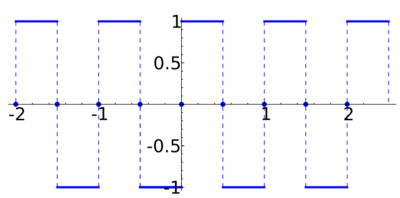
1.2.4 Gibbs phenomenon

After the ideas and graphics above, we infer that for regular functions a limitation of the frequencies to size less than δ^{-1} causes that we miss the function by something like δ and not better in general. Clearly we cannot perform a uniform approximation of a discontinuous function by continuous functions so the regularity plays a role. We can hope anyway that if we have a single jump discontinuity, except for the δ suggested by the uncertainty principle, the truncated Fourier series remains in the gap between the left and the right branches with good approximations when we are δ far apart from the discontinuity. We can hope or conjecture whatever we want and reality can be a slap in the face.

Before entering into details perhaps you are wondering “Why do we care now about discontinuous functions? Is it a mathematicians thing?” Not exactly, if you press a button you have a discontinuous signal, for instance switching on a fluorescent lamp is like considering a single discontinuity in the voltage source in (1.21). On the other hand a digital signal is plenty of jumps. One may say “OK, this is just an idealization because everything in Mother Nature is continuous, nothing can go from 0 to 1 in a short period of time without taking the middle values”. Probably many physicists would not agree and even if we admit this claim, we work with mathematical models and a discontinuous function is a good model for something that changes so quickly from a value to another very different value that we cannot detect it. Another natural question is why we bother to know whether the Fourier series approximately stays in the gap when it faces a jump discontinuity. If not, it could introduce serious artifacts. We expect that if we pass from light green to dark green a scarlet tone should not appear in the middle whatever method we use to represent the transition.

The really bad news for applications is that Fourier series and integrals are strongly non local. It means that a bad point “contaminates” the global behavior. Let us focus on the 1-periodic extension f of the sign function $\operatorname{sgn}(x)$ in the interval $[-1/2, 1/2]$, which is just the double of the square wave (1.58).

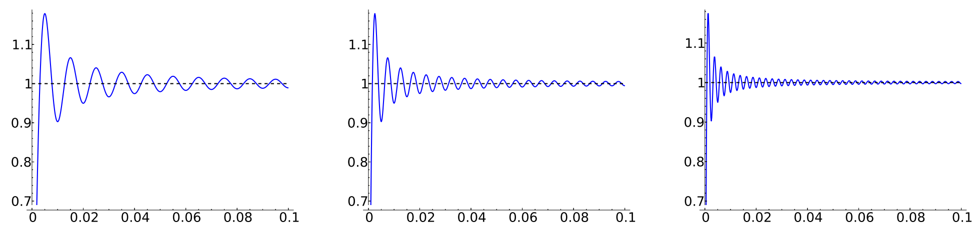
(1.94) $f =$



$$f(x) = \begin{cases} 1 & \text{if } \lfloor 2x \rfloor \text{ is even} \\ 0 & \text{if } 2x \in \mathbb{Z} \\ -1 & \text{if } \lfloor 2x \rfloor \text{ is odd} \end{cases}$$

where $\lfloor x \rfloor = \max\{n \le x \,:\, n \in \mathbb{Z}\}$. The Fourier series of f has coefficients that decay as $1/n$ because it is not possible to get rid of the boundary terms when integrating by parts. So we have a non absolutely convergent Fourier series in any compact subset of $(-1/2, 0)$ and $(0, 1/2)$ although f is regular, even constant, in these intervals.

In principle, this does not contradicts our claim, a conditionally convergent series can converge very quickly and uniformly in some regions but the following details of the graph of the truncated Fourier series for x in the left part of $(0, 1/2)$ terminates our hope. The convergence is lame, the series overshoots wildly the function and it is not approximately confined to the gap between the branches at both sides of the singularity. The Fourier series and the function are odd, so the behavior to the left is exactly symmetric.



frequencies$<$ 100 frequencies$<$ 200 frequencies$<$ 400

When we take frequencies less than δ^{-1} it seems that the interval $[0, \delta]$ contains a lump of size like 0.2 independently of how small is δ. Let us write and prove it as a mathematical result.

Proposition 1.2.11. *For f as before, we have*

$$\lim_{N\to\infty} \sup_{|x|<1/(2N+1)} \big|f(x) - S_N f(x)\big| = \int_{-1}^{1} \frac{\sin(\pi x)}{\pi x}\, dx - 1 = 0.17897974\ldots$$

There is nothing special about $\operatorname{sgn}(x)$. Think for instance that for $g : \mathbb{R} \longrightarrow \mathbb{C}$ regular, $g(x) + \sum_{j=1}^{J} \lambda_j \operatorname{sgn}(x - \mu_j)$ is the generic form of a regular real variable function except for a finite set of jump discontinuities. In the case of $\operatorname{sgn}(x)$ we have a lump of size $0.17\ldots$ for a jump of size 2. Scaling these values, in general the lumps are always like a 9% of the jump, this is called *Gibbs phenomenon*.

Proof. By the symmetry, we only consider the case $x \ge 0$. For any f we know that $S_N f = D_N * f$ and in our case we have

$$S_N f(x) = -\int_{x}^{x+1/2} D_N(t)\, dt + \int_{x-1/2}^{x} D_N(t)\, dt.$$

Substituting the explicit formula for D_N and using its parity, this is

$$S_N f(x) = \int_{-x}^{x} \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt - \int_{-x-1/2}^{x-1/2} \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt.$$

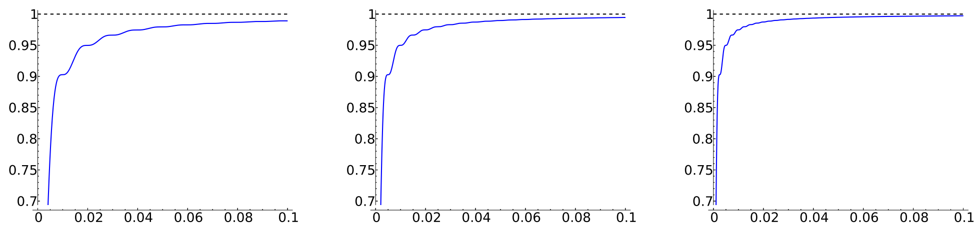
Integrating by parts (note that x is small), the second integral is $O\big(N^{-1}\big)$.

Clearly the supremum of the first integral is reached for $x = 1/(2N+1)$ because the function under the integral is positive. Finally,

$$\int_{-1/(2N+1)}^{1/(2N+1)} \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt = \int_{-1}^{1} \frac{\sin(\pi x)}{(2N+1)\sin(\pi x/(2N+1))}\, dx$$

is just a change of variables and the denominator tends to πx. $\square$

It seems that Gibbs phenomenon is a serious drawback in the applicability of Fourier analysis for discontinuous signals but it is not the end of the world because there are localized versions of Fourier analysis (for instance the wavelets that we shall treat in a future chapter) and because we can reduce the effect of the singularities with a regularization (this is related to the filters that we shall consider in the digital setting in another future section). For instance, compare the following figures with the previous ones, the only difference is replacing $S_N f$ by $\widetilde{S}_N f$ as in Theorem 1.2.5.



frequencies< 100 frequencies< 200 frequencies< 400

Although we still have large errors near zero (as it must be because we are approximating a discontinuous function), we do not see lumps, the general aspect is less shaky. This is important, think for instance in medical imaging. The big wobbles in the first case would transform in stains between sharp transitions, different tissues, that could be misinterpreted as a tumor while the second approach would give a blur smoother change between tissues.

Suggested Readings. Again Gibbs phenomenon belong to basic Fourier analysis and it is nicely explained in the monographs mentioned before, for instance [Kör88] and [DM72].

1.2.5 More flavors of harmonic analysis

Why cosine and sine? Because, as we have seen, basic electronic circuits lead to these functions. Another not technologically based reason is that for instance, the physiology of hearing introduces some filtering on the frequencies, then if we want to understand or to simulate hearing (yes, with technology!) it seems handy to read the content on each frequency through Fourier analysis[6].

In other situations, it might be more convenient to analyze in terms of other non trigonometrical functions having special properties. The bundle of techniques and problems derived from the analytical decomposition of a function into something that we would call "pure tones" or even better "harmonics", it is called *harmonic analysis*. What is really a harmonic? The honest answer is whatever you find convenient. Here we consider harmonic analysis somewhat related to symmetries to illustrate that there is something beyond the classic Fourier series and integrals.

Symmetry sounds to group theory, then we are going to consider mainly groups. Let us start with a finite abelian group G. The functions $f : G \longrightarrow \mathbb{C}$ that we want to analyze

[6] In [MK11, §4.1] we read "*Fourier analysis is like a glass prism, which splits a beam of light into frequency components corresponding to colors*".