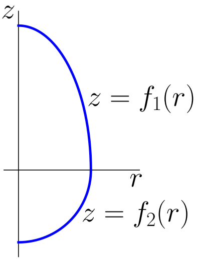


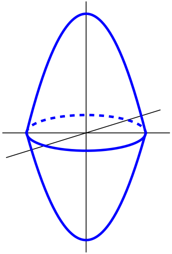
Theorem (Chamizo, P. 2017)

If the zeros of the functions f_1''' and f_2''' are of finite order, we also have $\alpha_K \le 11/8 = 1.375$.

Ball: $\alpha_K \le 1.3125$ Smooth convex bodies: $\alpha_K \le 1.4620\ldots$.

The most “pathological” case:





$f_i''' \equiv 0$

Theorem (Chamizo, P. 2017, ...*)

If K is a double revolution paraboloid then $\alpha_K \leq 1$.
If moreover all the coefficients of K are rationals, $\alpha_K = 1$.

*Krätzel (1997, 1991); Popov (1975).

For $d \geq 3$, let K be the d-dimensional double elliptic paraboloid

$$K = \{(\vec{x}, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, |y| \leq c - Q(\vec{x} + \vec{\beta})\},$$

where $c \in \mathbb{R}$, $\vec{\beta} \in \mathbb{R}^{d-1}$ and $Q(\vec{x}) = \sum_{ij} a_{ij} x_i x_j$.

Theorem (Chamizo, P. 2017)

- *If* $a_{12}/a_{11}, a_{22}/a_{11} \in \mathbb{Q}$, *then* $\alpha_K \leq d-2$.
- *If all the coefficients of* K *are rational numbers,* $\alpha_K = d-2$.

Moreover $\mathcal{N}(R) - \mathrm{vol}(R)R^d = \Omega\big(R^{d-2}\eta(R)\big)$ *where*

$$\eta(R) = \begin{cases} \exp\left(a\frac{\log R}{\log\log R}\right) & \text{for any } a < \log 2 \text{ when } d = 3, \\ \log\log R & \text{when } d = 4, \\ \sqrt{\log\log R} & \text{when } d = 5, \\ 1 & \text{when } d \geq 6. \end{cases}$$

In the case of the double revolution paraboloid:

$$\mathcal{N}(R) - \mathrm{vol}(R)R^3 \approx \sum_{m \le R} \frac{1}{m} \sum_{n_1^2+n_2^2 \le R^2} \exp\left\{2\pi i \frac{m}{R}(n_1^2+n_2^2)\right\}.$$

Hardy-Littlewood bound:

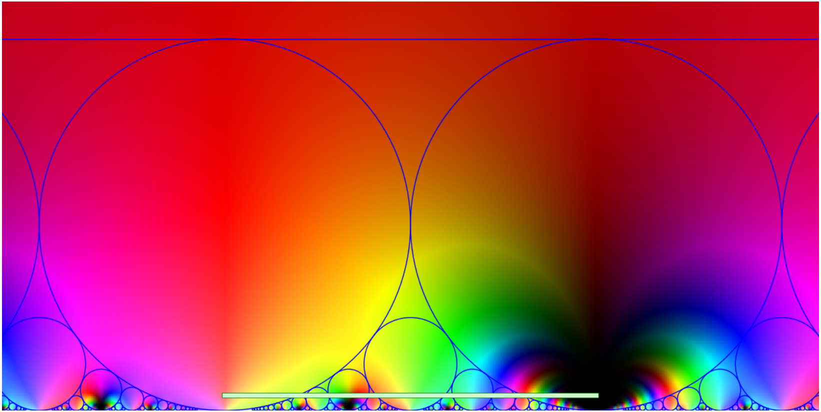
$$\sum_{|n|\le N} e^{2\pi i n^2 x} \ll \frac{N}{\sqrt{q}} \qquad \text{if} \quad \left|x - \frac{p}{q}\right| \le \frac{1}{qN} \quad \text{with} \quad q \le N.$$

- Squaring the bound: $\displaystyle\sum_{|n_i|\le N} \exp\left\{2\pi i x(n_1^2+n_2^2)\right\} \ll \frac{N^2}{q}$.
- "Typically" $q \gg N^{1-\epsilon}$.

5. Lattice points in bodies of revolution

Convolving $\theta^2(z) = \sum_{n_1,n_2} e^{\pi i(n_1^2+n_2^2)z}$ with the Dirichlet kernel:

$$\sum_{n_1^2+n_2^2 \le N} \exp\left\{2\pi i x(n_1^2+n_2^2)\right\} = \int_0^1 \theta^2(u+i/N) D_N(x-u-i/N)\, du.$$



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If the functions $\frac{1}{r}f_i'''(r)$ *(extended by continuity to* $r = 0$*) do not vanish, then* $\alpha_K \le 11/8 = 1.375$.

Theorem (Chamizo, P. 2017)

If the zeros of the functions f_1''' *and* f_2''' *are of finite order, we also have* $\alpha_K \le 11/8 = 1.375$.

- Slice the exponential sum depending on the location of the zeros.
- Van der Corput method for the pieces far away from the zeros.
- Kuzmin-Landau inequality for the pieces close to the zeros.
- Diophantine estimation of the phase when really close.

¡Thank you for your attention!