Primer parcial 28/Feb/2018   Variable Compleja II

Nombre y apellidos ....

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1) Decide si las siguientes afirmaciones son verdaderas o falsas incluyendo en cada caso una pequeña justificación.

a) [1 punto] Si f es una función par que es meromorfa en $\mathbb{C}$ y holomorfa en $\mathbb{C}-\{0\}$, entonces necesariamente $\int_{\{|z|=1\}} f(z)\,dz = 0$.

b) [1.5 puntos] Existe un polinomio P no idénticamente nulo tal que $P(\wp) = 0$.

c) [1.5 puntos] Si $\mathcal{P}$ es un paralelogramo fundamental con $0 \in \operatorname{Int}(\mathcal{P})$ y la función definida por $\sum_{\omega\in\Lambda}(z-\omega)^{-2018}$ no se anula en $\partial\mathcal{P}$, entonces tiene 2018 ceros en $\operatorname{Int}(\mathcal{P})$ contados con multiplicidades.

2) [3 puntos] Para $f(z) = (z^2+1)\cos(\pi z)$, halla todos los polos de $1/f$ y sus residuos.

3) [3 puntos] Demuestra la siguiente identidad debida a Gauss:

$$\sum_{n\in\mathbb{Z}} (-1)^n x^{n^2} = \prod_{n=1}^{\infty} \frac{1-x^n}{1+x^n} \qquad \text{para} \quad -1 < x < 1.$$

Indicación. ¿Por qué la igualdad $\prod(1+x^n)^{-1} = \prod(1-x^n)/(1-x^{2n}) = \prod(1-x^{2n-1})$ es "elemental" sin necesidad de apelar a propiedades de funciones elípticas?

Fórmulas sobre funciones elípticas

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\Big(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\Big) = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_{2k} z^{2k} \qquad \text{con} \qquad a_{2k} = (2k+1)\sum_{\omega\in\Lambda^*}\omega^{-2k-2}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \qquad \text{con} \qquad g_2 = 60\sum_{\omega\in\Lambda^*}\omega^{-4}, \quad g_3 = 140\sum_{\omega\in\Lambda^*}\omega^{-6}$$

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z} = \prod_{n=1}^{\infty}\big(1-q^{2n}\big)\big(1+q^{2n-1}e^{2\pi i z}\big)\big(1+q^{2n-1}e^{-2\pi i z}\big) \quad \text{con} \quad q = e^{\pi i \tau}, \quad \Im\tau > 0$$

$$\theta(z+\tau) = q^{-1}e^{-2\pi i z}\theta(z) \qquad \text{y} \qquad \wp(z;1,\tau) = A_\tau\frac{\theta^2(z+1/2)}{e^{2\pi i z}\theta^2(z+\tau^*)} + B_\tau = -\Big(\frac{\theta'(z+\tau^*)}{\theta(z+\tau^*)}\Big)' + C_\tau.$$

Name ....

....

1) Decide whether the following claims are true or false writing in each case a brief justification

a) [1 point] If f is an <u>even</u> function which is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C}-\{0\}$, then $\int_{\{|z|=1\}} f(z)\, dz = 0$.

b) [1.5 points] There exists a non-identically zero polynomial P such that $P(\wp) = 0$.

c) [1.5 points] If $\mathcal{P}$ is a fundamental parallelogram with $0 \in \operatorname{Int}(\mathcal{P})$ and the function defined by $\sum_{\omega\in\Lambda}(z-\omega)^{-2018}$ does not have zeros on $\partial\mathcal{P}$, then it has 2018 zeros in $\operatorname{Int}(\mathcal{P})$ counted with multiplicities.

2) [3 points] For $f(z) = (z^2+1)\cos(\pi z)$, compute all the poles of $1/f$ and their residues.

3) [3 points] Prove the following identity due to Gauss:

$$\sum_{n\in\mathbb{Z}}(-1)^n x^{n^2} = \prod_{n=1}^{\infty}\frac{1-x^n}{1+x^n} \qquad \text{for} \quad -1 < x < 1.$$

Hint. Why $\prod(1+x^n)^{-1} = \prod(1-x^n)/(1-x^{2n}) = \prod(1-x^{2n-1})$ is "elementary" without appealing to the theory of elliptic functions?

Some formulas on elliptic functions

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\Big(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\Big) = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_{2k}z^{2k} \qquad \text{with} \qquad a_{2k} = (2k+1)\sum_{\omega\in\Lambda^*}\omega^{-2k-2}$$

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \qquad \text{with} \qquad g_2 = 60\sum_{\omega\in\Lambda^*}\omega^{-4}, \quad g_3 = 140\sum_{\omega\in\Lambda^*}\omega^{-6}$$

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2\pi i n z} = \prod_{n=1}^{\infty}\big(1-q^{2n}\big)\big(1+q^{2n-1}e^{2\pi i z}\big)\big(1+q^{2n-1}e^{-2\pi i z}\big) \quad \text{with} \quad q = e^{\pi i\tau}, \quad \Im\tau > 0$$

$$\theta(z+\tau) = q^{-1}e^{-2\pi i z}\theta(z) \qquad \text{and} \qquad \wp(z;1,\tau) = A_\tau\frac{\theta^2(z+1/2)}{e^{2\pi i z}\theta^2(z+\tau^*)} + B_\tau = -\Big(\frac{\theta'(z+\tau^*)}{\theta(z+\tau^*)}\Big)' + C_\tau.$$
