By the linearity of d and re-naming the variables, we can assume without loss of generality

$$\omega = f\ dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k. \tag{1}$$

With the change of variables $x^i = x^i(y^1, y^2, \dots, y^n)$, $1 \le i \le n$, we have

$$\omega = f\frac{\partial x^1}{\partial y^{j_1}}\frac{\partial x^2}{\partial y^{j_2}}\cdots\frac{\partial x^k}{\partial y^{j_k}}dy^{j_1} \wedge dy^{j_2} \wedge \cdots \wedge dy^{j_k} \tag{2}$$

where the summation convention is in use and I do not indicate (by brevity) that f is substituted in the corresponding point. Note that this just follows from $dx^i = \frac{\partial x^i}{\partial y^j}dy^j$ and the multilinear properties of tensors. Note also that I am not imposing any ordering on the j_r.

If we compute $d\omega$ using (1) and (2), we obtain, respectively,

$$\frac{\partial f}{\partial x^r}\ dx^r \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k \qquad \text{and} \qquad \frac{\partial}{\partial y^r}\Big(f\frac{\partial x^1}{\partial y^{j_1}}\frac{\partial x^2}{\partial y^{j_2}}\cdots\frac{\partial x^k}{\partial y^{j_k}}\Big)dy^r \wedge dy^{j_1} \wedge dy^{j_2} \wedge \cdots \wedge dy^{j_k}.$$

Note that the definition of $d\omega$ given in the statement is still true if the i_s are not ordered.

It remains to check that these expressions are the same (up to the change of variables). The key point is that, after applying the product rule in the last formula, any term with second derivatives is canceled because

$$\frac{\partial^2 x^s}{\partial y^{j_s}\partial y^r}dy^r \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s} \wedge \cdots \wedge dy^{j_k} = -\frac{\partial^2 x^s}{\partial y^r \partial y^{j_s}}dy^{j_s} \wedge dy^{j_1} \wedge \cdots \wedge dy^r \wedge \cdots \wedge dy^{j_k}.$$

Recall that a differential form is antisymmetric in any pair of arguments. Hence

$$\frac{\partial}{\partial y^r}\Big(f\frac{\partial x^1}{\partial y^{j_1}}\frac{\partial x^2}{\partial y^{j_2}}\cdots\frac{\partial x^k}{\partial y^{j_k}}\Big)dy^r \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_k} = \frac{\partial f}{\partial y^r}\frac{\partial x^1}{\partial y^{j_1}}\frac{\partial x^2}{\partial y^{j_2}}\cdots\frac{\partial x^k}{\partial y^{j_k}}dy^r \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_k}.$$

Using again $\frac{\partial x^i}{\partial y^j}dy^j = dx^i$ and $\frac{\partial f}{\partial y^r}dy^r = df = \frac{\partial f}{\partial x^r}dx^r$ we deduce that this equals to the first expression.