Demonſtratio Gemina
THEOREMATIS NEUTONIANI
quo traditur relatio inter coëfficientes cujusvis æquationis algebraicæ & ſummas poteſtatum radicum ejusdem.

§. I.

Poſtquam æquatio algebraica tam a fractionibus quam ab irrationalitate fuerit liberata, atque ad hujusmodi formam reducta:

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \dots \pm N = 0$$

demonſtrari ſolet in analyſi, hujusmodi æquationem tot ſemper habere radices, ſive ſint reales ſive imaginariæ, quot unitates contineantur in poteſtatis ſummæ exponente n. Tum vero non minus certum eſt, ſi hujus æquationis omnes radices fuerint $\alpha, \beta, \gamma, \delta, \epsilon, \dots \nu$, coëfficientes terminorum æquationis A, B, C, D, E, &c. ex his radicibus ita conflari, ut ſit:

A = ſummæ omnium radicum $= \alpha + \beta + \gamma + \delta + \dots + \nu$

B = ſummæ productorum ex binis $= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \&c.$

C = ſummæ productorum ex ternis $= \alpha\beta\gamma + \&c.$

D = ſummæ productorum ex quaternis $= \alpha\beta\gamma\delta + \&c.$

E = ſummæ productorum ex quinis $= \alpha\beta\gamma\delta\epsilon + \&c.$

&c.

Ultimumque tandem terminum abſolutum $\pm$ N eſſe productum ex omnibus radicibus $\alpha\beta\gamma\delta \dots \nu$.

§. II

§. II. Quo jam theorema, cujus demonstrationem hic tradere constitui, facilius ac brevius enunciare queam; designet $\int\alpha$ summam omnium radicum; $\int\alpha^2$ summam quadratorum earumdem radicum; $\int\alpha^3$ summam cuborum radicum; $\int\alpha^4$ summam biquadratorum istarum radicum, & ita porro: ita ut sit:

$$\int\alpha = \alpha + \beta + \gamma + \delta + \varepsilon + \cdots + \nu$$
$$\int\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \cdots + \nu^2$$
$$\int\alpha^3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \cdots + \nu^3$$
$$\int\alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \cdots + \nu^4$$
$$\int\alpha^5 = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \cdots + \nu^5$$

&c.

§. II. Hac signandi ratione exposita Neutonus affirmat istas potestatum, quae ex singulis radicibus formantur, summas per coefficientes aequationis A, B, C, D, E, &c. ita definiri, ut sit.

$$\int\alpha = A$$
$$\int\alpha^2 = A\int\alpha - 2B$$
$$\int\alpha^3 = A\int\alpha^2 - B\int\alpha + 3C$$
$$\int\alpha^4 = A\int\alpha^3 - B\int\alpha^2 + C\int\alpha - 4D$$
$$\int\alpha^5 = A\int\alpha^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E$$
$$\int\alpha^6 = A\int\alpha^5 - B\int\alpha^4 + C\int\alpha^3 - D\int\alpha^2 + E\int\alpha - 6F$$

&c.

Cujus theorematis demonstrationem Neutonus non solum nullam tradit, sed etiam Ipse videtur ejus veritatem ex continua illatione conclusisse. Primum enim demonstratione non eget esse $\int\alpha = A$: & cum sit

$$A^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c. + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma +$$

$2\beta\delta + \&c.$ erit $A^2 = \int\alpha^2 + 2B$, ideoque $\int\alpha^2 = A^2 - 2B = A\int\alpha - 2B$: similique modo veritas sequentium formularum evinci potest; sed continuo majore opus erit labore.

§. IV. Cum plures jam hujus theorematis utilissimi veritatem ostenderint, eorum demonstrationes autem regulis combinationum plerumque innitantur, quae etiamsi verae sint, tamen ab inductione plurimum pendeant; duplicem hic afferam demonstrationem, in quarum utraque inductioni nihil tribuitur. Altera quidem ex analysi infinitorum est petita, quae etsi nimis longe remota videatur, tamen totum negotium perfecte conficit: verum tamen cum contra eam jure objici queat, hujus theorematis veritatem evictam esse oportere, antequam ad analysin infinitorum perveniatur; alteram demonstrationem adjungam, in qua nihil assumitur, nisi quod statim ab initio in explicatione naturae aequationum tradi solet.

Demonstratio I.

§. V. Ponatur $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \ldots + N = Z$ & cum aequationis $Z = 0$ radices seu valores ipsius x sint, $\alpha, \beta, \gamma, \delta, \ldots \nu$, quorum numerus est $= n$, erit ex natura aequationum:

$$Z = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots(x-\nu)$$

et

et logarithmis sumendis habebitur:

$$lZ = l(x-\alpha) + l(x-\beta) + l(x-\gamma) + l(x-\delta) + \cdots + l(x-\nu)$$

Quod si jam harum formularum differentialia capiantur erit:

$$\frac{dZ}{Z} = \frac{dx}{x-\alpha} + \frac{dx}{x-\beta} + \frac{dx}{x-\gamma} + \frac{dx}{x-\delta} + \&c. + \frac{dx}{x-\nu}$$

Ideoque per dx dividendo fiet:

$$\frac{dZ}{Zdx} = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \frac{1}{x-\delta} + \cdots + \frac{1}{x-\nu}$$

Convertantur nunc singulae hae fractiones more solito in series geometricas infinitas: ob

$$\frac{1}{x-\alpha} = \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \frac{\alpha^4}{x^5} + \frac{\alpha^5}{x^6} + \&c.$$

$$\frac{1}{x-\beta} = \frac{1}{x} + \frac{\beta}{x^2} + \frac{\beta^2}{x^3} + \frac{\beta^3}{x^4} + \frac{\beta^4}{x^5} + \frac{\beta^5}{x^6} + \&c.$$

$$\frac{1}{x-\gamma} = \frac{1}{x} + \frac{\gamma}{x^2} + \frac{\gamma^2}{x^3} + \frac{\gamma^3}{x^4} + \frac{\gamma^4}{x^5} + \frac{\gamma^5}{x^6} + \&c.$$

&c.

$$\frac{1}{x-\nu} = \frac{1}{x} + \frac{\nu}{x^2} + \frac{\nu^2}{x^3} + \frac{\nu^3}{x^4} + \frac{\nu^4}{x^5} + \frac{\nu^5}{x^6} + \&c.$$

His

His igitur seriebus colligendis, signisque ante expositis $\int \alpha$, $\int^2 \alpha$, $\int^3 \alpha$ &c. introducendis invenietur, quia numerus harum serierum est $= n$:

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2}\int\alpha + \frac{1}{x^3}\int^2\alpha + \frac{1}{x^4}\int^3\alpha + \frac{1}{x^5}\int^4\alpha + \&c.$$

§. VI. Cum autem statuerimus:

$$Z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \cdots \pm N$$

erit similiter differentialibus sumendis:

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.$$

hineque colligetur superior formula $\frac{dZ}{Zdx}$ ita expressa ut sit:

$$\frac{dZ}{Zdx} = \frac{nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.}$$

quae igitur fractio aequalis esse debet seriei supra inventae:

$$\frac{n}{x} + \frac{1}{x^2}\int\alpha + \frac{1}{x^3}\int^2\alpha + \frac{1}{x^4}\int^3\alpha + \frac{1}{x^5}\int^4\alpha + \&c.$$

Quare

Quare si utraque expressio pro $\frac{dZ}{Zdx}$ inventa per alterius denominatorem $x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + \&c.$ multiplicetur resultabit hæc æquatio:

$$
\begin{aligned}
&nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c. \\
&= nx^{n-1} + x^{n-2}\int\alpha + x^{n-3}\int\alpha^{2} + x^{n-4}\int\alpha^{3} + x^{n-5}\int\alpha^{4} + \&c. \\
&\qquad - nAx^{n-2} - Ax^{n-3}\int\alpha - Ax^{n-4}\int\alpha^{2} - Ax^{n-5}\int\alpha^{3} - \&c. \\
&\qquad\qquad + nBx^{n-3} + Bx^{n-4}\int\alpha + Bx^{n-5}\int\alpha^{2} + \&c. \\
&\qquad\qquad\qquad - nCx^{n-4} - Cx^{n-5}\int\alpha - \&c. \\
&\qquad\qquad\qquad\qquad + nDx^{n-5} + \&c.
\end{aligned}
$$

§. VII. Quemadmodum jam utrinque termini primi nx^{n-1} sunt æquales, necesse est ut & secundi, tertii, quarti &c. inter se seorsim æquentur; unde sequentes nascentur æquationes:

$$
\begin{aligned}
&-(n-1)A = \int\alpha - nA \\
&+(n-2)B = \int\alpha^{2} - A\int\alpha + nB \\
&-(n-3)C = \int\alpha^{3} - A\int\alpha^{2} + B\int\alpha - nC \\
&+(n-4)D = \int\alpha^{4} - A\int\alpha^{3} + B\int\alpha^{2} - C\int\alpha + nD \\
&\qquad \&c.
\end{aligned}
$$

harumque æquationum lex, qua progrediuntur, sponte est

 mani-

manifesta. Ex iis autem obtinentur formulæ illæ ipsæ, quibus theorema Neutonianum constat; scilicet:

$$\int\alpha = A$$
$$\int\alpha^2 = A\int\alpha - 2B$$
$$\int\alpha^3 = A\int\alpha^2 - B\int\alpha + 3C$$
$$\int\alpha^4 = A\int\alpha^3 - B\int\alpha^2 + C\int\alpha - 4D$$
$$\int\alpha^5 = A\int\alpha^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E$$

Quæ est altera theorematis propositi demonstratio.

Demonstratio II.

§. VIII. Quo hujus demonstrationis vis clarius perspiciatur, eam ad æquationem determinati gradus accommodabo, ita tamen ut ea intelligatur ad quosvis gradus æque patere. Sit ergo proposita æquatio quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

cujus quinque radices sint α, β, γ, δ, ε. Quia igitur quælibet radix loco x substituta æquationi satisfacit, erit:

$$\alpha^5 - A\alpha^4 + B\alpha^3 - C\alpha^2 + D\alpha - E = 0.$$
$$\beta^5 - A\beta^4 + B\beta^3 - C\beta^2 + D\beta - E = 0$$
$$\gamma^5 - A\gamma^4 + B\gamma^3 - C\gamma^2 + D\gamma - E = 0$$
$$\delta^5 - B\delta^4 + B\delta^3 - C\delta^2 + D\delta - E = 0$$
$$\varepsilon^5 - A\varepsilon^4 + B\varepsilon^3 - C\varepsilon^2 + D\varepsilon - E = 0$$

Colligentur hæ æquationes in unam summam, & ob signa supra recepta (§. 2.) habebitur:

$$\int\alpha^5 - A\int\alpha^4 + B\int\alpha^3 - C\int\alpha^2 + D\int\alpha - 5E = 0$$
seu $\int\alpha^5 = A\int\alpha^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E.$

§. IX.

§. IX. Hinc dilucide patet, si æquatio proposita fuerit gradus cujuscunque

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \dots \pm Mx \mp N = 0$$

ubi in ultimis terminis signorum ambiguorum superiora valent, si exponens summus n fuerit numerus impar, inferiora si par; fore pariter:

$$\int \alpha^n = A\int \alpha^{n-1} - B\int \alpha^{n-2} + C\int \alpha^{n-3} - \dots \mp M\int \alpha \pm nN$$

si quidem per α indicetur radix quælibet istius æquationis sicque veritas Theorematis Neutoniani jam pro uno casu est ostensa. Super est igitur, ut ejusdem veritatem tam pro altioribus quam pro inferioribus radicum potestatibus demonstremus.

§. X. Pro altioribus quidem potestatibus res pari modo patet, si enim valores $\alpha, \beta, \gamma, \delta, \varepsilon$ satisfaciant æquationi

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

satisfacient quoque sequentibus æquationibus:

$$x^6 - Ax^5 + Bx^4 - Cx^3 + Dx^2 - Ex = 0$$
$$x^7 - Ax^6 + Bx^5 - Cx^4 + Dx^3 - Ex^2 = 0$$
$$x^8 - Ax^7 + Bx^6 - Cx^5 + Dx^4 - Ex^3 = 0$$

&c.

Ac propterea si in unaquaque æquatione pro x singuli valores $\alpha, \beta, \gamma, \delta, \varepsilon$ substituantur, & aggregata colligantur, erit

$$\int\alpha^6 = A\int\alpha^5 - B\int\alpha^4 + C\int\alpha^3 - D\int\alpha^2 + E\int\alpha$$
$$\int\alpha^7 = A\int\alpha^6 - B\int\alpha^5 + C\int\alpha^4 - D\int\alpha^3 + E\int\alpha^2$$
$$\int\alpha^8 = A\int\alpha^7 - B\int\alpha^6 + C\int\alpha^5 - D\int\alpha^4 + E\int\alpha^3$$
&c.

§. XI. Si ergo α denotet radicem quamcunque hujus æquationis:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \dots + Mx \mp N = 0$$

erit non solum, uti jam invenimus:

$$\int\alpha^n = A\int\alpha^{n-1} - B\int\alpha^{n-2} + C\int\alpha^{n-3} - D\int\alpha^{n-4} \dots \mp M\int\alpha \pm nN$$

sed etiam ad altiores quoque potestates progrediendo erit:

$$\int\alpha^{n+1} = A\int\alpha^{n} - B\int\alpha^{n-2} + C\int\alpha^{n-3} - D\int\alpha^{n-4} \dots \mp M\int\alpha^2 \pm N\int\alpha$$
$$\int\alpha^{n+2} = A\int\alpha^{n+1} - B\int\alpha^{n} + C\int\alpha^{n-1} - D\int\alpha^{n-2} + \dots \mp M\int\alpha^3 \pm N\int\alpha^2$$
$$\int\alpha^{n+3} = A\int\alpha^{n+2} - B\int\alpha^{n+1} + C\int\alpha^{n} - D\int\alpha^{n-1} + \dots \mp M\int\alpha^4 \pm N\int\alpha^3$$
&c.

& in genere quidem, si ad n addatur numerus quicunque m, erit

$$\int\alpha^{n+m} = A\int\alpha^{n+m-1} - B\int\alpha^{n+m-2} + C\int\alpha^{n+m-3} - \dots \mp M\int\alpha^{m+1} \pm N\int\alpha^{m}.$$

Ubi quidem notandum est, si sit $m = 0$, ob singulas potestates $\alpha^0 = 1$, $\beta^0 = 1$, $\gamma^0 = 1$; &c. numerumque harum litterarum $= n$ fore

fore $\int \alpha^0 = n$, quo caſu formula primo inventa in hac expreſſione continetur.

§. XII. Quamquam autem hæc expreſſio æque veritati eſt conſentanea, ſi pro m accipiatur numerus negativus: hincque pro æquatione quinti gradus aſſumta

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

ſequentes formulæ pariter locum habent:

$$\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha^1 - D\int \alpha^0 + E\int \alpha^{-1}$$

$$\int \alpha^3 = A\int \alpha^2 - B\int \alpha^1 + C\int \alpha^0 - D\int \alpha^{-1} + E\int \alpha^{-2}$$

$$\int \alpha^2 = A\int \alpha^1 - B\int \alpha^0 + C\int \alpha^{-1} - D\int \alpha^{-2} + E\int \alpha^{-3}$$

&c.

tamen hæ formulæ ſunt diverſæ ab iis, quas theorema continet. Demonſtrandum enim eſt eſſe:

$$\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D$$

$$\int \alpha^3 = A\int \alpha^2 - B\int \alpha + 3C$$

$$\int \alpha^2 = A\int \alpha - 2B$$

$$\int \alpha = A$$

Harum igitur formularum veritatem ſequenti modo oſtendo.

§. XIII. Propoſita ſcilicet æquatione quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

Formentur retinendis iisdem coëfficientibus sequentes æquationes inferiorum graduum:

I. $x - A = 0$. Radix sit p

II. $x^2 - Ax + B = 0$. Radix quælibet sit q

III. $x^3 - Ax^2 + Bx - C = 0$. Sit radix quælibet r

IV. $x^4 - Ax^3 + Bx^2 - Cx + D = 0$. Radix quælibet s

Quarum æquationum radices, etiamsi inter se maxime discrepent, tamen in his singulis æquationibus eandem constituent summam $= A$. Deinde remota prima summa productorum ex binis radicibus ubique erit eadem $= B$: Tum summa productorum ex ternis radicibus ubique erit $= C$, præter æquationes scilicet I & II, ubi C non occurrit. Similiter in IV & proposita summa productorum ex quaternis radicibus erit eadem $= D$.

§. XIV. In quibus autem æquationibus non solum summa radicum est eadem, sed etiam summa productorum ex binis radicibus, ibi quoque summa quadratorum radicum est eadem. Sin autem præterea summa productorum ex ternis radicibus fuerit eadem, tum summa quoque cuborum omnium radicum erit eadem. Atque si insuper summa productorum ex quaternis radicibus fuerit eadem, tum quoque summa biquadratorum omnium radicum erit eadem atque ita porro. Hic scilicet assumo, quod facile concedetur, summam quadratorum per summam radicum & summam productorum ex binis determinari; summam cuborum autem præterea requirere summam factorum ex ternis radicibus; ac summam biquadratorum præterea summam factorum ex quaternis

ternis radicibus, & ita porro; quod quidem demonstratu non esset difficile.

§. XV. In æquationibus ergo inferiorum graduum, quarum radices denotantur respective per litteras p, q, r, s, dum ipsius propositæ quinti gradus quælibet radix littera u indicatur, erit:

$$\int u = \int s = \int r = \int q = \int p$$

$$\int u^2 = \int s^2 = \int r^2 = \int q^2$$

$$\int u^3 = \int s^3 = \int r^3$$

$$\int u^4 = \int s^4$$

At per ea quæ ante §. 9 demonstravimus est

$$\int p = A$$

$$\int q^2 = A\int q - 2B$$

$$\int r^3 = A\int r^2 - B\int r + 3C$$

$$\int s^4 = A\int s^3 - B\int s^2 + C\int s - 4D$$

Hinc ergo nanciscimur pro æquatione quinti gradus proposita: $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$ has formulas

$\int u$

$$\int a = A$$
$$\int a^{2} = A\int a - 2B$$
$$\int a^{3} = A\int a^{2} - B\int a + 3C$$
$$\int a^{4} = A\int a^{3} - B\int a^{2} + C\int a - 4D$$

§. XVI. In æquatione ergo cujuscunque gradus proposita:

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. \cdots \pm N = 0$$

si quælibet radix littera a indicetur erit:

$$\int a = A$$
$$\int a^{2} = A\int a - 2B$$
$$\int a^{3} = A\int a^{2} - B\int a + 3C$$
$$\int a^{4} = A\int a^{3} - B\int a^{2} + C\int a - 4D$$
$$\int a^{5} = A\int a^{4} - B\int a^{3} + C\int a^{2} - D\int a + 5E$$
&c.

Hocque modo veritas Theorematis Neutoniani pariter habetur demonstrata.

Ani-

E llip
ut
neis rect
expectar
jus integ
ab irratic
tegration
quidem e
tenendur
me sequi
trum elli
cularibus
curvæ a
ellipsis,
ri queant

§.
tam pori
lipsis per
sum etia
nes finita
hil admo
præter e
Euleri