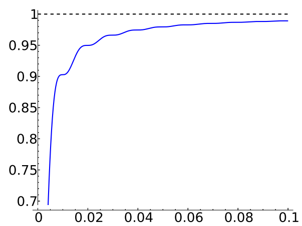
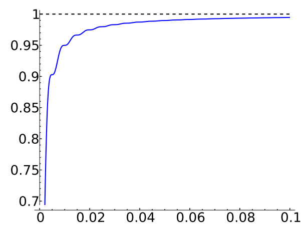
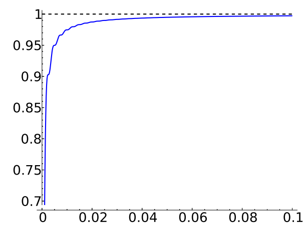
It seems that Gibbs phenomenon is a serious drawback in the applicability of Fourier analysis for discontinuous signals but it is not the end of the world because there are localized versions of Fourier analysis (for instance the wavelets that we shall treat in a future chapter) and because we can reduce the effect of the singularities with a regularization (this is related to the filters that we shall consider in the digital setting in another future section). For instance, compare the following figures with the previous ones, the only difference is replacing $S_N f$ by $\widetilde{S}_N f$ as in Theorem 1.2.5.



frequencies< 100



frequencies< 200



frequencies< 400

Although we still have large errors near zero (as it must be because we are approximating a discontinuous function), we do not see lumps, the general aspect is less shaky. This is important, think for instance in medical imaging. The big wobbles in the first case would transform in stains between sharp transitions, different tissues, that could be misinterpreted as a tumor while the second approach would give a blur smoother change between tissues.

Suggested Readings. Again Gibbs phenomenon belong to basic Fourier analysis and it is nicely explained in the monographs mentioned before, for instance [Kör88] and [DM72].

1.2.5 More flavors of harmonic analysis

Why cosine and sine? Because, as we have seen, basic electronic circuits lead to these functions. Another not technologically based reason is that for instance, the physiology of hearing introduces some filtering on the frequencies, then if we want to understand or to simulate hearing (yes, with technology!) it seems handy to read the content on each frequency through Fourier analysis[6].

In other situations, it might be more convenient to analyze in terms of other non trigonometrical functions having special properties. The bundle of techniques and problems derived from the analytical decomposition of a function into something that we would call "pure tones" or even better "harmonics", it is called *harmonic analysis*. What is really a harmonic? The honest answer is whatever you find convenient. Here we consider harmonic analysis somewhat related to symmetries to illustrate that there is something beyond the classic Fourier series and integrals.

Symmetry sounds to group theory, then we are going to consider mainly groups. Let us start with a finite abelian group G. The functions $f : G \longrightarrow \mathbb{C}$ that we want to analyze

[6] In [MK11, §4.1] we read "*Fourier analysis is like a glass prism, which splits a beam of light into frequency components corresponding to colors*".

can be considered as ordered finite lists of real numbers. Why do we want to analyze them? Aren't they already very simple? The case treated in (1.32) corresponds to $\mathbb{Z}_N$, the classes of integers modulo N, because $f(n) = f(n + kN)$ and it is important in digital signal processing. One may imagine that the digital world approximates the real (analog) world when N grows and suspect that it is the only interesting case but the fact is that even small groups are relevant in practice. Believe or not, when you store a photo in your cellular or your computer, harmonic analysis in $\mathbb{Z}_8 \times \mathbb{Z}_8$ is applied quite a number of times (if you are curious about it, continue reading these notes beyond the first chapter).

For G finite and abelian the analogue of cosine and sine unified in the complex exponential are the *characters*. These are maps

$$\chi : G \longrightarrow \mathbb{C}^* \qquad \text{with} \quad \chi(g_1 g_2) = \chi(g_1)\chi(g_2) \quad \forall g_1, g_2 \in G, \tag{1.95}$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$. This latter set has a multiplicative group structure and then we are just saying with formulas that χ is a *homomorphism*, a map preserving the group structure. It is easy to see that $\chi(g)$ is always a $|G|$-th root of unity $\chi(g) = e(k_{\chi,g}/|G|)$. Let $\widehat{G}$ denote the set of characters. It inherits a group structure from $\mathbb{C}^*$. For $G = \mathbb{Z}_N$ we have

$$\widehat{G} = \{\chi_0, \chi_1, \dots, \chi_{N-1}\} \qquad \text{with} \quad \chi_k(\overline{n}) = e(kn/N). \tag{1.96}$$

The usual notation in $\mathbb{Z}_N$ is additive then we have $\chi(\overline{n} + \overline{m}) = \chi(\overline{n})\chi(\overline{m})$. Note that $|\widehat{G}| = |G|$. This holds in general [Ter99], you can believe it promptly for instance appealing to the classification of finite abelian groups that allows to write G as a direct product of $\mathbb{Z}_N$'s and the characters are constructed as products of the characters of the direct factors.

The important point in this more general context, is that we have something like an analogue of (1.33) that was the precursor of (1.37).

Lemma 1.2.12. *If G is a finite abelian group the following* orthogonality relations *hold true for every $\chi_1, \chi_2 \in \widehat{G}$*

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi}_1(g)\chi_2(g) = \begin{cases} 1 & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2. \end{cases} \tag{1.97}$$

One could proof this with an explicit construction of the characters as indicated above but there is a shorter, more elegant and generalizable way to proceed.

Proof. As $\chi_1(g)$ is a root of unity $\overline{\chi}_1(g)\chi_1(g) = 1$ and the case $\chi_1 = \chi_2$ becomes trivial. If $\chi_1 \neq \chi_2$ let $h \in G$ such that $\chi_1(h) \neq \chi_2(h)$. We have

$$\sum_{g \in G} \overline{\chi}_1(g)\chi_2(g) = \sum_{g \in G} \overline{\chi}_1(hg)\chi_2(hg) = \overline{\chi}_1(h)\chi_2(h) \sum_{g \in G} \overline{\chi}_1(g)\chi_2(g). \tag{1.98}$$

The first equality follows because $g \mapsto hg$ permutes the elements of G. As $\overline{\chi}_1(h)\chi_2(h) = \chi_2(h)/\chi_1(h) \neq 1$, the sum must vanish. □

With an elementary calculation, from (1.97) we get a generalization of (1.32).

Theorem 1.2.13. *If G is a finite abelian group, any $f : G \longrightarrow \mathbb{C}$ can be written as*

$$f(g) = \sum_{\chi\in\widehat{G}} a_\chi \chi(g) \qquad \textit{with} \quad a_\chi = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi}(g). \tag{1.99}$$

If we think about a topological infinite abelian group as a kind of limit of finite groups, $|G|^{-1}\sum_{g\in G}$ may be interpreted as an equidistributed measure. It can be proved that for G locally compact there exists a formalization of this concept, it is the *Haar measure* μ that verifies $\mu(gS) = \mu(S)$ for every $g \in G$ and $S \subset G$ a Borel set. It is unique except for scaling it multiplying by a constant. For G abelian and locally compact the *characters*, in this more general context are defined as continuous bounded homomorphisms $\chi : G \longrightarrow \mathbb{C}^*$ (in the finite case the continuity was obvious because everything is continuous with the discrete topology). If G is compact we can state the orthogonality relations as

$$\int_G \overline{\chi}_1\chi_2 \, d\mu = \begin{cases} 1 & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2, \end{cases} \tag{1.100}$$

choosing the scaling in such a way that $\mu(G) = 1$. In this situation, (1.99) holds true with $a_\chi = \int_G f\overline{\chi}\, d\mu$ under conditions assuring the convergence. If G is not compact it may happen $\mu(G) = \infty$ and a kind of suspicious "infinite scaling" appears to be needed here. Actually it is possible to scale at the same time the sums over G and $\widehat{G}$ to keep the result. To sum up, if G is abelian and locally compact, we have [Kat68]

$$f(g) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(g)\, d\nu \qquad \text{with} \quad \widehat{f}(\chi) = \int_G f(g)\overline{\chi}(g)\, d\mu \tag{1.101}$$

where μ and ν are the Haar measures of G and $\widehat{G}$ respectively, with a convenient normalization, and f is assumed to be continuous with $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$ to assure the convergence.

The formula (1.101) embodies the totality of the flavors of harmonic analysis typically seen in mathematical degrees: If G is finite then μ and ν are scaled counting measures and we have (1.99); if $G = \mathbb{T}$ then $\widehat{G} = \{e(nx)\}_{n\in\mathbb{Z}} \cong \mathbb{Z}$, μ is the Lebesgue measure and ν the counting measure and we have (1.34); finally if $G = \mathbb{R}$ then $\widehat{G} = \{e(\xi x)\}_{\xi\in\mathbb{R}} \cong \mathbb{R}$, μ and ν are the Lebesgue measure and we have (1.36).

Is it still possible to generalize (1.99) or (1.101) to nonabelian groups? The answer is yes but it requires to deal with more difficult concepts. We restrict ourselves to finite nonabelian groups G with an ending comment about the compact case.

The previous approach completely fails because in a nonabelian group we have in general very few characters defined as before. The surrogate concept is that of unitary representation. With a little narrow sighted definition adapted to our context, a *representation* is a map π that associated to every $g \in G$ a complex non singular square matrix $\pi(g)$ preserving multiplication, $\pi(g_1)\pi(g_2) = \pi(g_1g_2)$ and it is called *unitary* if $\pi(g)$ is a unitary matrix. The dimension d_π of a representation indicates the size of the matrix. A very important and not so simple concept is that of *irreducible representation*. It means that it is not possible to find a proper subspace of $\mathbb{C}^{d_\pi}$ that remains invariant by $\pi(g)$ simultaneously for every $g \in G$. Given a representation π we can construct another given by a change of basis $C^{-1}\pi(g)C$ where C is a constant matrix. We are only interested in *non-equivalent representations*, those not related by a change of basis. Just to mention a nontrivial example of representation, the following table defines a unitary irreducible representation of the permutation group S_3:

(1.102)

g	Id	$(1,2,3)$	$(1,3,2)$	$(1,2)$	$(2,3)$	$(1,3)$
$\pi(g)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

In the context of finite nonabelian groups, one defines $\widehat{G}$ to be a maximal set of non-equivalent unitary irreducible representations. It can be proved that it is finite, in fact one has the curious relation [Ter99]

$$\sum_{\pi \in \widehat{G}} d_\pi^2 = |G|. \tag{1.103}$$

For instance, for S_3 we have the previous representation with d_π, the trivial one $\pi(g) = (1)$ and $\pi(g) = (\text{sgn}(g))$. The equality $2^2 + 1^2 + 1^2 = 3!$ assures that any other irreducible representation is one of these after a change of basis.

After all of these definitions, the generalization of (1.99) is

$$f(g) = \sum_{\pi \in \widehat{G}} d_\pi \text{Tr}\big(\widehat{f}(\pi)\pi(g)\big) \qquad \text{with} \quad \widehat{f}(\pi) = \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g)^\dagger. \tag{1.104}$$

Here $\pi(g)^\dagger$ means the transpose conjugate of $\pi(g)$ and Note that the "Fourier coefficients" $\widehat{f}(\pi)$ are now matrices. This formula actually generalizes (1.99) because for an abelian group the irreducible representations have $d_\pi = 1$ and then $\widehat{G}$ becomes the group of characters. If G is not finite but it is compact, (1.104) still holds changing the formula for $\widehat{f}(\pi)$ by the integral $\int_G f\pi^\dagger$ where one employs a conveniently normalized Haar measure to integrate on G.

In a compact Riemannian manifold M, there is a natural second order operator, the *Laplace-Beltrami operator* and the general theory assures that it possesses and orthonormal discrete system of eigenfunctions $\{\phi_j\}_{j=1}^\infty$ and any function $f \in L^2(M)$ can be expanded as a spectral Fourier series

$$f = \sum a_j \phi_j \qquad \text{with Fourier coefficients} \quad a_j = \int_M \overline{\phi}_j f. \tag{1.105}$$

For $M = \mathbb{T}$ the Laplace-Beltrami operator is $-d^2/dx^2$ and the eigenfunctions are the complex exponentials $e(nx)$. For a (semisimple) compact *Lie group*, a (Riemannian) manifold having a smooth group structure, this description and (1.104) coincide, meaning that the entries of the matrices of $\pi(g)$ are actually orthogonal eigenfunctions of the Laplace-Beltrami operator and so one can rearrange the illusory limit of (1.104) as $|G| \to \infty$ to get (1.105).

Suggested Readings. The finite abelian and nonabelian case is treated in [Ter99], a master piece in exposition that includes also some applications linked to more advanced topics. In [DM72] there is a detailed analysis of the compact Lie group SO(3) from scratch. Although [Zee16] does not address topics on harmonic analysis it is a good source to learn about representations, Lie groups and why physicists care about them. A classic for basic spectral theory and generalized Fourier expansions with applications is [CH53].