Crazy inverses

Introduction. When we were kids, we learned how to expand $(a+b)^2$ or $(a+b)^3$ into finite terms, but nobody taught us something similar for $(a+b)^{-1}$. If it does not exist for real numbers, you can hardly expect something for matrices. But there are some crazy formulas for the inverse of a sum of matrices if one of them has rank 1. For instance, if A is invertible and $\sigma \neq -1$ is the sum of the elements of A^{-1} we have $(A+\mathbf{1})^{-1} = A^{-1} - A^{-1}\mathbf{1}A^{-1}/(1+\sigma)$ with $\mathbf{1}$ the square matrix with all its entries 1. In general, we have the *Sherman-Morrison identity* saying that for c a column matrix and r a row matrix, with the same dimension as an invertible matrix B, we have that $B+cr$ is invertible when $\tau := rB^{-1}c \neq -1$ and its inverse equals $B^{-1} - B^{-1}crB^{-1}/(1+\tau)$.

Note that any matrix of rank one can be written as cr and that if c and r have all their coordinates 1, we recover the first result.

The proof of $(A+\mathbf{1})^{-1} = A^{-1} - A^{-1}\mathbf{1}A^{-1}/(1+\sigma)$ is as follows:

It is plain to check that $\mathbf{1}B = \mathbf{1}D(B)$ where $D(B)$ is the diagonal matrix with the jj entry being the sum of the j-th column of B. Expanding the product,

$$(A+\mathbf{1})\Big(A^{-1} - \frac{A^{-1}\mathbf{1}A^{-1}}{1+\sigma}\Big) = I + \frac{\sigma}{1+\sigma}\mathbf{1}A^{-1} - \frac{1}{1+\sigma}\mathbf{1}A^{-1}\mathbf{1}A^{-1} = I + \frac{\sigma\mathbf{1}A^{-1} - \mathbf{1}D(A^{-1}\mathbf{1})A^{-1}}{1+\sigma}.$$

The entry ij of $A^{-1}\mathbf{1}$ is the sum of the i-th row of A^{-1}, then $D(A^{-1}\mathbf{1}) = \sigma I$ and $\mathbf{1}D(A^{-1}\mathbf{1})A^{-1} = \sigma\mathbf{1}A^{-1}$. This shows that the previous expression is the identity, as expected.

To deduce the Sherman-Morrison identity, take U and V invertible such that $c = U1_c$ and $r = 1_rV$, where 1_c and 1_r are the column and row matrices having all the coordinates one. Note that $1_c1_r = \mathbf{1}$, then

$$(B+cr)^{-1} = (B + U1_c1_rV)^{-1} = \big(UU^{-1}BV^{-1}V + U\mathbf{1}V\big)^{-1} = V^{-1}\big(U^{-1}BV^{-1} + \mathbf{1}\big)^{-1}U^{-1}.$$

Taking $A = U^{-1}BV^{-1}$ in the formula for $(A+\mathbf{1})^{-1}$ we get the result with τ the sum of the entries of $VB^{-1}U$ and it is enough to note that this sum is $1_rVB^{-1}U1_c = rB^{-1}c$.

AIR MAIL