Deadline: September 29th

Let $\mathcal{H}_4$ be the set of 4×4 Hermitian matrices. Consider

$$M = \{A \in \mathcal{H}_4 \;:\; A \text{ has two distinct eigenvalues of multiplicity } 2\}$$

(in other words, $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$). Compute the dimension of M.

Note. You are expected to proceed as in the lecture: using intuitive (but correct!) arguments counting degrees of freedom without entering into coordinate charts.

Solution. (This is rather concise. If you need more explanations, please ask).

Let V and W, respectively, the eigenspaces corresponding to λ_1 and λ_3 for certain $A \in M$. Using basic linear algebra, they verify $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = 2$ and $W = V^{\perp}$ (distinct eigenvalues $\Rightarrow$ orthogonal eigenvectors). Proceeding as in class, we know that $\mathcal{A}\big|_V = \lambda_1 \mathrm{Id}$ and $\mathcal{A}\big|_W = \lambda_2 \mathrm{Id}$ where $\mathcal{A}$ is the linear map corresponding to A (this is just saying that the Jordan canonical form would give for the corresponding matrices $U^* \lambda I U = \lambda I$). Then $\mathcal{A}$ is determined uniquely by λ_1, λ_3 and V and the required dimension is 2+the number of the real parameters needed to specify a subspace of dimension 2 of $\mathbb{C}^4$.

There are several ways of computing this number of real parameters, perhaps the simplest is to use that in a subspace of dimension 2 of $\mathbb{C}^4$ we can use two coordinates, generically z_3 and z_4, as parameters to express the other two. Each possible linear relation between them gives a different subspace. Then we have to specify $2 \times 2 = 4$ complex numbers or 8 real numbers. Consequently $\dim M = 2 + 8 = 10$.