Theoretical challenge: **Breaking up a sawtooth**

Simple waves | Deadline: 7/March/2018

Motivation

The plots shown in §1.2.4 to illustrate Gibbs phenomenon for the doubled square wave suggests that the approximation is not so bad when we are not very close to the singularity. Our purpose is to analyze the situation for the sawtooth wave

$$s(t) = t - \lfloor t \rfloor - \frac{1}{2} \qquad \text{with } \lfloor t \rfloor \text{ the integral part.}$$

The coefficients of its Fourier series decay as $1/n$. Let us consider some cheap heuristics. If we think about alternating series like $1/1 - 1/2 + 1/3 - \dots$ as a model, we may guess that when we are far apart from the discontinuities the error term when approximating $s(t)$ by the Nth partial sum should be comparable to N^{-1} (the first disregarded term). On the other hand, Gibbs phenomenon seems to appear in its glory at distances comparable to N^{-1} giving an error like a constant. One could combine both claims guessing an error $O\big((Nt)^{-1}\big)$ for $1/N < t \le 1/2$. In fact it is not hard to believe in a t^{-1} decay looking the plots in §1.2.4. The interval $0 < t \le 1/N$ is still under the influence of Gibbs phenomenon, then a kind of periodic version of $(1 + Nt)^{-1}$ could give the order of magnitude of the error term.

The challenge

Let $\|x\|$ be the distance of x to the nearest integer (e.g. $\|3.1\| = \|0.9\| = 0.1$). If $s(t)$ is the sawtooth wave as before, prove that

$$s(t) = -\sum_{n=1}^{N} \frac{\sin(2\pi n t)}{\pi n} + O\Big(\frac{1}{1 + N\|t\|}\Big).$$