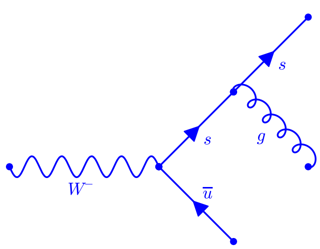
Particles in the standard model. The last column corresponds to bosons (force carriers) the other to fermions.

2.3M 2/3 1/2 u	1.3G 2/3 1/2 c	173G 2/3 1/2 t	0 0 1 g
4.8M -1/3 1/2 d	95M -1/3 1/2 s	4.2G -1/3 1/2 b	0 0 1 γ
0.5M -1 1/2 e	106M -1 1/2 μ	1.8G -1 1/2 τ	91.2G 0 1 Z
$<$ 2.2e 0 1/2 ν_e	$<$ 0.2M 0 1/2 ν_μ	$<$ 16M 0 1/2 ν_τ	80.4G $\pm$1 1 $W^\pm$

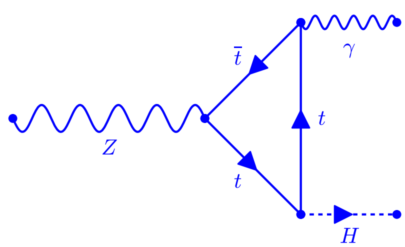
mass charge spin	Symbol

126G 0 0	H

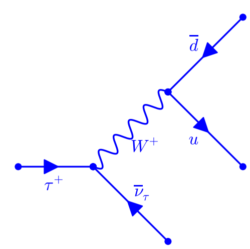
Examples of Feynman diagrams corresponding to decays. The general idea is that gluons just change the color of the quarks. They behave, in some sense as pair quark antiquark, with a color and its opposite. Bosons $W^\pm$ can change the flavor of a quark. The preferred changes are in the same generation but the CKM matrix (Cabibbo-Kobayashi-Maskawa) allows any other change with small probability. They can also turn a (left) lepton on its neutrino. The neutral bosons Z can be emitted by a neutrino or a fermion.



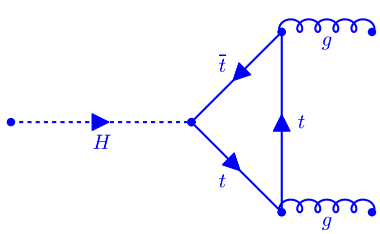
$W^- \longrightarrow \bar{u}sg$



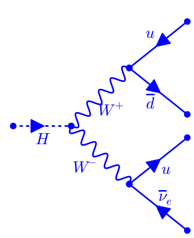
$Z \longrightarrow H\gamma$



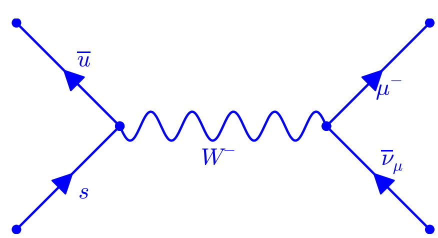
$\tau^+ \longrightarrow \bar{\nu}_\tau u\bar{d}$



$H \longrightarrow gg$



$H \longrightarrow u\bar{d}e^-\bar{\nu}_e$



$K^- = s\bar{u} \longrightarrow \mu^-\bar{\nu}_\mu$

Cross section and amplitude. For $P_1 + P_2 \longrightarrow P_3 + P_4$ the amplitude of probability is

$$T_{fi} = -i(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)F$$

where $-iF$ is the result of the Feynman diagrams. The relation with the cross section is

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\bar{F}|^2 \, d\text{Lips}$$

where dLips is the Lorentz invariant phase factor

$$d\text{Lips} = (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\frac{d^3\vec{p}_3}{2(2\pi)^3E_3}\frac{d^3\vec{p}_4}{2(2\pi)^3E_4}.$$

In the center of mass frame,

$$d\text{Lips} = \frac{1}{16\pi^2}\frac{|p_f|}{\sqrt{s}}d\Omega \qquad \text{and} \qquad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|p_f|}{|p_i|}|\bar{F}|^2 \tag{1}$$

where

$$|p_i| = \frac{1}{2\sqrt{s}}\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2} \qquad \text{and} \qquad |p_f| = \frac{1}{2\sqrt{s}}\sqrt{(s - m_3^2 - m_4^2)^2 - 4m_3^2m_4^2}$$

Recall that $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are the Mandelstam variables. In the CM frame $s = E_1 + E_2$, $p_i = |\vec{p}_1| = |\vec{p}_2|$ and $p_f = |\vec{p}_3| = |\vec{p}_4|$.

For a decay $P_1 \longrightarrow P_2 + P_3$ the decay rate is given by

$$d\Gamma = \frac{|\bar{F}|^2}{2m_1}d\text{Lips} \qquad \text{with} \qquad d\text{Lips} = \frac{|p_f|}{16\pi^2 m_1}d\Omega \qquad \text{and} \qquad |p_f| = |\vec{p}_2| = |\vec{p}_3|. \tag{2}$$

Gamma matrices. They are 4×4 (complex) matrices γ^μ satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{I}, \qquad \mu, \nu = 0, 1, 2, 3. \tag{3}$$

It is also defined $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ that anti-commutes with the rest. A possible choice (Dirac representation) are the Dirac matrices defined in 2×2 blocks as

$$\gamma^0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \text{and} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Another choice (chiral representation) are the Weyl matrices defined in 2×2 blocks as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \text{and} \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

The slash notation consists of abbreviating $a_\mu\gamma^\mu$ as $\not{a}$ and something similar for differential operators, especially $\not{\partial} = \gamma^\mu\partial_\mu$.

The gamma matrices are associated to a representation $D = D(\Lambda)$ of the Lorentz group[1] in the following way: If we want $x \mapsto \Lambda x$, $\Psi \mapsto D\Psi$ to preserve Dirac's equation, we need

$$D^{-1}\gamma^\mu D = \Lambda^\mu_\nu\gamma^\nu$$

because $(i\not{\partial} - m)\Psi = 0 \mapsto i(\Lambda^{-1})^\mu_\nu\partial_\mu\gamma^\nu D\Psi - mD\Psi = 0$ that is $i(\Lambda^{-1})^\mu_\nu D^{-1}\gamma^\nu D\partial_\mu\Psi - m\Psi = 0$. The matrix γ^5 commutes with products of two γ^μ and it proves that D is reducible giving rise in the chiral representation to the projectors $P_\pm = (\mathbf{I} \pm \gamma^5)/2$. We have the relations

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \qquad \text{and} \qquad \gamma^0 D^\dagger\gamma^0 = D^{-1}.$$

The second follows from $\gamma^0[\gamma^\mu, \gamma^\nu]^\dagger\gamma^0 = [\gamma^\nu, \gamma^\mu]$.

It is convenient to define

$$\overline{\Psi} = \Psi^\dagger\gamma^0.$$

With this definition

$\overline{\Psi}\Psi$ is a scalar, $\overline{\Psi}\gamma^\mu\Psi$ is a vector, $\overline{\Psi}\gamma^5\Psi$ is a pseudo-scalar, $\overline{\Psi}\gamma^\mu\gamma^5\Psi$ is an axial vector

The first quantity is a scalar because $\overline{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi \mapsto \Psi^\dagger D^\dagger\gamma^0 D\Psi = \overline{\Psi}D^{-1}\gamma^0 D\Psi$. For the second, use $D^{-1}\gamma^\mu D = \Lambda^\mu_\nu\gamma^\nu$. In the rest, pseudo-scalar and axial vector refer to the fact that they change sign under the parity transformation $\Psi(\vec{x}, t) \mapsto \gamma^0\Psi(-\vec{x}, t)$ (probably this only makes sense in the chiral representation).

In computations with Feynman diagrams, it is very convenient to employ identities involving traces of product of gamma matrices (Casimir's trick). The main ones are

$$\text{(4)} \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \qquad \text{that implies} \quad \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b;$$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big)$, or equivalently

$$\text{(5)} \qquad \mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4(a\cdot b)(c\cdot d) - 4(a\cdot c)(b\cdot d) + 4(a\cdot d)(b\cdot c)$$

and the product of an odd number of gamma matrices is traceless.

$$\text{(6)} \qquad \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\ldots\gamma^{\mu_{2k+1}}) = 0.$$

In the previous results, γ^5 is not admitted. Two identities involving γ_5 are

$$\text{(7)} \qquad \mathrm{Tr}\big(\gamma_5\not{a}\not{b}\not{c}\not{d}\big) = 4i\epsilon_{\mu\nu\lambda\sigma}a^\mu b^\nu c^\lambda d^\sigma \qquad \text{and} \qquad \mathrm{Tr}\big(\gamma_5\not{a}_1\cdots\not{a}_k\big) = 0 \quad \text{for } k < 4.$$

There are also some formulas not involving traces, but contractions. One of the most useful is

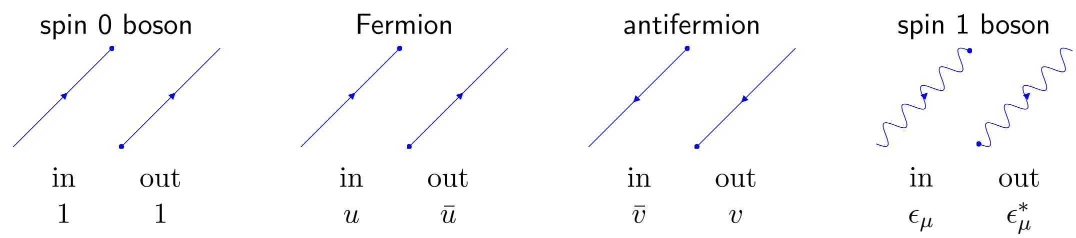
$$\text{(8)} \qquad \gamma_\mu\not{a}\gamma^\mu = -2\not{a}.$$

We also have

$$\text{(9)} \qquad \not{a}^2 = \gamma^\mu a_\mu\gamma^\nu a_\nu = \frac{1}{2}\big(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\big)a_\mu a_\nu = g^{\mu\nu}a_\mu a_\nu = p^2\mathbf{I}.$$

[1] It seems that $D(\Lambda) = \exp\big(\frac{1}{2}\lambda_{\mu\nu}\frac{1}{4}[\gamma^\mu, \gamma^\nu]\big)$ where $\lambda_{\mu\nu}$ are the coefficients for the usual generators.

Lines in Feynman diagrams. The factor for the internal lines:



For the internal lines we have the following propagators:

$$\frac{i}{q^2 - m^2} \qquad \frac{i}{\not{q} - m} = \frac{i(\not{q} + m)}{q^2 - m^2} \qquad -\frac{ig_{\mu\nu}}{q^2} \qquad -\frac{i(g_{\mu\nu} - q_\mu q_\nu/m^2)}{q^2 - m^2}$$

For the propagator of a gluon (spin 1, $m = 0$) we have to introduce also a factor δ^{ab} with $a, b = 1, 2, \ldots, 8$ are the eight color indexes.

Essentially the definition of u and v assures

$$(\not{p} - m)u = 0 \qquad \text{and} \qquad (\not{p} + m)v = 0. \tag{10}$$

We have also the relations

$$\sum_\sigma u(p,\sigma)\overline{u}(p,\sigma) = \not{p} + m \qquad \text{and} \qquad \sum_\sigma v(p,\sigma)\overline{v}(p,\sigma) = \not{p} - m \tag{11}$$

For massive vector (spin 1) particles, we have the completeness relation when summing over the three polarization states

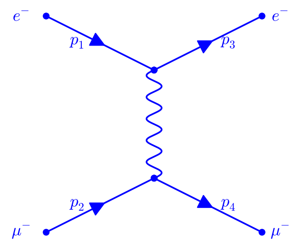
$$\sum_\sigma \epsilon^*_\mu(p,\sigma)\epsilon_\nu(p,\sigma) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{12}$$

For real photons, only two polarization states remains (the transverse ones). In Feynman diagrams we can replace when computing $|\bar{F}|^2$

$$\sum_\sigma \epsilon^*_\mu(p,\sigma)\epsilon_\nu(p,\sigma) \quad \text{by} \quad -g_{\mu\nu} \tag{13}$$

although this quantities are not equal.

The simplest Feynman diagram computation in QED. Consider $e^- + \mu^- \longrightarrow e^- + \mu^-$ in QED. There is only a possibility, the t channel (first particle connected to the third)



because e^- and μ^- cannot meet in any ingoing or outgoing situation by charge conservation or by the leptonic flavor conservation.

From the upper fermionic line (upper vertex), we have $ie\bar{u}_3\gamma^\mu u_1$ and from the lower $ie\bar{u}_4\gamma^\nu u_2$. The photon propagator is $-ig_{\mu\nu}/q^2$ where $q^2 = (p_1 - p_3)^2 = (p_2 - p_4)^2$. Then we have

$$-iF = ie\bar{u}_3\gamma^\mu u_1 \frac{-ig_{\mu\nu}}{(p_1 - p_3)^2} ie\bar{u}_4\gamma^\nu u_2.$$

by definition $|\bar{F}|^2$ is the average of $|F|^2$ over all possible initial spin states and the sum over the final states.

$$|\bar{F}|^2 = \frac{e^4}{(p_1 - p_3)^4}\frac{1}{2}\cdot\frac{1}{2}\sum_{\text{spin states}} \big|\bar{u}_3\gamma^\mu u_1 \bar{u}_4\gamma_\mu u_2\big|^2.$$

The term under $|\ |^2$ involves a sum, then

$$|\bar{F}|^2 = \frac{e^4}{4(p_1 - p_3)^4}\sum_{\text{spin states}} \bar{u}_3\gamma^\mu u_1\bar{u}_4\gamma_\mu u_2\bar{u}_2\gamma_\nu u_4\bar{u}_1\gamma^\nu u_3$$

where it is used $(\bar{u}_1\gamma^\nu u_2)^* = \bar{u}_2\gamma^\nu u_1$.

The part of the electron corresponds to

$$\sum_{\text{spin states}} \bar{u}_3\gamma^\mu u_1\bar{u}_1\gamma^\nu u_3 = \sum_{\text{spin states}} \mathrm{Tr}\big(\bar{u}_3\gamma^\mu u_1\bar{u}_1\gamma^\nu u_3\big).$$

Using the relations (11) and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we have

$$\sum_{\text{spin states}} \mathrm{Tr}\big(\bar{u}_3\gamma^\mu u_1\bar{u}_1\gamma^\nu u_3\big) = \sum_{\text{spin states}} \mathrm{Tr}\big(\bar{u}_3\gamma^\mu(\not{p}_1 + m)\gamma^\nu u_3\big) = \mathrm{Tr}\big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\big)$$

with m the electron mass, that is

$$\mathrm{Tr}\big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\big) = \mathrm{Tr}\big(\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\big) + m^2\mathrm{Tr}\big(\gamma^\mu\gamma^\nu\big),$$

where we have used $\operatorname{Tr}\big(\not p\gamma^\mu\gamma^\nu\big) = 0$ because (6).

Finally, we employ (4) and (5) to get

$$\operatorname{Tr}\big((\not p_3 + m)\gamma^\mu(\not p_1 + m)\gamma^\nu\big) = 4\big(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_1\cdot p_3)g^{\mu\nu} + m^2 g^{\mu\nu}\big)$$

The muon part gives a similar contribution and then

$$|\bar F|^2 = \frac{4e^4}{(p_1-p_3)^4}\big(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_1\cdot p_3 - m_e^2)g^{\mu\nu}\big)\big(p_{4\mu}p_{2\nu} + p_{4\nu}p_{2\mu} - (p_2\cdot p_4 - m_\mu^2)g_{\mu\nu}\big).$$

Neglecting the masses, after some calculations, this expression simplifies to

$$|\bar F|^2 = \frac{4e^4}{(p_1-p_3)^4}\big(2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\big).$$

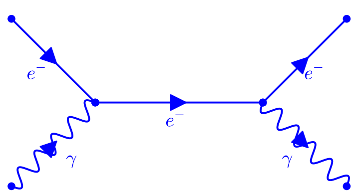
The Mandelstam variables for negligible masses are

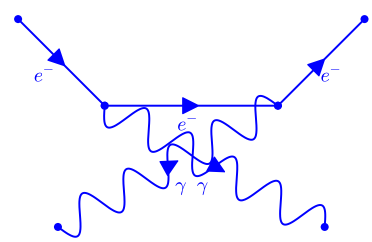
$$\begin{cases} s = (p_1+p_2)^2 = (p_3+p_4)^2 \sim 2p_1\cdot p_2 \sim 2p_3\cdot p_4 \\ t = (p_1-p_3)^2 = (p_4-p_2)^2 \sim -2p_1\cdot p_3 \sim -2p_4\cdot p_2 \\ u = (p_1-p_4)^2 = (p_3-p_2)^2 \sim -2p_1\cdot p_4 \sim -2p_3\cdot p_2 \end{cases} \tag{14}$$

Then we conclude

$$|\bar F|^2 = \frac{2e^4}{t^2}\big(s^2 + u^2\big)$$

A more involved diagram in QED. Consider Compton scattering $e^- + \gamma \longrightarrow e^- + \gamma$. In this cases there are two possible diagrams (apart from no interaction), the s channel and the u channel.





For the first diagram, the contributions of the first vertex, the second vertex and the propagator are

$$\epsilon_{2\mu} ie\gamma^\mu u_1, \qquad \epsilon^*_{4\nu} ie\bar u_3\gamma^\mu, \qquad \frac{i}{\not q - m} = \frac{i(\not q + m)}{q^2 - m^2}.$$

Putting it together, the first amplitude is

$$-iF_1 = -\frac{e^2}{s-m^2}\epsilon^*_{4\nu}\epsilon_{2\mu}\bar u_3\gamma^\nu(\not p_1 + \not p_2 + m)\gamma^\mu u_1.$$

In the same way, the amplitude corresponding to the second diagram is

$$-iF_2 = -\frac{e^2}{u-m^2}\epsilon^*_{4\mu}\epsilon_{2\nu}\bar{u}_3\gamma^\nu(\not{p}_1 - \not{p}_4 + m)\gamma^\mu u_1.$$

From here onwards, we are going to assume that the electron mass m is negligible. Even without his assumption, m can be omitted in the parentheses using the following trick involving (3) and (10)

$$(\not{p}+m)\gamma^\mu u = -p_\nu\gamma^\mu\gamma^\nu u + 2p_\nu g^{\mu\nu}u + m\gamma^\mu u = -\gamma^\mu(\not{p}-m)u + 2p_\nu g^{\mu\nu}u = 2p^\nu u$$

that does not depend on m.

The averaged square amplitude is

$$|\bar{F}|^2 = |\bar{F}_1|^2 + |\bar{F}_2|^2 + 2\Re\big(\overline{F_1F_2^*}\big).$$

There are two spin states for the ingoing electron and two polarization states for the photon, then

$$|\bar{F}_1|^2 = \frac{1}{2}\cdot\frac{1}{2}\frac{e^4}{s^2}\sum_{\text{spin \& pol.}}\epsilon^*_{4\nu}\epsilon_{2\mu}\epsilon_{4\rho}\epsilon^*_{2\tau}\bar{u}_3\gamma^\nu(\not{p}_1+\not{p}_2)\gamma^\mu u_1\bar{u}_1\gamma^\tau(\not{p}_1+\not{p}_2)\gamma^\rho u_3.$$

By the formula (11) and the recipe (13), we have

$$|\bar{F}_1|^2 = \frac{e^4}{4s^2}\mathrm{Tr}\big(\not{p}_3\gamma^\nu(\not{p}_1+\not{p}_2)\gamma^\mu\not{p}_1\gamma_\mu(\not{p}_1+\not{p}_2)\gamma^\nu\big) = \frac{e^4}{s^2}\mathrm{Tr}\big(\not{p}_3(\not{p}_1+\not{p}_2)\not{p}_1(\not{p}_1+\not{p}_2)\big)$$

where (8) was applied in the second equality. With (9), under $m = p_1^2 = p_3^2 = 0$, and (5) we get the compact expression

$$|\bar{F}_1|^2 = \frac{e^4}{s^2}\mathrm{Tr}\big(\not{p}_3\not{p}_2\not{p}_1\not{p}_2\big) = \frac{8e^4}{s^2}(p_1\cdot p_2)(p_2\cdot p_3) = -\frac{2e^4}{s}t$$

where the last equality follows from the approximation (14).

The result for $|\bar{F}_2|^2$ is similar with the change $p_2 \leftrightarrow -p_4$ and it is like $s \leftrightarrow t$. In the interference term $\overline{F_1F_2^*}$ we find the trace

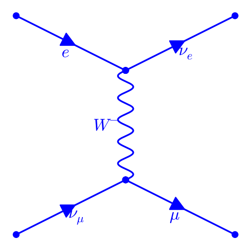
$$\mathrm{Tr}\big(\not{p}_3\gamma^\nu(\not{p}_1+\not{p}_2)\gamma^\mu\not{p}_1\gamma_\mu(\not{p}_1-\not{p}_2 4\gamma^\nu\big) = 4\mathrm{Tr}\big(\not{p}_3(\not{p}_1+\not{p}_2)\not{p}_1(\not{p}_1-\not{p}_4)\big) = -4\mathrm{Tr}\big(\not{p}_3\not{p}_2\not{p}_1\not{p}_4\big),$$

under $m = 0$. The identity (5) and the conservation of 4-momentum $p_4 = p_1 + p_2 - p_3$ prove that this trace vanishes.

In this way, there is no interference term and we conclude

$$|\bar{F}|^2 = |\bar{F}_1|^2 + |\bar{F}_2|^2 = -\frac{2e^4}{s}t - \frac{2e^4}{u}s.$$

The simplest Feynman diagram computation in EW. Consider $e^- + \nu_\mu \longrightarrow \nu_e + \mu^-$ in EW. As in QED, there is only a possibility, the t channel



The contributions of the upper and lower fermionic lines are

$$\bar{u}_3 \frac{ig}{2\sqrt{2}} \gamma^\mu (1-\gamma_5) u_1 \qquad \text{and} \qquad \bar{u}_4 \frac{ig}{2\sqrt{2}} \gamma^\mu (1-\gamma_5) u_2.$$

The W propagator in the unitary gauge is

$$\frac{-i}{q^2 - M_W^2}\big(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}\big) \quad \text{that we approximate by} \quad \frac{ig_{\mu\nu}}{M_W^2}$$

because M_W is large.

Putting these terms together

$$-iF = -i\frac{g^2}{8M_W^2}\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\bar{u}_4\gamma^\mu(1-\gamma_5)u_2.$$

Recall that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$, $\bar{\gamma}_5 = -\gamma_5$, and $\bar{\gamma}_\mu = \gamma_\mu$. Hence $\overline{(1-\gamma_5)\gamma^\mu} = \gamma^\mu(1+\gamma_5) = (1-\gamma_5)\gamma^\mu$. Then

$$|F|^2 = \frac{g^4}{64M_W^4}[\bar{u}_3\gamma_\mu(1-\gamma_5)u_1][\bar{u}_4\gamma^\mu(1-\gamma_5)u_2][\bar{u}_1\gamma_\nu(1-\gamma_5)u_3][\bar{u}_2\gamma^\nu(1-\gamma_5)u_4]$$

where the terms in the brackets are scalars and then it can be reordered. We have

$$[\bar{u}_3\gamma_\mu(1-\gamma_5)u_1][\bar{u}_1\gamma_\nu(1-\gamma_5)u_3] = \mathrm{Tr}\big(u_3\bar{u}_3\gamma_\mu(1-\gamma_5)u_1\bar{u}_1\gamma_\nu(1-\gamma_5)\big)$$

and a similar formula for the lower fermionic line. Using (11), we have that the sum of $64M_W^4 g^{-4}|F|^2$ is

$$\frac{g^4}{64M_W^4}\mathrm{Tr}\big(\not{p}_3\gamma_\mu(1-\gamma_5)(\not{p}_1+m_e)\gamma_\nu(1-\gamma_5)\big)\mathrm{Tr}\big((\not{p}_4+m_\mu)\gamma^\mu(1-\gamma_5)\not{p}_2\gamma^\nu(1-\gamma_5)\big)$$

and the last trace can be replaced by $\mathrm{Tr}\big(\not{p}_2\gamma^\nu(1-\gamma_5)(\not{p}_4+m_\mu)\gamma^\mu(1-\gamma_5)\big)$ because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Note that $\mathrm{Tr}\big(\not{p}_i\gamma_\mu(1-\gamma_5)\gamma_\nu(1-\gamma_5)\big) = 0$, $i = 2,3$, because $(1-\gamma_5)\gamma_\nu(1-\gamma_5) = (1-\gamma_5)(1+\gamma_5)\gamma_\nu = (1-\gamma_5^2)\gamma_\nu = 0$. It implies that we can omit m_e and m_μ below.

There are two spin states for the electron and only one for the neutrino (it is always left-handed). Then

$$|\bar{F}|^2 = \frac{1}{2}\cdot\frac{g^4}{64M_W^4}H_{\mu\nu}(p_1,p_3)H^{\mu\nu}(p_4,p_2) \qquad \text{with} \quad H_{\mu\nu}(p_1,p_3) = \mathrm{Tr}\big(\not{p}_3\gamma_\mu(1-\gamma_5)\not{p}_1\gamma_\nu(1-\gamma_5)\big).$$

Using that $\{\gamma_5,\gamma^\mu\}=0$, and that $(1-\gamma_5)/2$ is a projector, in particular $(1-\gamma_5)^2 = 2(1-\gamma_5)$,

$$H_{\mu\nu}(p_1,p_3) = \mathrm{Tr}\big(\not{p}_3\gamma_\mu(1-\gamma_5)^2\not{p}_1\gamma_\nu\big) = 2\mathrm{Tr}\big(\not{p}_3\gamma_\mu\not{p}_1\gamma_\nu\big) - 2\mathrm{Tr}\big(\gamma_5\not{p}_3\gamma_\mu\not{p}_1\gamma_\nu\big).$$

Now we employ (5) and (7) to get

$$H_{\mu\nu}(p_1,p_3) = 8\big(p_{1\mu}p_{3\nu} + p_{3\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot p_3)\big) - 8i\epsilon_{\mu\lambda\nu\sigma}p_1^\lambda p_4^\sigma.$$

Note that the term in the first parenthesis is symmetric and vanishes when contracted with an anti-symmetric expression. Keeping this idea in mind, after some calculations

$$H_{\mu\nu}(p_1,p_3)H^{\mu\nu}(p_4,p_2) = 64\big(2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\big) - 64\epsilon_{\mu\lambda\nu\sigma}\epsilon^{\mu\rho\nu\tau}p_1^\lambda p_3^\sigma p_{4\rho}p_{2\tau}.$$

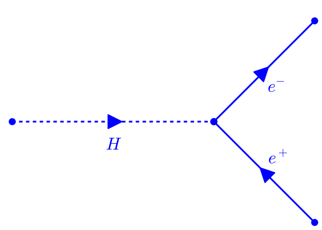
Using $\epsilon_{\mu\lambda\nu\sigma}\epsilon^{\mu\rho\nu\tau} = 2\delta_\lambda^\rho\delta_\sigma^\tau - 2\delta_\sigma^\rho\delta_\lambda^\tau$, we have

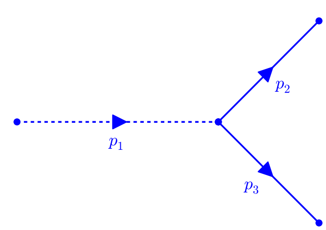
$$\epsilon_{\mu\lambda\nu\sigma}\epsilon^{\mu\rho\nu\tau}p_1^\lambda p_3^\sigma p_{4\rho}p_{2\tau} = 2(p_1\cdot p_4)(p_2\cdot p_3) - 2(p_1\cdot p_2)(p_3\cdot p_4).$$

Hence

$$|\bar{F}|^2 = \frac{g^4}{128M_W^4}H_{\mu\nu}(p_1,p_3)H^{\mu\nu}(p_4,p_2) = \frac{g^4}{128M_W^4}\cdot 256(p_1\cdot p_2)(p_3\cdot p_4) = 2\frac{g^4}{M_W^4}(p_1\cdot p_2)(p_3\cdot p_4).$$

A simple decay rate. Consider the decay of the Higgs particle $H \longrightarrow e^- + e^+$. The corresponding Feynman diagram and the naming of the momenta are





First of all, we compute the amplitude $-iF$. The Feynman rule assigns $-i\frac{gm}{2M_W}$ to the vertex, where m is the mass of the electron. The amplitude is this constant multiplied by $\bar{u}_2v_3$. Hence

$$|\bar{F}|^2 = \sum_{\text{spin states}}\frac{g^2m^2}{4M_W^2}\bar{u}_2v_3\bar{v}_3v_2 = \frac{g^2m^2}{4M_W^2}\mathrm{Tr}\big((\not{p}_2+m)(\not{p}_3-m)\big).$$

Using $\mathrm{Tr}\big(\gamma^\mu\gamma^\nu\big) = 4g^{\mu\nu}$ and that the matrices γ^μ are traceless,

$$|\bar{F}|^2 = \frac{g^2m^2}{4M_W^2}\big(4(p_2\cdot p_3) - 4m^2\big) = \frac{g^2m^2}{4M_W^2}\big(2M^2 - 8m^2\big),$$

where we have employed $2(p_2\cdot p_3) = p_1^2 - p_2^2 - p_3^2$ that follows from $p_1 = p_2 + p_3$.

By (2), we have

$$d\Gamma = \frac{|\bar{F}|^2}{2M}\cdot\frac{|\vec{p}_2|}{16\pi^2M}d\Omega \qquad \text{with } M \text{ the mass of the Higgs particle.}$$

In the CM frame, $p_1 = (M,\vec{0})$, $p_2 = (E_2,\vec{p}_2)$ and $p_3 = (E_2,-\vec{p}_2)$. By energy conservation $E_2 = \frac{1}{2}M$ and $|\vec{p}_2|^2 = E_2^2 - p_2^2 = \frac{1}{4}M^2 - m^2$. Substituting this and the previous value of the mean squared amplitude we have that $d\Gamma/d\Omega$ does not depend on Ω and we conclude

$$\Gamma(H \longrightarrow e^- + e^+) = \frac{g^2m^2M}{32\pi M_W}\Big(1 - \frac{4m^2}{M}\Big),$$

since $\int d\Omega = 4\pi$.

The Lagrangians of the SM. The full Lagrangian (density) of the SM is the combination of the QCD (quantum chromodynamics) and the EW (electroweak) Lagrangians modified with the Higgs field.

The Lagrangian of QED is an abelian gauge theory and then it is by far simpler than that of QCD and EW but still acts as a model. The Lagrangian of QED is obtained coupling Dirac Lagrangian $\bar{\psi}(i\partial\!\!\!/ - m)\psi$, corresponding to the free spinor field, and the electromagnetic field,

$$\text{(15)}\quad \mathcal{L} = i\bar{\psi}D\!\!\!\!/\,\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad \text{where} \quad D_\mu = \partial_\mu + ieA_\mu \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

It has local gauge under U(1) invariance given by $\psi \mapsto e^{-i\theta(x)e}\psi$ and $A_\mu \mapsto A_\mu + \partial_\mu\theta(x)$. In a displayed way, it is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - ej^\mu A_\mu$$

for the electromagnetic current $j^\mu = e\bar{\psi}\gamma^\mu\psi$. Fixing the gauge of the electromagnetic field may require a term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ with ξ a constant. For Feynman gauge, $\xi = 1$.

In QCD we have local gauge invariance under a more complicate group $\mathrm{SU}(N_c)$ where N_c is the number of colors, $N_c = 3$ in the standard theory. For a quark q, there are N_c spinorial wave functions ψ_q^A, $1 \le A \le N_c$. In this way, the free Lagrangian is

$$\mathcal{L} = \sum_{q=1}^{N_f}\sum_{A=1}^{N_c}\big(i\bar{\psi}_q^A\gamma^\mu\partial_\mu\psi_q^A - m_q\bar{\psi}_q^A\psi_q^A\big)$$

where N_f is the number of flavors, $N_f = 6$ in the actual theory (there are 6 quarks). It is common to write the inner sum as the Dirac Lagrangian $\bar{\psi}(i\partial\!\!\!/ - m_q)\psi$, then the gamma matrices act on each of the three (or N_c) color components. To mimic (15) with $\mathrm{SU}(N_c)$ invariance one considers as the "free" part

$$\mathcal{L}_0 = i\bar{\psi}D\!\!\!\!/\,\psi - m\bar{\psi}\psi \qquad \text{where} \qquad D_\mu = \partial_\mu - ig_s A_\mu.$$

Here g_s is a coupling constant (of the strong force) and A_μ a "vector potential" that for each Lorentzian index μ acts on the color space, it is a 3×3 matrix. The Lagrangian $\mathcal{L}_0$ is invariant under the gauge transformations $\psi \mapsto \Omega(x)\psi$ with $\Omega(x) \in \mathrm{SU}(N_c)$ (in color components $\psi^A = \Omega^A_B \psi^B$) when $A_\mu \mapsto \Omega A_\mu \Omega^\dagger - \frac{i}{g_s}\Omega\partial_\mu\Omega^\dagger$. We can always write $\Omega = e^{-i\theta_a(x)T^a}$ where T^a are the generators of $\mathfrak{su}(3)$. The standard choice is $T^a = \frac{1}{2}\lambda^a$ where λ^a are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note that $1 \le a \le N_c^2 - 1$ because $N^2 - 1$ is the dimension of $\mathfrak{su}(N)$. Its rank is $N - 1$ and in the previous case ($N = 3$) the Cartan subalgebra is generated by T^3 and T^8. The particles associated to the field A_μ are called gluons. There are $\dim \mathfrak{su}(3) = 8$ of them in the standard theory.

To complete the analogy with (15) one needs to introduce something similar to $F^{\mu\nu}F_{\mu\nu}$ but in this case the choice $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (and taking the trace to get an scalar) does not work because the Lagrangian would not be invariant. The obstruction is that A_μ and A_ν do not commute in general because we are in a not abelian group. The right definition, up to constants, is the curvature tensor associated to the covariant derivative D_μ

$$F_{\mu\nu} = \frac{i}{g_s}\big[D_\mu, D_\nu\big] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_s[A_\mu, A_\nu].$$

It is invariant in the sense that it transform with the adjoint representation $F_{\mu\nu} \mapsto \Omega F_{\mu\nu}\Omega^\dagger$ under gauge transformations.

The final Lagrangian for QCD is obtained from $\mathcal{L}_0$ adding the Yang-Mills term

$$\mathcal{L} = i\bar{\psi}D\!\!\!\!/\,\psi - m\bar{\psi}\psi - \frac{1}{2}\mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big). \tag{16}$$

Again one can add a term to fix the gauge.

When expanding this Lagrangian, one finds terms involving product or three or four A_μ (with derivatives in the first case), It implies that there are Feynman rules for 3 or 4 gluons meeting at a vertex.

The procedure can be copied to other groups of symmetries (Yang-Mills theories). In particular in the electroweak theory where the symmetry group is $SU(2)\times U(1)$, usually written $SU(2)_L \times U(1)_Y$. Note firstly that in the case of QCD we could define scalar functions $A^a_\mu(x)$ such that $A_\mu = \sum T^a A^a_\mu$ and express for instance $F_{\mu\nu}$ in terms of this functions (this is usually done). For SU(2), we could express in the same way any $W_\mu(x)$ as $\sigma_1 W^1_\mu + \sigma_2 W^2_\mu + \sigma_3 W^3_\mu = \frac{1}{2}\vec{\sigma}\cdot\vec{W}_\mu$. For U(1) the situation is like in QED and we have a vector potential denoted by B_μ.

The analog of (16) in the EW theory is

$$\mathcal{L} = i\sum_\psi \bar{\psi}\not{D}\psi - \frac{1}{4}W^i_{\mu\nu}W^{\mu\nu}_i - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{17}$$

where

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk}W^j_\mu W^k_\nu \qquad \text{and} \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

The, in principle easy, kinetic term deserve some comments. The covariant derivative is

$$D_\mu = \partial_\mu - ig\vec{T}\cdot\vec{W}_\mu - ig'\frac{Y}{2}B_\mu$$

The important point is that EW interaction is chiral and ψ involves a part, say L, that transforms with SU(2) and other, R, that transforms with U(1). We want to switch off $\vec{W}_\mu$ for L then we can say that $\vec{T}$ is 0 for R and $\frac{1}{2}\vec{\sigma}$ for L. We also take the weak hypercharge $Y=-2$ in R and $Y=-1$ in L for neutrinos and electrons (for quarks, it is $2q$ in R and $1/3$ in L). In a displayed form, for

$$\psi = \begin{pmatrix} L \\ R \end{pmatrix}, \qquad \text{with} \quad L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \quad R = e_R$$

where $e_L = \frac{1}{2}(1-\gamma^5)\psi_e$, $e_R = \frac{1}{2}(1-\gamma^5)\psi_e$ and ψ_e the wave function of the electron, we have that the kinetic term is

$$i\bar{L}\not{D}L + i\bar{R}\not{D}R \qquad \text{with} \qquad \begin{cases} D_\mu L = \big(\partial_\mu - i\frac{g}{2}\vec{\sigma}\cdot\vec{W} + i\frac{g'}{2}B_\mu\big)L \\ D_\mu R = \big(\partial_\mu + ig'B_\mu\big)R \end{cases}$$

The group $\mathrm{U}(1)_{\mathrm{em}}$ corresponding to the electromagnetic theory (QED) is included in a nontrivial way in $SU(2)_L \times U(1)_Y$. As the rank is 4, we have four force carriers, one is the photon and the other three are the vector bosons W^+, W^- and Z. These three particles has mass but the previous model does not allow it (terms of the form $W_\mu W^\mu$ are not invariant under the gauge transformations). To avoid this situation, a new term is introduced in the Lagrangian corresponding to the Higgs field. Firstly one makes the Lagrangian of a standard (double) scalar field with a Mexican hat potential

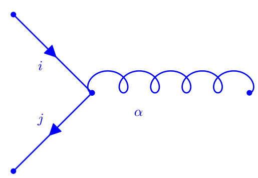
$$\mathcal{L}_\phi = (D^\mu\phi)^\dagger(D^\mu\phi) + \frac{1}{2}m_H^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2, \qquad \text{the usual notation is} \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix},$$

and one adds an interaction term with the particles of the form

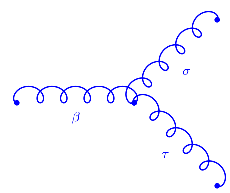
$$\mathcal{L}_{\text{int}} = -G_e(\bar{L}\phi R + \bar{R}\phi^\dagger L)$$

with G_e a coupling constant. In one consider quarks, new coupling constants are needed.

Color factors. In vertexes quark-quark-gluon and gluon-gluon-gluon, the Feynman rules of QCD introduce factors depending on color indexes:



$T^\alpha_{ij} = \frac{1}{2}\lambda^\alpha_{ij}$



$f^{\beta\sigma\tau}$

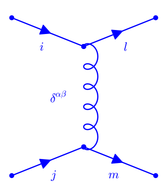
where $i, j \in \{1,2,3\}$ represent colors and $\alpha, \beta, \sigma, \tau \in \{1,\dots,8\}$ are indexes corresponding to the adjoint representation of SU(3), selecting Gell-Mann matrices. When computing amplitudes, after squaring, one has to average over the indexes of the incoming particles ans to sum over the indexes of the outgoing particles. The result corresponding to the previous factors is called the color factor.

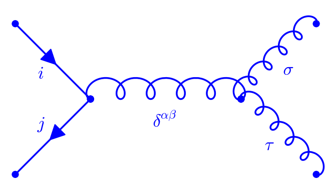
The basic formulas are

$$\delta_{\alpha\beta} C_A = \sum_{\gamma,\delta} f_{\alpha\gamma\delta} f_{\beta\gamma\delta}, \qquad \delta_{ik} C_F = \sum_{\alpha,l} T^\alpha_{il} T^\alpha_{lk} \qquad \text{and} \qquad \delta_{\alpha\beta} T_F = \sum_{k,i} T^\alpha_{ik} T^\beta_{ki} = \mathrm{Tr}(T^\alpha T^\beta)$$

with $C_A = 3$, $C_F = 4/3$ and $T_F = 1/2$ with the usual normalization.

Let us compute the color factors for the following diagrams:





In the first diagram, there are $9 = 3 \cdot 3$ possible values of (i, j), then when we compute $|\bar{F}|^2$ the color factor is

$$C = \frac{1}{9} \sum_{i,j,l,m} \left(T^\alpha_{il} \delta_{\alpha\beta} T^\beta_{jm}\right) \left(T^\sigma_{il} \delta_{\sigma\tau} T^\tau_{jm}\right)^*$$

where the Kronecker deltas come from the propagator and we assume the summation convention in their indexes. The Gell-Mann matrices are Hermitian, then $(T^\alpha)^* = (T^\alpha)^t$ and we have

$$C = \frac{1}{9} \sum_{\alpha,\sigma} \sum_{i,j,l,m} T^\alpha_{il} T^\sigma_{li} T^\alpha_{jm} T^\sigma_{mj} = \frac{1}{9} \sum_{\alpha,\sigma} T_F \delta_{\alpha\sigma} T_F \delta_{\alpha\sigma} = \frac{8}{9} T_F^2 = \frac{2}{9}.$$

For instance the scattering $u + d \longrightarrow u + d$ can be treated as $e^- + \mu^- \longrightarrow e^- + \mu^-$ in QED but one has to replace e by g_s and to introduce this factor, i.e.

$$|\bar{F}|^2 = \frac{4g_s^4}{9t^2}\big(s^2 + u^2\big).$$

In this case, the differential cross section (1) is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|p_f|}{|p_i|}|\bar{F}|^2 = \frac{|\bar{F}|^2}{64\pi^2 s} = \frac{g_s^4}{144\pi^2 st^2}\big(s^2 + u^2\big).$$

For the second diagram, we have a color factor

$$C = \frac{1}{9}\sum_{i,j,\sigma,\tau}\big(T^\alpha_{ij}\delta_{\alpha\beta}f^{\beta\sigma\tau}\big)\big(T^\rho_{ij}\delta_{\rho\lambda}f^{\lambda\sigma\tau}\big)^*.$$

Using the properties

$$C = \frac{1}{9}\sum_{\alpha,\rho}\sum_{i,j,\sigma,\tau}T^\alpha_{ij}T^\rho_{ji}f^{\alpha\sigma\tau}f^{\rho\sigma\tau} = \frac{1}{9}\sum_{\alpha,\rho}\delta_{\alpha\rho}T_F\delta_{\alpha\rho}C_A = \frac{8}{9}T_F C_A = \frac{4}{3}.$$
