Almost everybody has solved correctly this problem. Instead of repeating the proof given by most of you, I try to add something here emphasizing some points about mathematical rigor. For instance, at some point $\frac{d}{dt}\frac{\partial y}{\partial x} = \frac{\partial \dot{y}}{\partial x}$ is employed. What does it mean? At the right hand side, $\dot{y}$ suggest a function of t, but then how can we take the derivative with respect to x?

Recall that $L : TM \longrightarrow \mathbb{R}$, then $(x, \dot{x})$ and $(y, \dot{y})$ are coordinate maps and they have nothing to do with derivatives from the mathematical point of view. To avoid any confusion we shall write (x, v) and (y, w). If we change variables in the Lagrangian, we have $L(x, v) = \widetilde{L}(y, w)$. If $y = y(x)$, we have $w = y'(x)v$ (by Lemma 1.2.1 in the notes, if you wish). By the chain rule:

$$\begin{aligned} L_v &= \widetilde{L}_y \cdot 0 + \widetilde{L}_w \frac{\partial w}{\partial v} \\ &= \widetilde{L}_w y'(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} L_x &= \widetilde{L}_y y'(x) + \widetilde{L}_w \frac{\partial w}{\partial x} \\ &= \widetilde{L}_y y'(x) + \widetilde{L}_w y''(x) v \end{aligned}$$

Recall that x and v are independent variables (coordinate maps) and y only depends on x.

When we write the Euler Lagrange equations, we look for a parametrized curve $(x(t), v(t))$ with $x'(t) = v(t)$. If you prefer to see it so, you can say that $\frac{d}{dt} L_v = L_x$ means the system of ODE's $L_{vx}x' + L_{vv}v' = L_x$, $x' = v$.

Then we have

$$\frac{d}{dt} L_v = L_x \quad \Leftrightarrow \quad y'(x)\frac{d}{dt}\widetilde{L}_w + \widetilde{L}_w y''(x) x' = \widetilde{L}_y y'(x) + \widetilde{L}_w y''(x) v \quad \Leftrightarrow \quad y'(x)\frac{d}{dt}\widetilde{L}_w = \widetilde{L}_y y'(x)$$

and $y'(x)$ cancels out because $y = y(x)$ is a change of variables (in particular with C^∞ inverse).