Tachyonic instabilities in Yang-Mills theories and number theory

Fernando Chamizo

Presentation of the Master Thesis

July 4, 2016

It is not only QCD...

It is not only QCD. . .

Understanding the Yang-Mills theories

is literally a 1 million dollar problem

Yang-Mills mass gap problem is one of the seven so-called *Millennium Problems* and Clay Mathematics Institute offers $1million for a solution

The master thesis

Main topic

Study the possibility of phase transitions (tachyonic instabilities) in a 2+1 model of $SU(N)$ pure Yang-Mills theory when N and the volume vary

The master thesis

Main topic

Study the possibility of phase transitions (tachyonic instabilities) in a 2+1 model of $SU(N)$ pure Yang-Mills theory when N and the volume vary

Keywords

Large N, volume independence, center symmetry, self-energy, symmetry breaking, twisted boundary conditions, lattice gauge theory, Diophantine approximation

Novelty

Surprisingly, the existence of instabilities translates into highly nontrivial problems in number theory

Scheme of the memoir

- Gauge theories
- Lattice gauge theory
- Large N
- EK and TEK models

Preliminary ideas and motivation

- Yang-Mills in $\mathbb{T}^2 \times \mathbb{R}$
- Perturbation theory

The model

- Regularization of the self-energy
- A number theoretical approach to tachyonic instabilities

Original contribution

Scheme of the memoir

- Gauge theories
- Lattice gauge theory
- Large N
- EK and TEK models

Preliminary ideas and motivation

- Yang-Mills in $\mathbb{T}^2 \times \mathbb{R}$
- Perturbation theory

The model

- Regularization of the self-energy
- A number theoretical approach to tachyonic instabilities

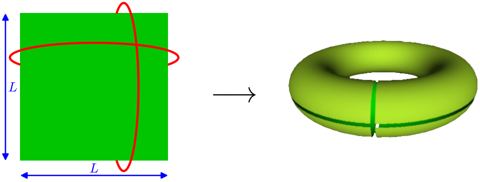
Original contribution

The model

Configuration space

How to glue the space

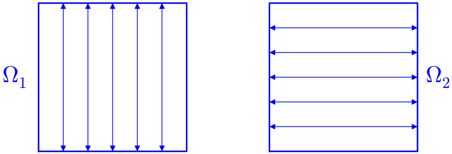
Flat torus $\mathbb{T}^2$



$SU(N)$ bundle

How to glue the field

Transition functions



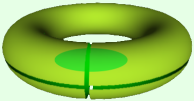
Three important integral numbers

1. The *magnetic flux* $m \in \mathbb{Z}$

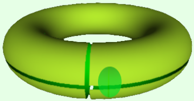
Three important integral numbers

1. The *magnetic flux* $m \in \mathbb{Z}$

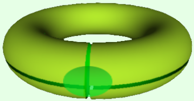
Meaning: It expresses a compatibility condition.



Transition function Ω_1



Transition function Ω_2



Compatibility condition

$$\Omega_1\big(x + (0, L)\big)\Omega_2(x) = e^{2\pi i m/N}\Omega_2\big(x + (L, 0)\big)\Omega_1(x)$$

Values of $m \leftrightarrow$ topological sectors in the space of gauge fields

Three important integral numbers

1. The *magnetic flux* $m \in \mathbb{Z}$
2. The *(chromo-) electric flux* $\vec{e} = (e_1, e_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$

Three important integral numbers

1. The *magnetic flux* $m \in \mathbb{Z}$
2. The *(chromo-) electric flux* $\vec{e} = (e_1, e_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$

Meaning: It represents the *center symmetry*: The gluon field does not "see" the center of $SU(N)$, $Z_N = \{e^{2\pi ik/N}\mathbb{I}\}$

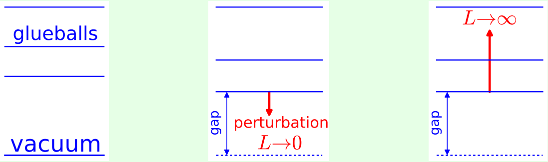
$$\begin{aligned} \Omega\big(x + (L,0)\big) &= e^{2\pi i k_1/N}\Omega_1(x)\Omega(x)\Omega_1(x)^\dagger \\ \Omega\big(x + (0,L)\big) &= e^{2\pi i k_2/N}\Omega_2(x)\Omega(x)\Omega_2(x)^\dagger \end{aligned}$$

(k_1, k_2) and (e_1, e_2) are dual

The quantum problem

Center symmetry $\rightarrow$ division into sectors with Hamiltonian $H_{\vec{e}}$
Vacuum $\rightarrow$ ground state of $H_{\vec{0}}$

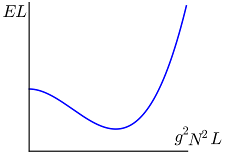
"Normalized" energy levels



$\vec{e} = \vec{0}$ $\qquad$ $\vec{e} \neq \vec{0}$ $\qquad$ $\vec{e} \neq \vec{0}$

What happens when we let N and L vary?

Conjectural behavior of the "mass" gap



Possible problem: the graph cross zero $\to$ the vacuum becomes unstable and decays (phase transition) $\to$ **tachyonic instabilities**

The fundamental question

Given a (large) N can we choose m to prevent tachyonic instabilities for any $\vec{e}$ and L?

Timeline and motivation

(1974)	't Hooft	large N
(1979)	't Hooft	twist, center symmetry
(1982)	Eguchi-Kawai	reduced models [lattice]
		$\downarrow$
(1983)	González-Arroyo Okawa	*Symmetry breaking!?*
	$+$	
(2014)	García Pérez	Continuous models

Timeline and motivation

(1974)	't Hooft	large N
(1979)	't Hooft	twist, center symmetry
(1982)	Eguchi-Kawai	reduced models [lattice]
		↓
(1983)	González-Arroyo Okawa	*Symmetry breaking!?*
	+	
(2014)	García Pérez	Continuous models

To avoid tachyonic instabilities, the following quantity has to be positive for every $\vec{e}$

$$\underbrace{x^{-2}|\vec{n}|^2 + \alpha x^{-1} \sum_{\vec{k}} \frac{\sin^2(\pi \vec{k}\cdot\vec{e}/N)}{|\vec{k}|}}_{\text{1-loop perturbative regime}} + \underbrace{\beta + \gamma N^{-2} x^2 |\vec{e}|^2}_{\substack{\text{Non perturbative} \\ \text{(confinement)}}}$$

$$x = \frac{g^2 N^2 L}{4\pi}, \qquad \alpha, \beta, \gamma \asymp 1, \qquad \vec{n} = m(e_2, -e_1) + N\vec{n}_0$$

Important point

$$\sum_{\vec{k}} \frac{\sin^2(\pi \vec{k}\cdot\vec{x})}{|\vec{k}|} \xrightarrow{\text{regularization}} \sim -\frac{1}{2}\begin{pmatrix} \text{distance of } \vec{x} \text{ to the} \\ \text{closer integer vector} \end{pmatrix}^{-1}$$

The mathematical interpretation

$\mathrm{d}(\vec{x}) =$ distance of $\vec{x}$ to the closer point in $\mathbb{Z} \times \mathbb{Z}$
$\vec{e} = (e_1, e_2), \qquad \vec{e} \in \mathbb{Z}_N \times \mathbb{Z}_N - \{(0,0)\}$
$\vec{e}^{\perp} = (e_2, -e_1)$
$c_0 =$ universal constant

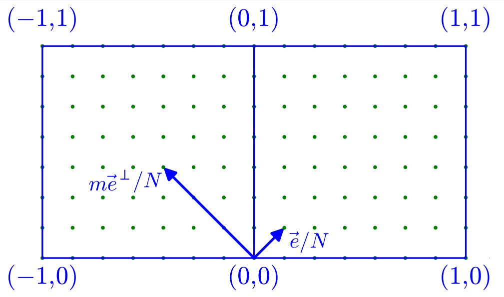
Given N (large) if there exists m such that for every $\vec{e} \neq \vec{0}$

$N \, \mathrm{d}\big(\frac{\vec{e}}{N}\big) \, \mathrm{d}\big(\frac{m\vec{e}^{\perp}}{N}\big) \geq c_0 \implies$ No tachyonic instabilities

$N^2 \, \mathrm{d}\big(\frac{\vec{e}}{N}\big) \, \mathrm{d}^2\big(\frac{m\vec{e}^{\perp}}{N}\big) \geq c_0 \implies$ No tach. inst. at perturbative regime

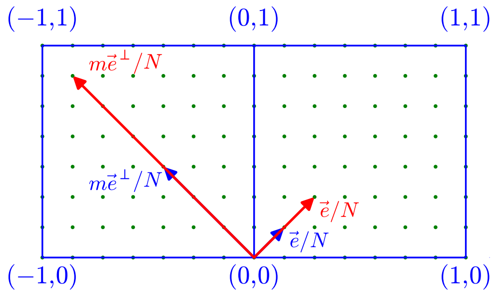
Example: Can we take $m = (N-1)/2$?

$$\begin{array}{cccccc} & N & \mathrm{d}\left(\frac{\vec{e}}{N}\right) & \mathrm{d}\left(\frac{m\vec{e}^{\perp}}{N}\right) & \overset{?}{>} & c_0 \\ & \downarrow & \downarrow & \downarrow & & \\ \vec{e} = (1,1) & N & \frac{\sqrt{2}}{N} & \frac{N-1}{\sqrt{2}N} & \approx & 1 \end{array}$$



Example: Can we take $m = (N-1)/2$?

$$
\begin{array}{rccccc}
 & N & \mathrm{d}\left(\frac{\vec{e}}{N}\right) & \mathrm{d}\left(\frac{m\vec{e}^{\perp}}{N}\right) & \overset{?}{>} & c_0 \\
 & \downarrow & \downarrow & \downarrow & & \\
\vec{e} = (1,1) & N & \frac{\sqrt{2}}{N} & \frac{N-1}{\sqrt{2}N} & \approx & 1 \\
\vec{e} = (2,2) & N & \frac{2\sqrt{2}}{N} & \frac{\sqrt{2}}{N} & \approx & \frac{4}{N} \not> c_0
\end{array}
$$



Our results

$\boxed{\text{Study of tachyonic instabilities}} \longleftrightarrow \boxed{\text{Approximation of } k/N \text{ by } k'/N' \text{ with } N' < N}$

Our results

$$\begin{tikzcd} \boxed{\text{Yang-Mills theories}} \arrow[r, dashed] \arrow[d] & \boxed{\text{Number theory}} \\ \boxed{\text{Study of tachyonic instabilities}} \arrow[r, leftrightarrow] & \boxed{\text{Approximation of } k/N \text{ by } k'/N' \text{ with } N' < N} \arrow[u] \end{tikzcd}$$

Our results

$$
\begin{tikzcd}
\boxed{\text{Yang-Mills theories}} \arrow[r, dashed] \arrow[d] & \boxed{\text{Number theory}} \\
\boxed{\text{Study of tachyonic instabilities}} \arrow[r, leftrightarrow] & \boxed{\text{Approximation of } k/N \text{ by } k'/N' \text{ with } N' < N} \arrow[u]
\end{tikzcd}
$$

Results

- Total absence of instabilities $\leftrightarrow$ Conjecture in number theory
- Optimal situation $\leftrightarrow$ N = Fibonacci number and prime
- Algorithm to limit electric fluxes $\leftrightarrow$ continued fractions
- No instabilities in pertub. regime $\leftrightarrow$ Dioph. approximation

Bibliography I

F. Chamizo and A. González-Arroyo.
Tachyonic instabilities in $2+1$ dimensional Yang-Mills and its connection to number theory results and conjectures.
(In preparation).

M. García Pérez, A. González-Arroyo, and M. Okawa.
Spatial volume dependence for $2+1$ dimensional $SU(N)$ Yang-Mills theory.
JHEP, 2013(9), 2013.

M. García Pérez, A. González-Arroyo, and M. Okawa.
Volume independence for Yang-Mills fields on the twisted torus.
Int. J. Mod. Phys. A, 29(1445001), 2014.

Bibliography II

T. Eguchi and H. Kawai.
Reduction of dynamical degrees of freedom in the large-N gauge theory.
Phys. Rev. Lett., 48:1063–1066, Apr 1982.

A. González-Arroyo and M. Okawa.
The twisted-Eguchi-Kawai model: A reduced model for large-N lattice gauge theory.
Phys. Rev. D, 27(10):2397, 1983.

A. González-Arroyo and M. Okawa.
A twisted model for large N lattice gauge theory.
Physics Letters B, 120:174–178, jan 1983.

Bibliography III

G. 't Hooft.
A planar diagram theory for strong interactions.
Nuclear Phys. B, 72:461–473, 1974.

G. 't Hooft.
A property of electric and magnetic flux in non-Abelian gauge theories.
Nuclear Phys. B, 153(1-2):141–160, 1979.

G. 't Hooft.
Large *N*.
arXiv:hep-th/0204069, 2002.

A. Zee.
Quantum field theory in a nutshell.
Princeton University Press, Princeton, NJ, 2003.