Problem set 2

Introduction to digital signals — Deadline: 18/Apr/2018

Problems

1) Let f be a smooth band limited function with $\widehat{f}(\xi) = 0$ for $|\xi| \ge B$ and w a bounded integrable function such that $w(\xi) = 1$ when $\widehat{f}(\xi) \ne 0$ and $w(\xi) = 0$ for $|\xi| \ge B$. Prove

$$f(x) = \frac{1}{2B} \sum_{n=-\infty}^{\infty} f\big(\frac{n}{2B}\big)\widehat{w}\big(\frac{n}{2B} - x\big).$$

2) A signal takes values in $J = [-2, 2]$ with density of probability proportional to the distance to the closest extreme of J and we want to quantize the signal to get only two possible values. Design a quantizer minimizing the mean square error.

3) For the nodes $x_j = -1 + j/2$, $0 \le j \le 3$ and the images $y_0 = 5$, $y_1 = y_3 = 2$, $y_2 = 0$ consider the natural cubic spline s with $s(x_j) = y_j$. Compute explicitly the cubic polynomials $s\big|_{[x_j, x_{j+1}]}$ for $j = 0, 1, 2$.

4) Prove that the entries of the (dyadic) Bayer matrices M_k, as defined in the notes, are a rearrangement of $\{j2^{-2k}\}_{j=0}^{2^{2k}-1}$.

5) Prove the formula employed in the experimental challenge *Moiré, qu'est-ce que c'est?*

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n-\nu t)^2} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2} e(\nu n t).$$

Notes and hints

1) Note that Shannon sampling theorem can be got as a consequence. It corresponds to choose as w the characteristic function of the interval $[-B, B]$. This exercise allows to choose different windows to recover the signal.

2) "Design" means "find a formula". Using the symmetry you will simplify a lot. By sheer curiosity I have tried the calculations not assuming any symmetry and I can assure they are a nightmare.

3) Use whatever you like to help you with the computations (fingers, slide rule, calculator, supercomputer…) but saying "I have a wise computer package that putting the nodes and values give me the spline" is not a valid answer. In other words, you have to display the sequence of calculations you (or your calculator or your computer) are doing. The final result involves small numbers and it is possible to proceed completely by hand without any assistance. If you want to think about it, there is a subtle symmetry trick to simplify the calculations with bare hands but I did not assume it in my last claim.

4) Yes, this is simple cheap combinatorics (no offense intended if there is a combinatorist in the room). Consider it an Easter gift.

5) The most expeditious way to proceed is to apply the Poisson summation formula. You can also try the direct Fourier expansion that is almost as quick and somewhat includes the proof of the Poisson summation formula.