A functional equation and some series evaluations

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December 26, 2021

Abstract

We study a functional equation closely related to the work of Ramanujan, its interpretation in terms of continued fractions and its application to evaluate some series. This is an expository paper, the presentation is original but the results have been appeared elsewhere in an equivalent form.

1 Introduction

In his celebrated first letter to Hardy [4], Ramanujan claimed

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7}=\frac{19\pi^7}{56700}. \tag{1.1}$$

This and other related evaluations appear in his notebooks [2, 293–299] as well as general summation formulas [3, 411–424]. They can be taught to undergraduate students as applications of the residue theorem. Without entering into details, let us generalize (1.1) with an argument of this kind. For $m\in\mathbb{Z}^+$, the function $f(z)=\pi z^{1-4m}\coth(\pi z)\cot(\pi z)$ clearly has poles at $z=n$ and $z=ni$ with $n\in\mathbb{Z}$. For $n\neq 0$ they are simple and

$$\operatorname{Res}(f,n)=\operatorname{Res}(f,ni)=n^{1-4m}\coth(\pi n).$$

Then by the residue theorem applied in a large centered square

$$4\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{4m-1}}=-\operatorname{Res}(f,0).$$

The formula (1.1) corresponds to the case $m=2$. The calculation of $\operatorname{Res}(f,0)$ may require a great computational effort when m grows (for instance, for $m=5$ we obtain a denominator of 17 digits) but, conceptually, it is elementary. This proof can be found in [16] and, according to the comments in [2, p. 293], already Cauchy gave hints to get sums like this. In [14] there is a less computationally demanding proof of (1.1) assuming the partial fraction expansion of $\coth(x)$ and some special values of the Riemann zeta function.

It may sound strange that one of the most conspicuous "sum evaluators" in the history of Mathematics considered noticeable sums which are somewhat standard. Arguably, the explanation relies on the unusual mathematical education (or the lack of academic education beyond high school) of Ramanujan. In the book [7], which was very influential to him, some typical complex variable arguments are replaced by real variable arguments (see for instance the treatment of the Γ function there). According to Hardy [10], even in Cambridge, Ramanujan did a scarce use of Cauchy's integral formula (cf. [1, p. 296]). In opposition to these considerations, he mastered the theory of elliptic and modular functions which requires a highly non-trivial knowledge of complex analysis. It is clear that he knew part of the classic theory before visiting Cambridge. We can only conjecture how much he knew about it and what books were his primary sources (see [4] and [5] for more information).

2 The functional equation

Consider the function

$$F(z) = \sum_{n=1}^{\infty} \frac{\cot(\pi n z)}{n^{2m+1}} \qquad \text{with} \quad \Im(z) > 0 \text{ and } m \in \mathbb{Z}^+.$$

It is plain that it converges absolutely and $F(z) = F(z+1)$. It turns out that it also behaves well with respect to the other standard generator of the modular group, the inversion. Namely, there is a functional equation relating $F(z)$ and $F(-1/z)$. It is closely connected, in fact equivalent, to a famous identity involving $\zeta(2n+1)$ due to Ramanujan [2, p. 276], [6, §3].

Proposition 2.1. *Let F be as before. We have*

$$F(z) = z^{2m} F(-1/z) + G(z) \qquad \textit{for} \quad \Im(z) > 0$$

where

$$G(z) = (-1)^m (2\pi)^{2m+1} z^{-1} \sum_{n=0}^{m+1} f_n f_{m+1-n} z^{2n}$$

with $f_n = B_{2n}/(2n)!$ and B_k the Bernoulli numbers.

Taking $z = i$, $m = 3$ and looking up in a table the first even indexed Bernoulli numbers $\{B_{2k}\}_{k=0}^{4} = \{1, 1/6, -1/30, 1/42, -1/30\}$, we obtain (1.1).

A reader not acquainted with the Bernoulli numbers could find more informative that $(-1)^n f_n$ are the Taylor coefficients of an elementary function [9, 1.411]. Namely, we have in a neighborhood of the origin

$$\frac{t}{2}\cot\frac{t}{2} = \sum_{n=0}^{\infty}(-1)^n f_n t^{2n}. \tag{2.1}$$

The proof of Proposition 2.1 is an application of the residue theorem in the same lines as the previous proof of (1.1). This result has been reproved by several authors since the beginning of the 20th century to the present.

Proof. For z fixed in the upper half complex plane, consider the meromorphic function

$$f(w) = -\pi z^{-m} w^{-1-2m} \cot(\pi w) \cot(\pi z w).$$

It has poles at $w = n$ and $w = n/z$ for $n \in \mathbb{Z}$. If $n \neq 0$ the poles are simple and we have

$$\operatorname{Res}(f, n) = -\frac{\cot(\pi z n)}{z^m n^{2m+1}} \qquad \text{and} \qquad \operatorname{Res}(f, n/z) = \frac{z^m \cot(-\pi n/z)}{n^{2m+1}}.$$

Both formulas are invariant by $n \mapsto -n$, then the sum of the residues is

$$2z^m F(-1/z) - 2z^{-m} F(z) + \operatorname{Res}(f, 0).$$

By the residue theorem applied to a family of enlarging regions avoiding the poles on the boundary, for instance centered parallelograms with sides parallel to 1 and z, this sum must vanish and it remains to prove $\operatorname{Res}(f, 0) = 2z^{-m} G(z)$. Using (2.1) we have in a neighborhood of the origin

$$f(w) = -\frac{\pi}{z^m w^{2m+1}} \cdot \frac{1}{\pi w} \sum_{k=0}^{\infty} (-1)^k f_k (2\pi w)^{2k} \cdot \frac{1}{\pi z w} \sum_{\ell=0}^{\infty} (-1)^\ell f_\ell (2\pi z w)^{2\ell}.$$

Selecting the coefficient of w^{-1}, which corresponds to $k + \ell = m + 1$, we get the expected equality for $\operatorname{Res}(f, 0)$. $\square$

With the following short `sagemath` code we get, for a given m, the polynomial corresponding to the sum in Proposition 2.1 with the normalizing factor $(2m + 4)!$ to reduce the size of the denominators:

```
z = var('z')
f = [bernoulli(2*k)/factorial(2*k) for k in srange(m+2)]
P = sum([f[k]*f[m+1-k]*z^(2*k) for k in srange(m+2)])
print(P*factorial(2*m+4))
```

Running it for $1 \le m \le 6$ we get the table:

m	$(2m+4)! \sum_{n=0}^{m+1} f_n f_{m+1-n} z^{2n}$
1	$-z^4 + 5z^2 - 1$
2	$\frac{4}{3}z^6 - \frac{14}{3}z^4 - \frac{14}{3}z^2 + \frac{4}{3}$
3	$-3z^8 + 10z^6 + 7z^4 + 10z^2 - 3$
4	$10z^{10} - 33z^8 - 22z^6 - 22z^4 - 33z^2 + 10$
5	$-\frac{691}{15}z^{12} + \frac{455}{3}z^{10} + \frac{1001}{10}z^8 + \frac{286}{3}z^6 + \frac{1001}{10}z^4 + \frac{455}{3}z^2 - \frac{691}{15}$
6	$280z^{14} - \frac{2764}{3}z^{12} - \frac{1820}{3}z^{10} - 572z^8 - 572z^6 - \frac{1820}{3}z^4 - \frac{2764}{3}z^2 + 280$

For the connoisseurs, these polynomials measure, in some sense, how much F differs from being a modular form of weight $-2m$. This may sound very artificial but Siegel's celebrated short proof of the modular relation for the Dedekind η-function [15] is formally a variant of the previous proof for $m = -1/4$.

3 The real case and continued fractions

Plainly $F(x)$ is not well defined for $x \in \mathbb{Q}$ and it is apparent that the convergence for other real values is an issue linked to Diophantine properties. For instance, if x is a Liouville number there exist infinitely many values of n such that nx differs from the nearest integer less than an arbitrary negative power of n. Hence the series defining $F(x)$ diverges regardless the value of m.

Some of the main results in [13] and [8] tackle the problem of the convergence for real values in a somewhat more general situation providing a complete answer in terms of their continued fractions. We recall briefly here the standard modern notation (see for instance [12]). Any real irrational value x admits a unique continued fraction expansion. If a_j are the partial quotients, we write

$$x = [a_0, a_1, a_2, a_3, \dots].$$

The convergents of x are the irreducible fractions

$$\frac{p_j}{q_j} = [a_0, \dots, a_j] \qquad \text{with} \quad j \ge 0.$$

A basic result in the theory is that for $j \in \mathbb{Z}^+$

$$(3.1) \qquad |q_j x - p_j|^{-1} = \alpha_{j+1} q_j + q_{j-1} \qquad \text{with} \quad \alpha_{j+1} = [a_{j+1}, a_{j+2}, \dots].$$

As p_j and q_j grow exponentially, it proves that the convergents of x actually converge very quickly to x.

To introduce our result it is convenient to introduce the partial products of the sequence $\{\alpha_j\}_{j=1}^\infty$, namely

$$A_0 = 1 \qquad \text{and} \qquad A_j = \prod_{k=1}^{j} \alpha_k \quad \text{for } j > 0.$$

Coming back to the convergence of F, if the sequence $\{q_j\}_{j=0}^\infty$ has a moderate growth, then $F(x)$ converges, in fact it is possible to obtain a complete characterization of the convergence. This is done in [13] and [8] in a broader context. It is also possible to save the functional equation for real values. The following result summarizes all of this information.

Theorem 3.1 ([13, Th. 1.1], [8, Cor. 3.3], [8, Prop. 4.3])**.** *For $x \in \mathbb{R}$ the series $F(x)$ converges if and only if $\sum_{j=0}^\infty (-1)^j q_j^{-2m-1} q_{j+1}$ converges. In this case, $F(-1/x)$ also converges and the functional equation of Proposition 2.1 holds for $z = x$.*

We take this result as granted and, once the convergence is assured, we will derive some explicit evaluations from the functional equation.

It is well known [12] that the real irrational quadratic numbers have periodic continued fractions (Legendre theorem) and the convergence condition becomes automatic by the general bound $q_{j+1}/q_j < 1 + a_{j+1}$. With the language of modular forms, the real quadratic irrationals are fixed points of hyperbolic elements and the quasi-modular relation given by the functional equation allows a complete evaluation.

Let us work out the case of pure periodic continued fractions which corresponds to $x \in \mathbb{R}_{>1}$ irrational quadratic with real conjugate in $(-1, 0)$. The rest of the quadratic cases reduce to it. The following result is included in an equivalent form in [11] but the convergence is not studied there. It could also be derived with some effort from [8, Th. 1.3].

Proposition 3.2. *If the partial quotients in the continued fraction of x satisfy $a_j = a_{j+n}$ for certain $n \in \mathbb{Z}^+$ and every $j \geq 0$, then*

$$F(x) = \frac{1}{1 - B_n} \sum_{j=0}^{n-1} B_j G\big(\alpha_{j+1}^{-1}\big) \qquad \textit{with} \quad B_j = \frac{(-1)^j}{A_j^{2m}}.$$

Proof. Taking in the functional equation $z = 1/\alpha_1$, which is allowed by Theorem 3.1, and recalling $A_0 = 1$, we have

$$F(x) = F(\alpha_0) = F(\alpha_1^{-1}) = -\alpha_1^{-2m} F(\alpha_1) + A_0^{-2m} G\big(\alpha_1^{-1}\big).$$

With successive application of the functional equation with $x = \alpha_j^{-1}$ it is proved by induction

$$F(x) = (-1)^n A_n^{-2m} F(\alpha_n) + \sum_{j=0}^{n-1} (-1)^j A_j^{-2m} G\big(\alpha_{j+1}^{-1}\big).$$

We have $F(\alpha_n) = F(\alpha_0) = F(x)$ since $a_j = a_{j+n}$ and the result follows eliminating $F(x)$. $\square$

Clearly $F(\lfloor\sqrt{k}\rfloor + \sqrt{k}) = F(\sqrt{k})$ and, by the aforementioned criterion, the first argument has a pure periodic continued fraction. Using this with $k = 2$ for $m = 1$ and $m = 2$ we get the evaluations

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{2})}{n^3} = \frac{\pi^3\sqrt{2}}{360} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{2})}{n^5} = \frac{\pi^5\sqrt{2}}{1890}.$$

In general, for a 1-period continued fraction expansion $[\ell, \ell, \ell \dots]$, $\ell \in \mathbb{Z}^+$, Proposition 3.2 gives

$$F\Big(\frac{\ell + \sqrt{\ell^2+4}}{2}\Big) = \frac{(\ell + \sqrt{\ell^2+4})^{2m}}{(\ell + \sqrt{\ell^2+4})^{2m} + 2^{2m}} G\Big(\frac{\sqrt{\ell^2+4} - \ell}{2}\Big).$$

The calculations with $k = 14$ are more demanding because $3 + \sqrt{14}$ has period four. In this case we get

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{14})}{n^3} = -\frac{23\pi^3\sqrt{14}}{2520} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{14})}{n^5} = -\frac{11\pi^5\sqrt{14}}{13230}.$$

The longest period for $k < 100$ happens for $k = 94$ and the length is 16. Then the evaluation

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\sqrt{94})}{n^7} = -\frac{2396429986305621361\pi^7\sqrt{94}}{1002303110932090982658000},$$

which is the analog of (1.1) for $k = 94$, definitively requires non human help for the calculations.

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