

Lattice point counting and harmonic analysis

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Abstract

We explain the application of harmonic analysis to count lattice points in large regions. We also present some of our recent results in the three-dimensional case.

1. Introduction

The problem of counting lattice points in enlarging regions has a long history in number theory that can be traced back to some problems considered by Gauss and Dirichlet. In general terms, given a region $\mathcal{B} \subset \mathbb{R}^d$ we consider the number of lattice points in large homothetic regions

$$\mathcal{N}(R) = \#\{\vec{n} \in \mathbb{Z}^d : \vec{n}/R \in \mathcal{B}\}.$$

We expect that this number is well approximated by the volume of $R\mathcal{B}$ and we want to bound the *lattice error term* (*lattice point discrepancy*)

$$E(R) = \mathcal{N}(R) - \text{Vol}(\mathcal{B})R^d.$$

Usually one looks for the *error exponent*

$$\theta_d = \inf\{\alpha : E(R) = O(R^\alpha)\}.$$

The meaning of “region” is not fixed here. The most natural setting is to consider \mathcal{B} to be a convex compact set such that its boundary is a smooth compact and connected $(d - 1)$ -submanifold of \mathbb{R}^d with positive curvature,

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but there are simple classical examples that do not match these requirements.

Sometimes lattice point problems appear naturally when averaging arithmetical functions. For instance the formulas

$$(1.1) \quad \sum_{d=1}^N h(-d) = \frac{\pi}{18\zeta(3)} N^{3/2} - \frac{3}{2\pi^2} N + O(N^{\alpha_1})$$

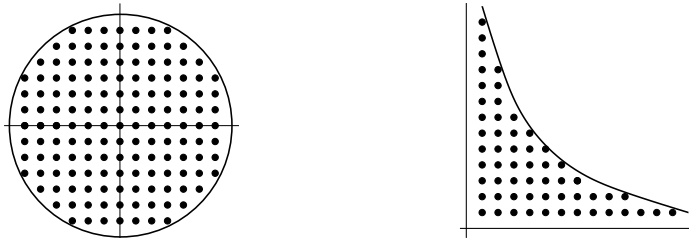
where $h(-d)$ is the class number (of binary quadratic forms of negative discriminant $-d$),

$$(1.2) \quad \sum_{n=1}^N r(n) = \pi N + O(N^{\alpha_2})$$

where $r(n) = \#\{(a, b) \in \mathbb{Z}^2 : n = a^2 + b^2\}$, and

$$(1.3) \quad \sum_{n=1}^N d(n) = N \log N + (2\gamma - 1)N + O(N^{\alpha_3})$$

where $d(n)$ is the number of divisors of n , lead to probably the oldest lattice point problems by chronological order. Clearly (1.2) and (1.3), the so-called *Gauss circle problem* (Gauss 1834) and *Dirichlet divisor problem* (Dirichlet 1849), reduce to count lattice points in a circle and under a hyperbola.



With respect to (1.1) (Gauss 1801), we postpone further comments to §6.2 and we only mention here that the discriminant of a reduced form $ax^2 + bxy + cy^2$ is given by $d = b^2 - 4ac$, and this can be used to interpret (1.1) as a lattice point count in some hyperboloid.

Denoting by β_i the greatest lower bound for the α_i 's such that these formulas are valid, the best known upper bounds are $\beta_1 \leq 21/32$ [6] (see also [7], [15]), and $\beta_2, \beta_3 \leq 131/416$ [19]. On the other hand the conjectured values are $\beta_1 = 1/2$ and $\beta_2 = \beta_3 = 1/4$ matching with the proved Ω -results [33] (see also [13]) and [32]. The best known general upper bound for θ_d in the many dimensional case (as formulated before) is due to W. Müller

[23] (see [25] for Ω -results). The problem in dimension $d \geq 4$ is completely solved for spheres and for rational ellipsoids of higher dimensions ($d \geq 9$) [4]. We address the reader to the survey [20] for the state of the art in the field and to the monographs [18] and [22] for more technical information.

2. Harmonics: the building blocks

The goal of harmonic analysis, at least the etymological goal, is to decompose (analyze) large function spaces in terms of simple functions called *harmonics*. The following table is a list containing some of the more often employed examples in number theory

Space	Harmonics
1-periodic, $L^2([0, 1])$	$e^{2\pi i n x}$ with $n \in \mathbb{Z}$
$L^2(\mathbb{R}^d)$	$e^{2\pi i \vec{\xi} \cdot \vec{x}}$ with $\vec{\xi} \in \mathbb{R}^d$
$f : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{C}$	$e^{2\pi i n x/m}$ with $n = 0, 1, \dots, m - 1$
$f : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}$	Dirichlet characters
$\mathrm{SL}_2(\mathbb{Z})$ -periodic, $L^2(\mathbb{H})$	Maass forms, Eisenstein series
$f : \mathbb{A} \rightarrow \mathbb{C}$, $\mathbb{A} = \text{adèles}$	$e^{2\pi i \lambda(\vec{\xi} \cdot \mathbf{x})}$, $\lambda(\mathbf{t}) = \sum \pm \mathrm{Tr}_{K_v/\mathbb{Q}_p}(\mathbf{t})$

The first row is the classical Fourier analysis for periodic functions. A smooth enough 1-periodic function $f : \mathbb{R} \rightarrow \mathbb{C}$ admits the Fourier expansion

$$f(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x} \quad \text{where} \quad a_m = \int_0^1 f(x) e^{-2\pi i m x} dx.$$

The second row is still classical Fourier analysis, in this case a rapidly decreasing function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is a continuous combination of the harmonics modulated by the *Fourier transform* \widehat{f} through the *inversion formula*

$$f(\vec{x}) = \int_{\mathbb{R}^d} \widehat{f}(\vec{\xi}) e^{2\pi i \vec{\xi} \cdot \vec{x}} d\vec{\xi} \quad \text{where} \quad \widehat{f}(\vec{\xi}) = \int_{\mathbb{R}^d} f(\vec{x}) e^{-2\pi i \vec{\xi} \cdot \vec{x}} d\vec{x}.$$

Let us see an example to peek at the connection with arithmetic. The Fourier expansion of the fractional part $\mathrm{Frac}(x) = x - [x]$ gives

$$\mathrm{Frac}(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{\pi m}$$

which is in fact only true for $x \notin \mathbb{Z}$ by regularity matters (note that it is not continuous at $x \in \mathbb{Z}$). The functions $\sin(2\pi m x)$ are simpler and nicer

than $\text{Frac}(x)$ from the analytic point of view. If we want to apply analytic operators, like averaging, this expansion benefits us. For instance, in the divisor problem one has to deal with $\sum_{n < N} \text{Frac}(N/n)$ and this is clearly an arithmetical problem, but in $\sum_{n < N} \sin(2\pi mN/n)$ it seems that there exists a chance to use analytic methods (and it is so). Of course identities are identities and in some sense the arithmetical complications of $\text{Frac}(x)$ are hidden in the infinite summation of the simple non-arithmetical harmonics, but analysis provides us with tools to mollify or cut this infinite summation paying a fair prize (see §4). In connection with this it is possible to prove

$$\text{Frac}(x) = \frac{1}{2} + \sum_{m < M} \frac{\sin(2\pi mx)}{\pi m} + O(\epsilon^{-1}M^{-1})$$

whenever the distance of x to the nearest integer is greater than ϵ .

3. Summation formulas

One of the most important tools in analytic number theory is the Poisson summation formula. It transforms sums in a non trivial way and we hope the transformed sum to be easier than the original one.

Its proof is simple and illustrative: If $f : \mathbb{R} \rightarrow \mathbb{C}$ is a smooth function with a good decay at infinity, then $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ makes sense in $L^2([0, 1])$ and it is 1-periodic. An easy computation proves that its n th-Fourier coefficient is $\widehat{f}(n)$ and Fourier expansion reads

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{m=-\infty}^{\infty} \widehat{f}(m)e^{2\pi imx}.$$

By aesthetic reasons (symmetry), usually one takes $x = 0$ and the result is the simplest form of the *Poisson summation formula*

$$(3.1) \quad \sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \widehat{f}(m).$$

With the same proof one deduces the variant

$$(3.2) \quad \sum_{\vec{n} \in \mathbb{Z}^d} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^d} \widehat{f}(\vec{m}).$$

The crux of the argument is that the one-dimensional torus \mathbb{T} (this is $[0, 1]$ gluing the boundary points) can be unwrapped on \mathbb{R} , in other words, we force the 1-periodicity of f summing it over all translations. In the same way

when one considers harmonic analysis on a locally compact abelian group G , for any discrete subgroup H one can define $F(x) = \sum_{h \in H} f(x + h)$ and repeat the argument. We shall not enter into details but we shall mention that this can be very deep, for instance, J. Tate found in his thesis a fruitful application to number theory when G is the adèle group [24].

One of the most celebrated examples of (3.1) is the modular relation for Jacobi θ -function

$$\theta(x) = x^{-1/2}\theta(1/x) \quad \text{where} \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}, \quad x > 0.$$

A calculation shows $\int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx = \pi^{-s/2} \Gamma(s/2) n^{-s}$, then

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{2} \int_0^{\infty} x^{s/2-1} \theta^*(x) dx \quad \text{with} \quad \theta^*(x) = \theta(x) - 1.$$

Dreaming that θ^* satisfies the same functional equation as θ , after the change of variables $x \mapsto 1/x$ one would obtain the functional equation for ζ

$$(3.3) \quad \Lambda(s) = \Lambda(1-s) \quad \text{where} \quad \Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

The actual proof is more involved but the ideas are the same. These three topics, summation formulas, modular forms and functional equations are related. One could prove for instance (3.1) or $\theta(x) = x^{-1/2}\theta(1/x)$ out of (3.3), although it is not the most economical way.

4. The uncertainty principle

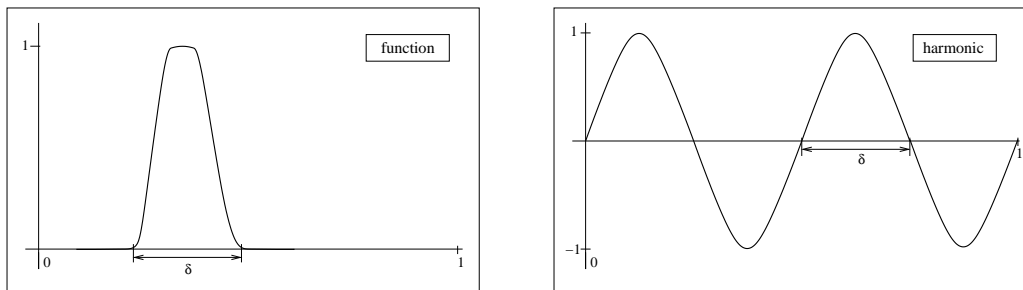
Thanks to the good expository abilities of our physicist colleagues, when we hear the words *uncertainty principle* we surely think of quantum mechanics, flying photons hitting electrons, non-deterministic paths, Copenhagen interpretation... Too difficult. Let us try to explain it geometrically.

A function and a point (physicists would say a wave and a particle) are quite different, but still a point can be represented as a very special kind of function: its characteristic function (physicists would prefer, no doubt, Dirac delta, that is in fact more convenient).

A point is too small to be detected by harmonic analysis and in fact in L^2 sense its characteristic function is identically zero. Let us think, instead, in a blur point, for instance the acoustic intensity of a rapid train passing at 100 mph in front of us, say that we represent it by a bump function and the width of the “effective” support is δ . We assume that the support is inside

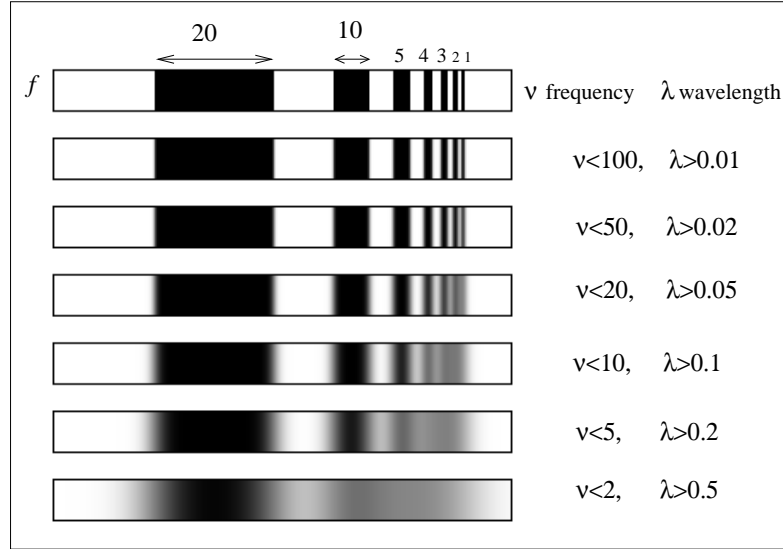
of $(0, 1)$ indeed, then we can use classical Fourier analysis for 1-periodic functions.

If we have a look to the shape of the graphics representing the harmonics we quickly realize that the bumps of width comparable to δ occur when the number of bumps (that is proportional to the frequency) is comparable to δ^{-1} . A bump function of width δ (as our blur point) cannot be well approximated by harmonics of frequency much less than δ^{-1} , in general harmonics with frequency less than δ^{-1} are not suitable to see details at scale less than δ . Imagine that the function has a certain small anomaly in some interval of length $\epsilon < \delta$ (for instance a discontinuity), then to study it we must use at least harmonics of frequencies less than something comparable to ϵ^{-1} .



You cannot use optical microscopes in crystallography because the wavelength of light ($4 \cdot 10^{-7}m - 7 \cdot 10^{-7}m$) skips the atoms, in fact is too coarse to penetrate the crystals (this is an oversimplification because there are absorption processes giving the colors or molecular structures producing the transparency), you need something like X-rays, with a tiny wavelength ($3 \cdot 10^{-11}m - 3 \cdot 10^{-9}m$) and still you will not “see” clearly the atoms (their “size” can be less than $1 \text{ \AA} = 10^{-10}m$) but they are good enough to observe some nice diffraction patterns showing bright points.

In the next figure there is a visual example. We represent a function as a strip of gray tones depending on the absolute value at each point (black = 1, white = 0). The first strip corresponds to f , a sum of thinner and thinner characteristic functions, and the rest of the strips are the approximations using harmonic analysis with a smooth filtering in the indicated range of frequencies (far beyond this range the cut of the frequencies is severe). Note that the initial sharp vertical bands become blur and eventually disappear, it is impossible to represent sharp edges with a short range of frequencies. We have lost localization and gained uncertainty.



Summarizing: with a range of frequencies like F your sight is limited to scale F^{-1} and similarly, if the threshold of your sight is X then you cannot get information about a range of frequencies larger than X^{-1} . In quantum mechanics the frequency is related to the momentum p and physicists write $\Delta x \cdot \Delta p \geq C$ and even say that $C = \hbar/2 = 5.27285 \cdot 10^{-35}$. This value just informs you that quantum effects only appear at submicroscopic scale, so small that you cannot usually distinguish the “real” blur particles of quantum mechanics from the nonexistent classical ones.

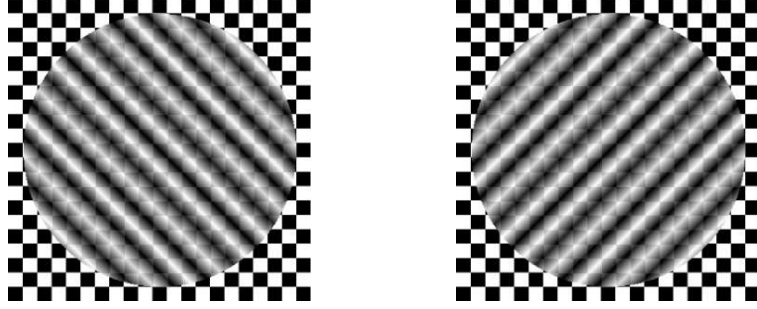
Is this relevant in real world? Absolutely yes. First of all quantum effects are employed nowadays in some real machines, noticeably on NMR (nuclear magnetic resonance) or in the scanning tunneling microscope. Secondly uncertainty principle is associated to harmonic analysis and this explains its ubiquity when analyzing signals. Therefore the Shannon sampling theorem, artifacts in low quality *jpeg* image files, anti-aliasing filters and Heisenberg inequality are relatives. In the last decades wavelets have flourished to push the balance between the localization in space and frequency [9] and they have real and conspicuous applications in engineering.

5. The basic lattice point analysis

Let us say that we want to study the circle problem (1.2), then we take \mathcal{B} as the unit disk in \mathbb{R}^2 and $\mathcal{N}(R)$ counts the number of integral solutions of $x^2 + y^2 \leq R^2$ (here $R = \sqrt{N}$). If f equals the characteristic function of $R\mathcal{B}$,

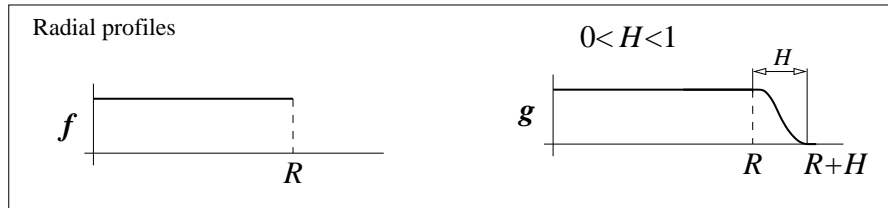
$$\mathcal{N}(R) = \sum_{\vec{n} \in \mathbb{Z}^2} f(\vec{n})$$

and it is tempting to apply Poisson summation formula (3.2). The Fourier transform of f is a radial function. Geometrically this follows because the value of the integral of f against a planar wave does not depend on the direction of the wave



Hence $\widehat{f}(\vec{m}) = \widehat{f}(\|\vec{m}\|, 0)$ and this latter can be expressed in terms of a Bessel function giving $\widehat{f}(\vec{m}) = O(R^{1/2}\|\vec{m}\|^{-3/2})$. Intuitively, if we divide the plane in squares of length $\|\vec{m}\|^{-1}$, then the integral of $f(x, y)e^{-2\pi i\|\vec{m}\|x}$ is zero on each of them except in the $O(R\|\vec{m}\|)$ squares on the boundary, in which the value oscillates between $-\|\vec{m}\|^{-2}$ and $\|\vec{m}\|^{-2}$. Assuming that it behaves as a random variable, central limit theorem would give $\widehat{f}(\vec{m}) = O((R\|\vec{m}\|)^{1/2}\|\vec{m}\|^{-2}) = O(R^{1/2}\|\vec{m}\|^{-3/2})$, the same bound as before without appealing to special functions (of course this is not a rigorous proof).

Note that $\sum_{\vec{m} \in \mathbb{Z}^2 - \{\vec{0}\}} \|\vec{m}\|^{-3/2} = \infty$, so that this approach is useless with this f . A harmonic analyst would explain us that it is a consequence of the low regularity of f . Let choose a new function g , a smoothed version of f . If $g \in C_0^\infty$ then $\widehat{g}(\vec{x}) = O(\|\vec{x}\|^{-N})$ holds for any N and convergence is assured.



We want to keep g close to f , say that they only differ in the annulus $R < r < R + H$ as represented above. We need H to be small because the new volume term $\widehat{g}(\vec{0}) = \pi(R + O(H))^2$ should be very close to the previous one $\widehat{f}(\vec{0}) = \pi R^2$.

What is the influence of the smoothing on the Fourier transform side? The uncertainty principle says that harmonic analysis does not see the smoothing of size H before the frequency range reaches H^{-1} then we are

forced to keep this part of the previous analysis. On the other hand regularization will take care of the high frequencies. Roughly speaking we get

$$\mathcal{N}(R) = \pi(R + O(H))^2 + O(R^{1/2} \sum_{\|\vec{m}\| < H^{-1}} \|\vec{m}\|^{-3/2}).$$

If H is very small the sum is large and if H is large the main term is affected. The balance is reached for $H = R^{-1/3}$ giving

$$\mathcal{N}(R) = \pi R^2 + O(R^{2/3}).$$

This is (1.2) for $\alpha_2 = 1/3$.

The advantage of having a radial Fourier transform is that Bessel functions appear. This becomes a technical minor point because good approximations for Fourier transforms of characteristic functions of smooth convex bodies are known ([17], [16]). In particular it is known that for \mathcal{B} as in §1, the characteristic function f of $R\mathcal{B}$ satisfies $\widehat{f}(\vec{x}) = O(R^{(d-1)/2} \|\vec{x}\|^{-(d+1)/2})$. The previous arguments give

$$\mathcal{N}(R) = \text{Vol}(\mathcal{B}) + O(R^{(d-1)d/(d+1)})$$

and hence $\theta_d \leq (d-1)d/(d+1)$ [17]. The expected values are $\theta_2 = 1/2$ and $\theta_d = d-2$ for $d \geq 3$.

5.1. The method of exponential sums

In the previous analysis we have disregarded the influence of the sign of $\widehat{f}(\vec{m})$. After dividing into dyadic intervals in one of the variables and extracting the amplitude by partial summation, the basic problem is to beat the trivial bound for an exponential sum

$$S = \sum_{N \leq n < M} e(f(n)) \quad \text{for} \quad M \leq 2N,$$

where typically the derivative of f satisfies $c_1 N^\gamma < f' < c_2 N^\gamma$ for some $\gamma > 0$. One looks for a κ -saving with respect to the trivial estimate, this means

$$S = O(N^{1-\kappa}).$$

Ideally one conjectures that $e(f(n))$ behave as identically distributed independent random variables and that something close to the 1/2-saving is reachable, but the actual proved values of κ are commonly far from it. For instance, $\kappa = (1 - \gamma)/2$ is true with some extra requirements on f , this is

worse than trivial for $\gamma > 1$ and several methods have been devised to treat this case. One of the highlights of the theory is the method of exponent pairs [12] based on the work of J.G. van der Corput and formulated in an operative and systematic form by E. Phillips [27] in 1933.

In some other applications (for instance in the prime number theorem) extra large oscillations appear, and the best methods are due to I.M. Vinogradov [35].

It is natural to think that we should double our winnings when two variables participate in exponential sums and in general to get more cancellation in many dimensional sums but so far there is not a clear theory to quantify this idea.

6. Recent research

In the following subsections we consider four lattice point problems in which harmonic analysis is complemented with arithmetical ideas to improve previous known results. They are related to our recent research: the first and the last are works in progress while the second and the third are [7] and [5].

6.1. Points in the sphere and L -functions

According to the basic analysis studied in §5 we can prove the error exponent $\theta_3 \leq 3/2$, that corresponds to $H = R^{-1/2}$. To improve this result for the sphere we examine the corresponding exponential sum. Indeed we have to control a sum like

$$S_M = \sum_{\|\vec{m}\| < M} e(R\|\vec{m}\|) = \sum_{m_1^2 + m_2^2 + m_3^2 < M} e(R\sqrt{m_1^2 + m_2^2 + m_3^2}).$$

Conjecturally one expects $S_M = O(M^{3/4+\epsilon})$ in natural ranges due to an $1/2$ -saving on each variable but this is out of reach with current technology (it would prove the conjecture $\theta_3 = 1$). If we “glue” two variables into one, $n = m_1^2 + m_2^2$, and write

$$S_M = \sum_{n < M} r(n) \sum_{m_3^2 < M-n} e(R\sqrt{n + m_3^2}),$$

we see that S_M is a kind of long average of exponential sums. It turns out that in this case one can prove $1/2$ -saving on average (essentially this is the approach in [8] and [34]) giving in some ranges the bound

$$(6.1) \quad S_M = O(M \cdot (M^{1/2})^{1/2+\epsilon}) = O(M^{5/4+\epsilon}).$$

Introducing it in the basic analysis, one deduces $\theta_3 \leq 4/3$ for the sphere choosing optimally $H = R^{-2/3}$.

Any improvement on (6.1) seems unrealistic without assuming big advances in the method of exponential sums. In other words, uncertainty principle establishes a strong barrier that limits our vision to variations of the radius like $H = R^{-2/3}$, then an error term like $\#\{\vec{m} \in \mathbb{Z}^3 : R < \|\vec{m}\| < R + R^{-2/3}\}$, which should be like $R^{4/3}$, is unavoidable if we use Fourier analysis.

How can we defeat the uncertainty principle? In [11] Gauss proved a deep result that, after the class number formula, allow to express the number of representation as a sum of three squares in terms of real Dirichlet characters. New harmonics allow us new techniques and new uncertainty ranges. Without entering into details, the character sum approach is advantageous in short distances and can be combined with Fourier analysis to break the uncertainty barrier. The best known result obtained by this method is $\theta_3 \leq 21/16$ due to D.R. Heath-Brown [15].

In a work in progress (it will be a part of the Ph.D. thesis of E. Cristóbal) we try to exploit in this setting the conjectural properties of Dirichlet L -functions

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

associated to real characters χ . Complex analysis establishes a relation between the location of the zeros of $L(s, \chi)$ and its growth (think for instance of Jensen formula as a model). On the other hand if the growth of $L(s, \chi)$ on vertical lines is under control then some cancellation in character sums $\sum_m \chi(m)$ should appear. Introducing this idea in the previous scheme one can prove that if the distance from the line $\Re s = 1$ to the zeros of $L(s, \chi)$ is greater than 0.08, then the best known bound for θ_3 in the sphere can be improved. Note that under Generalized Riemann Hypothesis the distance is $1/2$. We can read this kind of result as a link between counting primes in arithmetic progressions and lattice points in the sphere.

An insidious difficulty with this approach is that (6.1) is only known in some ranges that must be extended, with involved techniques, to match the ranges associated to character sum estimates.

6.2. Class number average

The origin of the problem (1.1) is Art.302 of [11] in which one can read:

we have found by a theoretical investigation that the average number of classes around the determinant $-D$ can be expressed approximately by $\gamma\sqrt{D} - \delta$ where $\gamma = 0.7467183115 = 2\pi/7e$ where e is the sum of the series $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \text{etc.}$ $\delta = 0.2026423673 = 2/\pi^2$.

In modern notation this can be written as

$$\sum_{d=1}^N h(-4d) \sim \frac{4\pi}{21\zeta(3)} N^{3/2} - \frac{2}{\pi^2} N.$$

(Note that $\sum_{D=1}^N (\gamma\sqrt{D} - \delta) \sim 2\gamma N^{3/2}/3 - \delta N$). Gauss only considered quadratic forms of the type $ax^2 + 2bxy + cy^2$, so that for him the discriminant was always a multiple of 4, but nowadays we also consider odd middle coefficients and the sum in (1.1) becomes more natural.

Where is the lattice point problem here? Gauss proved that each class of (primitive) quadratic forms of discriminant $-d < 0$ is represented by a reduced form $ax^2 + bxy + cy^2$ satisfying

$$\gcd(a, b, c) = 1, \quad 4ac - b^2 = d \quad \text{and} \quad -a < b \leq a < c \quad \text{or} \quad 0 \leq b \leq a = c.$$

In modern language we assign to each primitive quadratic form $ax^2 + bxy + cy^2$ of discriminant $-d < 0$ the complex number $(-b + i\sqrt{d})/2a$ in the upper half-plane \mathbb{H} and the action of $\text{PSL}_2(\mathbb{Z})$ on these complex numbers is equivalent to the action of unimodular matrices on quadratic forms. Reduced forms correspond to points in the standard fundamental domain.

Gauss also studied the average of class number for positive discriminants multiplied by the logarithm of the fundamental unit. In Art.304 of [11], he writes:

the average value of that product is approximately expressed by a formula like $m\sqrt{D} - n$. However, we have not this far been able to determine the values of the constant quantities m, n theoretically.

We can guess that Gauss did not succeed in this case because this average resists a simple interpretation as a lattice point problem. In 1944 C.L. Siegel [31] interpreted the first main term in Gauss' assertion expressing $h(d) \log \epsilon_d$ as a weighted sum of the number of geodesics in \mathbb{H} (endowed with Poincaré's metric) crossing the fundamental domain and having $(-b \pm \sqrt{d})/2a$ (with $d = b^2 - 4ac$) as end-points. Identifying the geodesics with their end-points we recover a lattice point interpretation.

If we try to parallel the scheme applied for the sphere we shall find strong difficulties to use Fourier analysis because the regions are complicated: The lack of regularity and curvature of the boundary causes serious problems in the application of the Poisson summation formula.

One elegant solution is to employ the relation between functional equations and summation formulas. In 1975 T. Shintani [29] proved a vector functional equation involving $\sum h(-d)d^{-s}$ and $\sum h(d)\log \epsilon_{4d}d^{-s}$. One of the consequences obtained by Shintani is that Gauss' claim is not completely correct. The corresponding summation formulas and the analysis of the associated lattice point problem is worked out in [7] and the method described in the previous section for the sphere is adapted to get

$$\sum_{d=1}^N h(4d)\log \epsilon_{4d} = \frac{4\pi^2}{21\zeta(3)}N^{3/2} - \frac{4}{\pi^2}(C + \log N)N + O(N^\alpha)$$

for any $\alpha > 21/32$, where $C = \log(2\pi) + 8(\log 2)/3 - \zeta'(2)/\zeta(2) - 1$.

Gauss gave in the notes to [11] the right coefficient of $N^{3/2}$ but his experimental computation led him to confuse $\log N$ with a constant.

6.3. Visible points

We say that a point $P \in \mathbb{Z}^2 - \{(0, 0)\}$ is *visible* (from the origin) if $O = (0, 0)$ and P are the only lattice points on the segment \overline{OP} . It is straightforward to show that $P = (n, m)$ is visible if and only if n and m are coprime. The analog of circle problem for visible points (counting visible points in large circles) is a hard problem and usually the Riemann Hypothesis is assumed to get nontrivial error exponents [26]. By the way, there is a nice variant for thick points due to Pólya, named the *orchard problem*, that in [28] p.150 is stated as: "How thick must the trunks of the trees in a regularly spaced circular orchard grow if they are to block completely the view from the center?". The elementary solution probably will challenge the reader (see [1]).

The visible lattice point problems in higher dimensions does not add anything new to the usual ones. For instance if the number of lattice points in the sphere is given by $4\pi R^3/3 + O(R^\alpha)$ then the number of visible points is $4\pi R^3/3\zeta(3) + O(R^\alpha)$ and vice versa, assuming in both cases $\alpha > 1$. The assumption is known to be necessarily true and leads to study Ω -results for the lattice error term, a topic having a prominent place in the theory.

We say that $f = \Omega(g)$ (and refer to it as an Ω -result) if $f = o(g)$ is not true. Intuitively this means that it is impossible to improve the bound $O(g)$ if it holds. Integration is usually an easy method to prove

Ω -results for lattice point problems. For instance, studying the phase of the Fourier transform in our basic analysis of the circle problem in §5, one can prove that the error term $E(R)$ is closely related to the real part of $cR^{1/4} \sum_{\|\vec{m}\| < R} \|\vec{m}\|^{-3/2} e^{2\pi i R \|\vec{m}\|}$ for a non-zero (complex) constant c (H is taken to be R^{-1} to reduce to $O(1)$ the uncertainty in the main term). Squaring and integrating, the non-diagonal terms are proved to be negligible (integration kills oscillatory terms) and it follows

$$\int_1^X |E(R)|^2 dR \sim CX^{3/2}$$

for a certain explicit positive constant C . This gives immediately $E(R) = \Omega(R^{1/4})$.

In higher dimensions more variables participate in the summation giving a huge number of non-diagonal terms. For the sphere one can still manage them [21] (winning just a power of logarithm over the diagonal contribution). For visible points in the sphere a new summation appears involving the μ function, in this case the error term $\tilde{E}(R)$ is related to the real part of

$$cR \sum_{d \leq R} \sum_{\|\vec{m}\| < M} \mu(d) \frac{e^{2\pi i R \|\vec{m}\|}}{d \|\vec{m}\|^2}$$

where for technical reasons M is chosen slightly less than R . A direct approximation to the second power moment is not possible, in part because of the unpredictable sign changes of $\mu(d)$.

In [5] we have overcome the problem introducing the auxiliary function $g(R) = \sum_{n < M^2} n^{-1/2} \cos(2\pi R \sqrt{n})$ that we expect to resonate with $\tilde{E}(R)$ because in the product $\tilde{E}(R)g(R)$ we find terms of the form (write $k = \|\vec{m}\|^2$)

$$R \sum_k \sum_{d^2 | k} \mu(d) \frac{r_3(k)}{k^{3/2}}$$

and this is non-oscillatory and large because of the formulas

$$\sum_{d^2 | k} \mu(d) = \mu^2(k) \quad \text{and} \quad \sum_{k < x} \mu^2(k) \frac{r_3(k)}{k^{3/2}} \sim \frac{14}{\pi} \log x.$$

The positivity is crucial. The analytic resonance has been deduced from the simple arithmetic relation between μ and μ^2 .

The rest of the terms are controlled after integration with a detailed analysis. In this way we can conclude that

$$\int_R^{2R} \tilde{E}(R)g(R) dR = \Omega(R^2 \log R).$$

As g is $\sqrt{\log \bar{R}}$ on (L^2) -average, it follows that $\tilde{E}(R) = \Omega(R\sqrt{\log \bar{R}})$.

6.4. Rational ellipsoids

We have seen that the lattice point problem associated to the sphere admits a special treatment. The relation between $r_3(n)$ and the class number allows to employ successfully multiplicative harmonics (character sums) in a thin layer when additive harmonics (exponential sums) are useless by the uncertainty principle.

In a work in progress we extend the known bound for the error exponent of the sphere [15] to rational ellipsoids: $\mathcal{B} = \{\vec{x} \in \mathbb{R}^3 : \vec{x}^t A \vec{x} \leq 1\}$ where $A \in \text{GL}_3(\mathbb{Q})$ is a positive definite matrix.

An interesting aspect of our approach is that some properties of modular forms appear intrinsically in the proof. To our knowledge this is a novelty in classical lattice point problems.

One can assume that A is an integral matrix because θ_d is trivially invariant by homothecies of \mathcal{B} , then $Q(\vec{x}) = \vec{x}^t A \vec{x}$ is a positive integral ternary form and the problem boils down to estimate very sharply short sums of $r_Q(n) = \#\{\vec{x} \in \mathbb{Z}^3 : Q(\vec{x}) = n\}$.

If, instead of $r_Q(n)$, we consider the number of representations of n by the genus of Q , the Siegel mass formula [30] allows to express it as a product of local (p -adic) factors that is susceptible to be treated by multiplicative analysis. The computation of the product of these local factors is a difficult task (A. Arenas got in her Ph.D. thesis some explicit formulas in connection with an arithmetical problem [2], [3]) but fortunately we do not need complete evaluations. Having in mind Dirichlet L -functions we change the usual terminology and name this number $\mathcal{L}(n)$.

For quadratic forms in many variables $\mathcal{L}(n)$ is a good approximation of $r_Q(n)$ but the ternary case is somewhat exceptional and W. Duke and R. Schulze-Pillot have studied it showing a more involved situation [10] (see also [14]). The key observation is that $r_Q(n)$ and $\mathcal{L}(n)$ are Fourier coefficients of modular forms of weight $3/2$ with the same behavior at the cusps, then

$$r_Q(n) = \mathcal{L}(n) + a_n$$

where $\sum a_n e^{2\pi i n z}$ is a cusp form of weight $3/2$. For individual n 's, a_n could exceed $\mathcal{L}(n)$, but using the best known bounds for coefficients of modular forms of half-integral weight and the structure of the Shimura lift, we can keep under control of its contribution to short sums of $r_Q(n)$.

In some sense our approach is based on the combination of three kinds of analysis: the bulk of the lattice points are counted with additive harmonics and for the rest of them (those in a thin layer) we use multiplicative harmonics.

ics and coefficients of modular forms, whose bounds come from Kloosterman sums that are the harmonics in the double coset decomposition of $\mathrm{PSL}_2(\mathbb{Z})$.

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