# On homological stability for configuration spaces on closed background manifolds

Federico Cantero Morán
Martin Palmer

Universität Münster

# Homological stability

We study the configuration space of $k$ points in a smooth manifold $M$:

$$C_k(M) = \{\mathbf{q} \subset M \mid |\mathbf{q}| = k\}.$$

If the manifold is open, a proper embedding of a half disc determines a map

$$s\colon C_k(M) \longrightarrow C_{k+1}(M)$$

that moves the configuration away the half disc and inserts a point in the disc.

**Theorem (McDuff'75-Segal'79, Church'12, Randal-Williams'13, Knudsen'14, Kupers-Miller'14)**

*The "stabilisation map" s is a homology iso with A-coeff. in the range $* \leqslant sr(k)$:*

| $sr(k) \geqslant$ | *coefficients* | *conditions* |
|---|---|---|
| $\frac{k}{2}$ | $A = \mathbb{Z}$ | *if* $\dim(M) \geqslant 2$, |
| $k$ | $A = \mathbb{Q}$ | *either* $\dim(M) \geqslant 3$ *or* $M$ *is non-orientable,* |
| $k-1$ | $A = \mathbb{Q}$ | *if* $\dim(M) = 2$ *and* $M$ *is orientable,* |
| $k$ | $A = \mathbb{Z}[\frac{1}{2}]$ | *if* $\dim(M) \geqslant 3$, |
| $\frac{2(p-1)}{p} - 1$ | $A = \mathbb{Z}_{(p)}$ | *if* $\dim(M) \geqslant 3$ *and* $H_1(M) = 0$. |

*Further improvements are possible under extra hypotheses.*

# Closed background manifolds

If the manifold is closed, the stabilisation map is not defined, so there is no direct way of comparing configuration spaces of different cardinality. In fact,

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# Torsion stability on closed background manifolds

**Theorem (Bendersky–Miller'13, C.–Palmer'14)**

*If $M^n$ is even dimensional, $* \leqslant sr(k)$ and either $\chi$ is even or $p \neq 2$, then*

- $H_*(C_k(M); \mathbb{Z}_{(p)})$ *depends only on the $p$-adic valuation of $2k - \chi$.*
  *Bendersky–Miller'13 for $p \geqslant \frac{n+3}{2}$*

*If $M^n$ is odd dimensional and $* \leqslant sr(k)$, then*

- $H_*(C_k(M); \mathbb{Z}[\frac{1}{2}])$ *is independent of $k$.*
  *Bendersky–Miller'13 for $\mathbb{Z}\left[\frac{1}{(\frac{n+3}{2})!}\right]$*
- $H_*(C_k(M); \mathbb{Z})$ *depends only on the parity of $k$.*

The isomorphisms are induced by a zig-zag of two maps.

# The Bendersky–Miller approach

There is a map (the "scanning map")

$$\mathscr{S} : C_k(M) \longrightarrow \Gamma_c(\dot{T}M)_k$$

to the space of degree $k$ compactly supported sections of the fibrewise one-point compactification of the tangent bundle of $M$.

**Theorem (McDuff'75)**

*The scanning map is an homology isomorphism in the stable range.*

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## Theorem (McDuff'75)

*The scanning map is an homology isomorphism in the stable range.*

Hence, if we are able to provide fibrewise endomorphisms $\phi\colon \dot{T}M \to \dot{T}M$ of degree $r$ invertible in $\mathbb{Z}_{(p)}$, then by the diagram

$$\begin{array}{ccccccc}
C_k(M) & \xrightarrow{\mathscr{S}} & \Gamma_c(\dot{T}M)_k & \xrightarrow[\simeq_{(p)}]{\text{localisation}} & (\Gamma_c(\dot{T}M)_k)_{(p)} & \xrightarrow[\simeq]{\text{Møller'87}} & \Gamma_c(\dot{T}M_{(p)})_k \\
 & & \downarrow \phi & & & & \simeq \downarrow \phi_{(p)} \\
C_j(M) & \xrightarrow{\mathscr{S}} & \Gamma_c(\dot{T}M)_j & \xrightarrow[\simeq_{(p)}]{\text{localisation}} & (\Gamma_c(\dot{T}M)_j)_{(p)} & \xrightarrow[\simeq]{\text{Møller'87}} & \Gamma_c(\dot{T}M_{(p)})_j
\end{array}$$

we have that $H_*(C_k(M);\mathbb{Z}_{(p)}) \cong H_*(C_j(M);\mathbb{Z}_{(p)})$.

# Existence of maps

We have to construct a section of the fibre bundle,

$$\mathrm{Map}^r(\dot{T}M, \dot{T}M) := \{(x, f_x) \mid x \in M, f_x \in \mathrm{Map}^r(\dot{T}_x M, \dot{T}_x M)\}$$

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and

- there is a bundle map

$$\begin{array}{ccc} V_2(\mathbb{R}^{n+1}) & \longrightarrow & \mathrm{Map}^r(S^n, S^n) \\ \downarrow & & \downarrow \\ V_2(TM \oplus \epsilon) & \longrightarrow & \mathrm{Map}^r(\dot{T}M, \dot{T}M) \\ & \searrow & \downarrow \\ & & M \end{array}$$

A point in $V_2(T_x M \oplus \mathbb{R})$ gives a splitting $T_x M \oplus \mathbb{R} \cong V \oplus \mathbb{R}^2$, and hence a splitting $S(T_x M \oplus \mathbb{R}) = S(V) * S^1$, from which we can define a map

$$\dot{T}_x M = S(T_x M \oplus \mathbb{R}) = S(V) * S^1 \overset{\mathrm{Id} * e^{2\pi i r}}{\longrightarrow} S(V) * S^1 = S(T_x M \oplus \mathbb{R}) = \dot{T}_x M$$

# The replication map

## Theorem (C.–Palmer'14)

*If the manifold $M$ admits a non-vanishing vector field $v\colon M \to S(TM)$, then the previous maps between spaces of sections can be upgraded to actual maps between configuration spaces making the following diagram commute:*

$$\begin{array}{ccc} C_k(M) & \xrightarrow{\mathscr{S}} & \Gamma_c(\dot{T}M)_k \\ \downarrow \rho_r & & \downarrow \text{fibrewise map of degree } r \\ C_{rk} & \xrightarrow{\mathscr{S}} & \Gamma_c(\dot{T}M)_j \end{array}$$

$$\{q_1, q_2, \dots, q_k\} \overset{\rho_r}{\longmapsto} \left\{ q_i + \frac{j\epsilon}{r} v \;\middle|\; \begin{array}{l} i = 1, \dots, k \\ j = 0, \dots, r-1 \end{array} \right\}$$

*Therefore, the replication map $\rho_r$ induces homology isomorphisms with $\mathbb{Z}[\frac{1}{r}]$-coefficients in the stable range.*

# Finite field stability on closed background manifolds

**Theorem (Löffler–Milgram'86, B–C–T'89, R-W'13, C.–Palmer'14)**

*If $M$ is even dimensional and $* \leqslant sr(k)$, then*

- $H_*(C_k(M); \mathbb{F}_2)$ *is independent of $k$ (Bödigheimer–Cohen–Taylor'89, R-W'13).*
- $H_*(C_k(M); \mathbb{F}_p)$ *depends only on* $\min\{\nu_p(2k-\chi), \nu_p(\chi/2)+1\}$.
- $H_*(C_k(M); \mathbb{F}_p)$ *is independent of $k$ if $\chi = 1$.*

*If $M$ is odd-dimensional and $* \leqslant sr(k)$, then*

- $H_*(C_k(M); \mathbb{F}_p)$ *is independent of $k$ (Bödigheimer–Cohen–Taylor'89, R-W'13).*

These are abstract isomorphisms.

If $D$ is a small disc in $M$, there is a homotopy cofibre sequence

$$C_k(M \setminus D) \longrightarrow C_k(M) \longrightarrow \frac{\dot{D} \times C_{k-1}(M \setminus D)}{\{*\} \times C_{k-1}(M \setminus D)} \simeq \Sigma^n C_{k+1}(M \setminus D)_+$$

where

- the first map is the inclusion,
- the second map divides the configuration into $k-1$ points outside $D$ and a point in the disc (only well defined on a certain subspace, to which we can restrict by excision).

If $D$ is a small disc in $M$, there is a partial map of exact sequences

$$\begin{array}{ccccccc}
\longrightarrow & H_i(C_k(M \setminus D);\mathbb{F}_p) & \longrightarrow & H_i(C_k(M);\mathbb{F}_p) & \longrightarrow & H_i(\Sigma^n C_{k-1}(M \setminus D)_+;\mathbb{F}_p) & \longrightarrow \\
 & \downarrow \rho_r & & & & \downarrow & \\
\longrightarrow & H_i(C_{rk}(M \setminus D);\mathbb{F}_p) & \longrightarrow & H_i(C_{rk}(M);\mathbb{F}_p) & \longrightarrow & H_i(\Sigma^n C_{rk-1}(M \setminus D)_+;\mathbb{F}_p) & \longrightarrow
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and if 1) both vertical maps are isomorphisms and 2) the square

$$\begin{array}{ccccccccc}
0 \to & \ker^k_{i+1} & \longrightarrow & H_{i+1}(\Sigma^n C_k(M\setminus D)_+;\mathbb{F}_p) & \longrightarrow & H_i(C_k(M\setminus D);\mathbb{F}_p) & \to & \mathrm{coker}^k_i & \to 0 \\
 & & & \downarrow & & \downarrow & & & \\
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commutes,then the dimension of both kernels and cokernels agree, hence

$$H_i(C_k(M);\mathbb{F}_p) \cong \ker^k_i \oplus \operatorname{coker}^k_i \cong \ker^{rk}_i \oplus \operatorname{coker}^{rk}_i \cong H_i(C_{rk}(M);\mathbb{F}_p)$$

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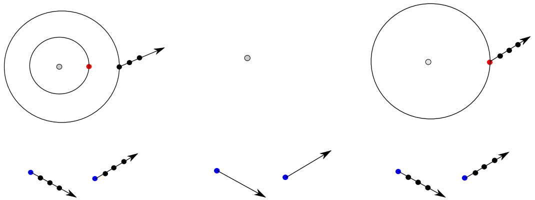
We will prove 1) if $p \nmid r$ and 2) if $p \mid (r-1)(\chi-1)$.

$$
\begin{array}{ccc}
S^{n-1} \times C_{k-1}(M \setminus D) & \xrightarrow{\text{add a point in the direction given by } S^{n-1}} & C_k(M \setminus D) \\
\Big\downarrow {\scriptstyle r\text{-replicate the } k-1 \text{ points}} & & \\
S^{n-1} \times C_{r(k-1)}(M \setminus D) & & \Big\downarrow {\scriptstyle r\text{-replicate the } k \text{ points}} \\
\Big\downarrow {\scriptstyle \text{Add a point in the direction given by } S^{n-1} \text{ and } (r-1)\text{-replicate it}} & & \\
S^{n-1} \times C_{rk-1}(M \setminus D) & \xrightarrow{\text{add a point in the direction given by } S^{n-1}} & C_{rk}(M \setminus D)
\end{array}
$$



The difference between both ways in the diagram is

$$(r-1)(\chi-1)\tau \in H_{n-1}(C_{rk}(M \setminus D)),$$

where $\tau$ is the class that swaps two points

# The two ways around the square

$\chi = 3 \qquad r = 6$

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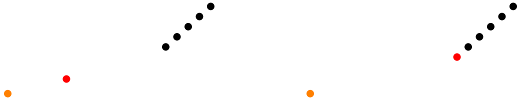

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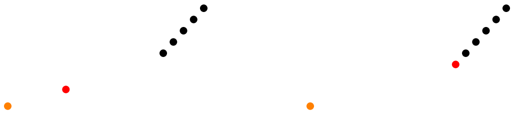

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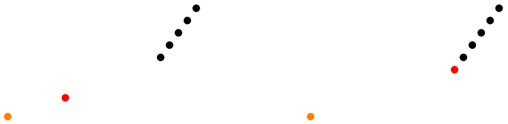

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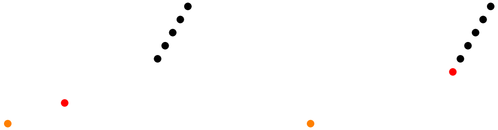

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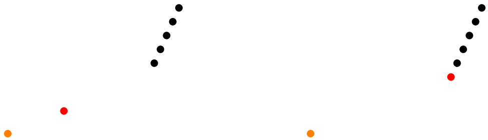

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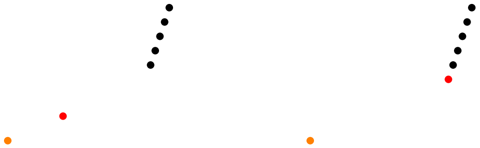

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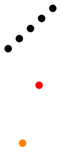

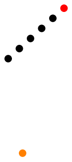

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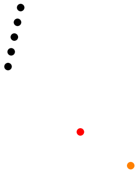

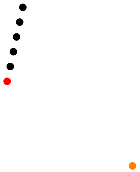

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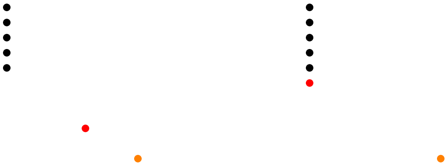

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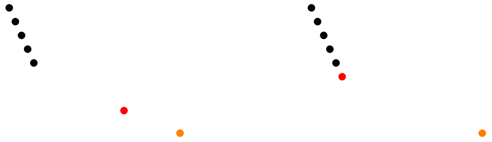

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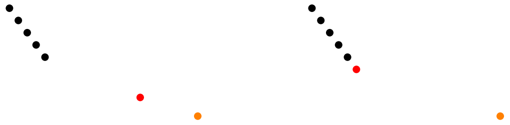

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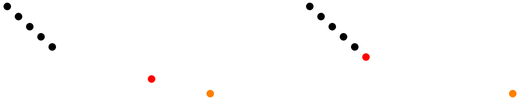

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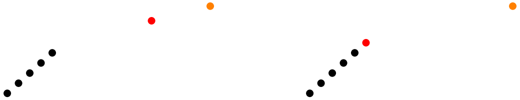

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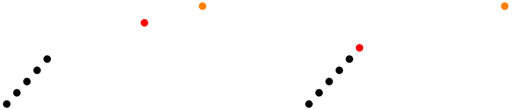

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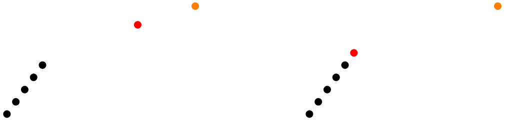

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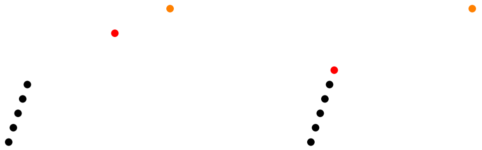

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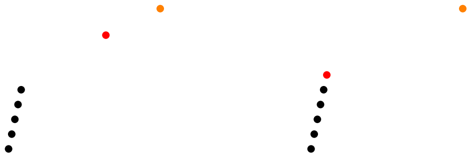

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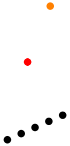

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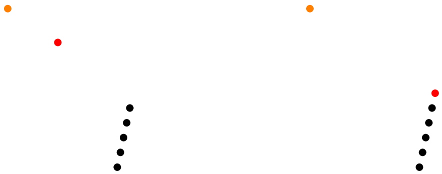

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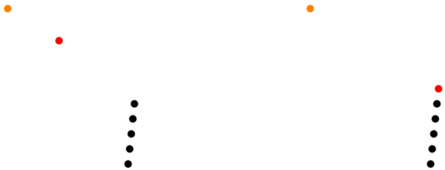

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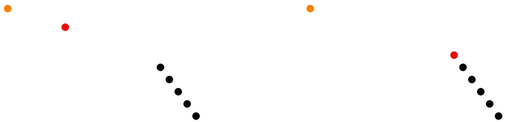

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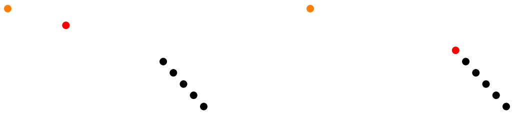

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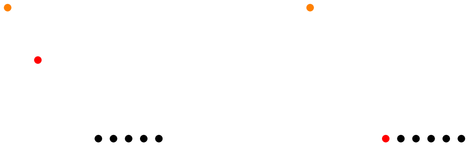

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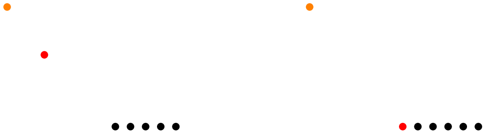

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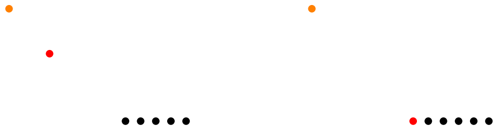

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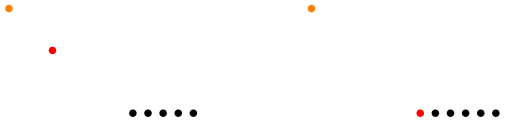

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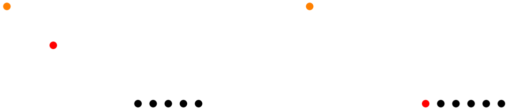

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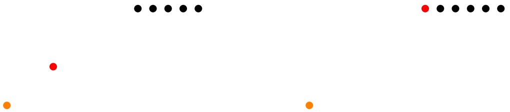

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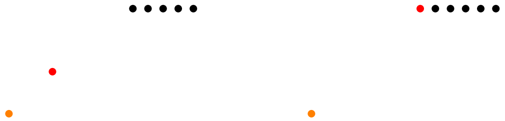

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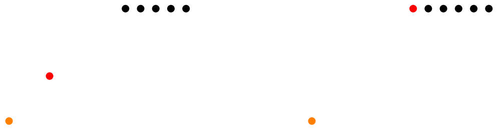

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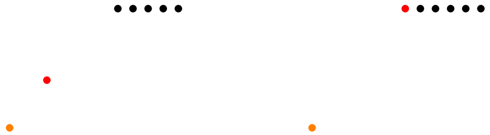

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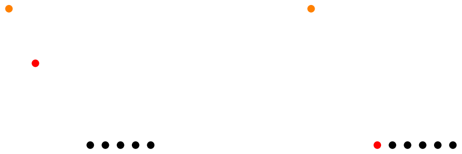

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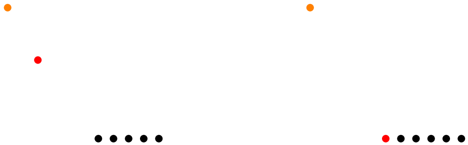

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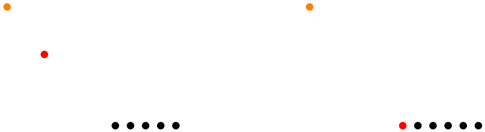

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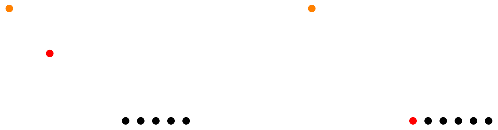

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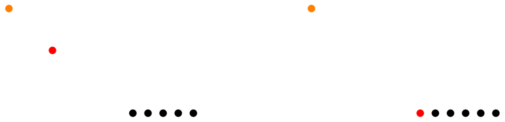

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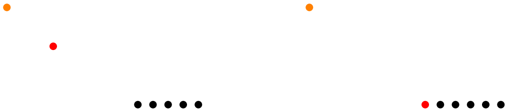

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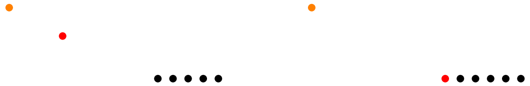

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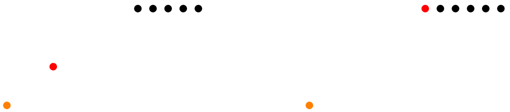

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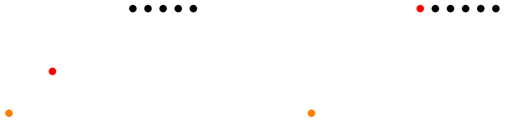

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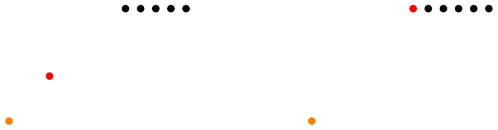

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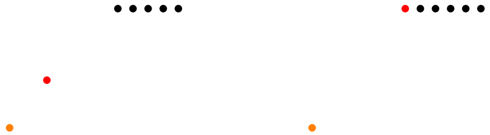

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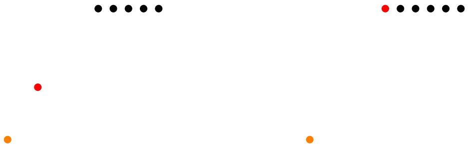

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