Stable homology of spaces of embedded surfaces: Closed background manifolds

Federico Cantero Morán
Universität Münster

January 14, 2014

The space of embedded surfaces in a manifold

Our purpose is to study the space of all *oriented* subsurfaces of a given manifold.

The space of embedded surfaces in a manifold

Our purpose is to study the space of all *oriented* subsurfaces of a given manifold. For this, we fix a compact, connected, *oriented* surface Σ_g and a "background" manifold M, and consider the set

$$\mathcal{E}_g(M) = \left\{ W \subset M \;\middle|\; \begin{array}{l} W \text{ is an } \textit{oriented} \text{ surface} \\ \text{in } M \text{ diffeomorphic to } \Sigma_g \end{array} \right\}$$

The space of embedded surfaces in a manifold

Our purpose is to study the space of all *oriented* subsurfaces of a given manifold. For this, we fix a compact, connected, *oriented* surface Σ_g and a "background" manifold M, and consider the set

$$\mathcal{E}_g(M) = \left\{ W \subset M \,\middle|\, \begin{array}{l} W \text{ is an \textit{oriented} surface} \\ \text{in } M \text{ diffeomorphic to } \Sigma_g \end{array} \right\}$$

We endow it with the quotient topology of the quotient map

$$\mathrm{Emb}(\Sigma_g, M) \longrightarrow \mathcal{E}_g(M)$$

given by sending an embedding f to its image $f(\Sigma_g)$.

The space of embedded surfaces in a manifold

Our purpose is to study the space of all *oriented* subsurfaces of a given manifold. For this, we fix a compact, connected, *oriented* surface Σ_g and a "background" manifold M, and consider the set

$$\mathcal{E}_g(M) = \left\{ W \subset M \,\middle|\, \begin{array}{l} W \text{ is an } \textit{oriented} \text{ surface} \\ \text{in } M \text{ diffeomorphic to } \Sigma_g \end{array} \right\}$$

We endow it with the quotient topology of the quotient map

$$\mathrm{Emb}(\Sigma_g, M) \longrightarrow \mathcal{E}_g(M)$$

given by sending an embedding f to its image $f(\Sigma_g)$.

We'll need to consider the space $\mathcal{E}_g^{\nu}(M)$ of pairs (W, u) where $W \in \mathcal{E}_g(M)$ and $u\colon NW \to U \subset M$ is a tubular neighbourhood.

The space of embedded surfaces in a manifold

Our purpose is to study the space of all *oriented* subsurfaces of a given manifold. For this, we fix a compact, connected, *oriented* surface Σ_g and a "background" manifold M, and consider the set

$$\mathcal{E}_g(M) = \left\{ W \subset M \;\middle|\; \begin{array}{l} W \text{ is an } \textit{oriented} \text{ surface} \\ \text{in } M \text{ diffeomorphic to } \Sigma_g \end{array} \right\}$$

We endow it with the quotient topology of the quotient map

$$\mathrm{Emb}(\Sigma_g, M) \longrightarrow \mathcal{E}_g(M)$$

given by sending an embedding f to its image $f(\Sigma_g)$.

We'll need to consider the space $\mathcal{E}_g^{\nu}(M)$ of pairs (W, u) where $W \in \mathcal{E}_g(M)$ and $u\colon NW \to U \subset M$ is a tubular neighbourhood. Forgeting the tubular neighbourhood gives a fibre bundle

$$\mathcal{E}_g^{\nu}(M) \longrightarrow \mathcal{E}_g(M)$$

which is also a weak homotopy equivalence.

Theorem A (C. – Randal-Williams)

If M is simply connected and of dimension at least 5, and $\partial M \neq \emptyset$, then the scanning map

$$\mathscr{S}_g \colon \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

induces an isomorphism in integral homology in degrees $k \leq \frac{2}{3}(g-1)$.

Theorem C (C. – Randal-Williams)

If M is simply connected and of dimension at least 5, then the scanning map

$$\mathscr{S}_g \colon \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

induces an isomorphism in integral homology in degrees $k \leq \frac{2}{3}(g-1)$.

The fibre bundle $\mathcal{S}(TM)$

From an inner product vector space V, we can construct the following:

- The Grassmannian of oriented linear 2-planes in V,

$$\mathrm{Gr}_2^+(V) = \{\text{oriented linear 2-planes in } V\}.$$

The fibre bundle $\mathcal{S}(TM)$

From an inner product vector space V, we can construct the following:

- The Grassmannian of oriented linear 2-planes in V,

$$\mathrm{Gr}_2^+(V) = \{\text{oriented linear 2-planes in } V\}.$$

- The Grassmannian of oriented affine 2-planes in V,

$$\begin{aligned}\gamma_2^\perp(V) &= \{\text{oriented affine 2-planes in } V\}\\ &= \{(P, v) \mid P \text{ is an oriented linear 2-plane and } v \in P^\perp\}.\end{aligned}$$

The fibre bundle $\mathcal{S}(TM)$

From an inner product vector space V, we can construct the following:

- The Grassmannian of oriented linear 2-planes in V,

$$\mathrm{Gr}_2^+(V) = \{\text{oriented linear 2-planes in } V\}.$$

- The Grassmannian of oriented affine 2-planes in V,

$$\begin{aligned}\gamma_2^\perp(V) &= \{\text{oriented affine 2-planes in } V\} \\ &= \{(P, v) \mid P \text{ is an oriented linear 2-plane and } v \in P^\perp\}.\end{aligned}$$

  Forgetting the vector v we obtain a vector bundle of rank $\dim V - 2$:

$$\gamma_2^\perp(V) \longrightarrow \mathrm{Gr}_2^+(V)$$

- The Thom space of this vector bundle,

$$\mathcal{S}(V) := \mathrm{Th}(\gamma_2^\perp(V) \to \mathrm{Gr}_2^+(V)).$$

The fibre bundle $\mathcal{S}(TM)$

Consider now a vector bundle $E \to M$ endowed with a metric.

Definition

The fibre bundle $\mathcal{S}(E) \to M$ is the result of applying the construction $\mathcal{S}$ fibrewise to the fibre bundle $E \to M$.

If E_p is the fibre of E over $p \in M$, then we obtain a fibre bundle

$$\mathcal{S}(E_p) \longrightarrow \mathcal{S}(E) \longrightarrow M.$$

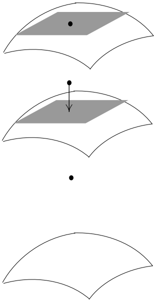
In particular, for the tangent bundle of a *Riemannian* manifold M, we obtain a fibre bundle

$$\mathcal{S}(T_pM) \longrightarrow \mathcal{S}(TM) \longrightarrow M.$$

The scanning map $\mathscr{S}_g\colon \mathcal{E}^{\nu}_g(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$

The scanning map approximates each oriented surface W ⊂ M with its tangent bundle.

$$
\begin{array}{ccc}
W & \xrightarrow{\;p \mapsto T_pW \subset T_pM\;} & \mathrm{Gr}_2^+(TM) \\
{\scriptstyle\pi}\uparrow\;\downarrow & & \downarrow \\
U & \xrightarrow{\;p \mapsto T_{\pi(p)}W \subset T_pM\;} & \gamma_2^{\perp}(TM) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\;p \mapsto \infty \in \mathcal{S}(T_pM)\;} & \mathcal{S}(TM).
\end{array}
$$



First, if $p \in W$, we have the Gauss map. Second, if $\pi\colon U \to W \subset U$ is a tubular neighbourhood of W, we can identify T_pM as a translation of $T_{\pi(p)}M$, and $T_{\pi(p)}W$ as an affine subspace of T_pM. Third, we may send any other point to the point at infinity (interpreted as the empty subspace).

The scanning map

We have obtained the *scanning map*:

$$\begin{aligned}\mathscr{S}_g \colon \mathcal{E}_g^{\nu}(M) &\longrightarrow \Gamma_c(\mathcal{S}(TM) \longrightarrow M)\\ (W, u) &\longmapsto \mathscr{S}_g(W, u).\end{aligned}$$

We have obtained the *scanning map*:

$$\mathscr{S}_g : \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \longrightarrow M)$$
$$(W, u) \longmapsto \mathscr{S}_g(W, u).$$

Lemma

If M is simply connected and of dimension at least 5, then

$$\pi_0(\Gamma_c(\mathcal{S}(TM) \to M)) \cong H_2(M; \mathbb{Z}) \times 2\mathbb{Z}.$$

The scanning map

We have obtained the *scanning map*:

$$\mathscr{S}_g\colon \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \longrightarrow M)$$
$$(W,u) \longmapsto \mathscr{S}_g(W,u).$$

Lemma

If M is simply connected and of dimension at least 5, then

$$\pi_0(\Gamma_c(\mathcal{S}(TM) \to M)) \cong H_2(M;\mathbb{Z}) \times 2\mathbb{Z}.$$

The space of compactly supported genus g sections $\Gamma_c(\mathcal{S}(TM) \to M)_g$ is the union of those components labeled by $H_2(M;\mathbb{Z}) \times \{2-2g\}$.

The scanning map

We have obtained the *scanning map*:

$$\mathscr{S}_g : \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \longrightarrow M)$$
$$(W, u) \longmapsto \mathscr{S}_g(W, u).$$

Lemma

If M is simply connected and of dimension at least 5, then

$$\pi_0(\Gamma_c(\mathcal{S}(TM) \to M)) \cong H_2(M; \mathbb{Z}) \times 2\mathbb{Z}.$$

The space of compactly supported genus g sections $\Gamma_c(\mathcal{S}(TM) \to M)_g$ is the union of those components labeled by $H_2(M; \mathbb{Z}) \times \{2 - 2g\}$.

Lemma

The image of $\mathscr{S}_g$ is contained in $\Gamma_c(\mathcal{S}(TM) \to M)_g$.

Theorem A (C. – Randal-Williams)

If M is simply connected and of dimension at least 5, and $\partial M \neq \emptyset$, then the scanning map

$$\mathscr{S}_g \colon \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

induces an isomorphism in integral homology in degrees $k \leq \frac{2}{3}(g-1)$.

Theorem C (C. – Randal-Williams)

If M is simply connected and of dimension at least 5, then the scanning map

$$\mathscr{S}_g \colon \mathcal{E}_g^{\nu}(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

induces an isomorphism in integral homology in degrees $k \leq \frac{2}{3}(g-1)$.

Relation to previous works

$B\Sigma_n$	$\mathrm{C}_n(M) := \mathrm{Emb}([n], M)/\Sigma_n$
Thm B Nakaoka '60 **Thm A** Barratt–Priddy '72	**Thm B** McDuff '75 **Thm A** McDuff '75
$B\mathrm{Diff}^+(\Sigma_g)$ **Thm B** Harer '85 **Thm A** Madsen–Weiss '07	$\mathcal{E}_g(M) := \mathrm{Emb}(\Sigma_g, M)/\mathrm{Diff}^+(\Sigma_g)$

Thm B Martin Palmer: Stability for embedded disconnected submanifolds.

Resolutions I

Definition

A semi-simplicial space $X_\bullet$ is a simplicial space without degeneracies, that is, a functor $X_\bullet \colon \Delta_{\mathrm{inj}} \to \mathrm{Spaces}$ from the full subcategory $\Delta_{\mathrm{inj}} \subset \Delta$ whose morphisms are the inclusions. A maps of semi-simplicial spaces is a natural transformation.

As usual, we denote by X_n the image of $[n]$.

Resolutions I

Definition

A semi-simplicial space $X_\bullet$ is a simplicial space without degeneracies, that is, a functor $X_\bullet \colon \Delta_{\mathrm{inj}} \to \mathrm{Spaces}$ from the full subcategory $\Delta_{\mathrm{inj}} \subset \Delta$ whose morphisms are the inclusions. A maps of semi-simplicial spaces is a natural transformation.

As usual, we denote by X_n the image of $[n]$.

Definition

An augmented semi-simplicial space is a triple consisting of

- *a space X,*
- *a semi-simplicial space $X_\bullet$ and*
- *a map $\epsilon \colon X_0 \to X$ (called augmentation) that equalizes the face maps $\partial_0 \colon X_1 \to X_0$ and $\partial_1 \colon X_1 \to X_0$.*

We denote by $\epsilon_i \colon X_i \to X$ the unique composition of face maps and ϵ. A map between augmented semi-simplicial spaces is a pair $(X \to Y, X_\bullet \to Y_\bullet)$ that commutes with the augmentation maps.

An augmented semi-simplicial space $(X, X_\bullet, \epsilon)$ is the same as a map from $X_\bullet$ to the constant semi-simplicial space X whose face maps are identities.

Resolutions I

Definition

A semi-simplicial space $X_\bullet$ is a simplicial space without degeneracies, that is, a functor $X_\bullet\colon \Delta_{\mathrm{inj}} \to \mathrm{Spaces}$ from the full subcategory $\Delta_{\mathrm{inj}} \subset \Delta$ whose morphisms are the inclusions. A maps of semi-simplicial spaces is a natural transformation.

Definition

An augmented semi-simplicial space is a triple consisting of

- *a space X,*
- *a semi-simplicial space $X_\bullet$ and*
- *a map $\epsilon\colon X_0 \to X$ (called augmentation) that equalizes the face maps $\partial_0\colon X_1 \to X_0$ and $\partial_1\colon X_1 \to X_0$.*

We denote by $\epsilon_i\colon X_i \to X$ the unique composition of face maps and ϵ. A map between augmented semi-simplicial spaces is a pair $(X \to Y, X_\bullet \to Y_\bullet)$ that commutes with the augmentation maps.

An augmented semi-simplicial space $(X, X_\bullet, \epsilon)$ is the same as a map from $X_\bullet$ to the constant semi-simplicial space X whose face maps are identities.

Resolutions II

Example (Hatcher, *Algebraic Topology*)

A semi-simplicial space with values in discrete spaces (aka sets) is called a Δ-set.

Resolutions II

Example (Hatcher, *Algebraic Topology*)

A semi-simplicial space with values in discrete spaces (aka sets) is called a Δ-set.

There is a functor (the *realization*)

$$\| - \| : \text{Semi-simplicial spaces} \longrightarrow \text{Spaces},$$

that sends the constant semi-simplicial space X to X, hence an augmentation map $X_0 \to X$ induces a map $\|X_\bullet\| \to X$, which we call *realized augmentation*.

Resolutions II

Example (Hatcher, *Algebraic Topology*)

A semi-simplicial space with values in discrete spaces (aka sets) is called a Δ-set.

There is a functor (the *realization*)

$$\| - \| : \text{Semi-simplicial spaces} \longrightarrow \text{Spaces},$$

that sends the constant semi-simplicial space X to X, hence an augmentation map $X_0 \to X$ induces a map $\|X_\bullet\| \to X$, which we call *realized augmentation*.

Definition

We say that a semi-simplicial space $X_\bullet$ is a resolution of a space X if $X_\bullet$ is augmented over X and the realized augmentation is a weak homotopy equivalence. A resolution of a map $f : X \to Y$ is a pair $X_\bullet, Y_\bullet$ of resolutions of X, Y and a map $f_\bullet : X_\bullet \to Y_\bullet$ that extends the map f.

Techniques I: How to prove that something is a resolution

Let $(X, X_\bullet, \epsilon)$ be an augmented semi-simplicial space.

Lemma

If $x \in X$, then there is a homotopy fibre sequence

$$\|\mathrm{hofib}_x(\epsilon_\bullet)\| \to \|X_\bullet\| \to X.$$

Techniques I: How to prove that something is a resolution

Let $(X, X_\bullet, \epsilon)$ be an augmented semi-simplicial space.

Lemma

If $x \in X$, then there is a homotopy fibre sequence

$$\|\mathrm{hofib}_x(\epsilon_\bullet)\| \to \|X_\bullet\| \to X.$$

We say that $(X, X_\bullet, \epsilon)$ is an augmented *topological flag complex* if in addition

- the product map $X_i \to X_0 \times_X \ldots \times_X X_0$ is an open embedding;
- a tuple $(x_0, \ldots, x_i)$ is in $X_i \Leftrightarrow (x_j, x_k) \in X_1$ for all $0 \leq j < k \leq i$.

Techniques I: How to prove that something is a resolution

Let $(X, X_\bullet, \epsilon)$ be an augmented semi-simplicial space.

Lemma

If $x \in X$, then there is a homotopy fibre sequence

$$\|\mathrm{hofib}_x(\epsilon_\bullet)\| \to \|X_\bullet\| \to X.$$

We say that $(X, X_\bullet, \epsilon)$ is an augmented *topological flag complex* if in addition

- the product map $X_i \to X_0 \times_X \ldots \times_X X_0$ is an open embedding;
- a tuple $(x_0, \ldots, x_i)$ is in $X_i \Leftrightarrow (x_j, x_k) \in X_1$ for all $0 \le j < k \le i$.

Lemma (Galatius–Randal-Williams '12)

Suppose in addition that

1. $\epsilon \colon X_0 \to X$ *has local sections;*
2. *given any finite collection* $\{x_1, \ldots x_n\} \subset X_0$ *in a single fibre of* ϵ *over some* $x \in X$*, there is a* x_∞ *in that fibre such that each* $(x_j, x_\infty) \in X_1$*.*

Then $\|\epsilon_\bullet\| \colon \|X_\bullet\| \to X$ *is a weak homotopy equivalence.*

Techniques II: How to prove that something is a fibration

Definition (Palais '60, Cerf '61)

If G is a (topological) group acting on X, we say that X is G-locally retractile if, for each point $x \in X$, the orbit map $G \times \{x\} \to X$ that sends $g \mapsto g \cdot x$ has local sections (in the weak sense).

Techniques II: How to prove that something is a fibration

Definition (Palais '60, Cerf '61)

If G is a (topological) group acting on X, we say that X is G-locally retractile if, for each point $x \in X$, the orbit map $G \times \{x\} \to X$ that sends $g \mapsto g \cdot x$ has local sections (in the weak sense).

Lemma (Palais '60, Cerf '61)

If X and Y are G-spaces, and $f : X \to Y$ is G-equivariant and Y is G-locally retractile, then f is a locally trivial fibration.

Techniques II: How to prove that something is a fibration

Definition (Palais '60, Cerf '61)

If G is a (topological) group acting on X, we say that X is G-locally retractile if, for each point $x \in X$, the orbit map $G \times \{x\} \to X$ that sends $g \mapsto g \cdot x$ has local sections (in the weak sense).

Lemma (Palais '60, Cerf '61)

If X and Y are G-spaces, and $f : X \to Y$ is G-equivariant and Y is G-locally retractile, then f is a locally trivial fibration.

Proposition (Palais '60, Cerf '61, Lima '63, Binz–Fischer '81)

The space of embeddings of a compact manifold into a manifold M and the space $\mathcal{E}_g(M)$ are $\mathrm{Diff}(M)$-locally retractile.

Techniques III: Homology connectivity

Lemma

If $X_\bullet \to X$ is an m-resolution, X_i is homologically $(n-i)$-connected, and $m \geq n$, then X is homologically n-connected.

Techniques III: Homology connectivity

Lemma

If $X_\bullet \to X$ is an m-resolution, X_i is homologically $(n-i)$-connected, and $m \geq n$, then X is homologically n-connected.

Lemma

If a bundle map over B

$$\begin{array}{ccc} F_p & \longrightarrow & F'_p \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

satisfies that for each $p \in B$ the induced map of fibres $F_p \to F'_p$ is homologically k-connected, then the map between total spaces is also homologically k-connected.

Proof: The two steps

1. construct **resolutions** of the source and target of the scanning map

$$\mathcal{F}_g(M)_\bullet \longrightarrow \mathcal{E}_g^\nu(M), \qquad \mathcal{G}_g(M)_\bullet \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

and a resolution of the scanning map

$$\begin{array}{ccc} \mathcal{F}_g(M)_\bullet & \longrightarrow & \mathcal{G}_g(M)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}_g^\nu(M) & \longrightarrow & \Gamma_c(\mathcal{S}(TM) \to M)_g. \end{array}$$

Proof: The two steps

1. construct **resolutions** of the source and target of the scanning map

$$\mathcal{F}_g(M)_\bullet \longrightarrow \mathcal{E}_g^\nu(M), \qquad \mathcal{G}_g(M)_\bullet \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

and a resolution of the scanning map

$$\begin{array}{ccc} \mathcal{F}_g(M)_\bullet & \longrightarrow & \mathcal{G}_g(M)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}_g^\nu(M) & \longrightarrow & \Gamma_c(\mathcal{S}(TM) \to M)_g. \end{array}$$

2. Construct vertical maps (called **approximations**)

$$\begin{array}{ccc} \mathcal{E}_g^\nu(M \setminus \{p_1, \ldots, p_i\}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \{p_1, \ldots, p_i\}) \to M \setminus \{p_1, \ldots, p_i\})_g \\ \downarrow & & \downarrow \\ \mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \end{array}$$

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically $\frac{2}{3}(g-1)$-connected.

Proof: Resolution of $\mathcal{E}_g^\nu(M)$

Let $\mathcal{F}_g(M)_i$ be the space of tuples $(W, a, d_0, \dots, d_i)$ where

1. $(W, u) \in \mathcal{E}_g^\nu(M)$
2. $d_0, \dots, d_i \colon D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all j.

These spaces form a semi-simplicial space $\mathcal{F}_g(M)_\bullet$ where the jth face map forgets the jth disc, and there is an augmentation to $\mathcal{E}_g^\nu(M)$ that forgets all the discs.

Proof: Resolution of $\mathcal{E}_g^\nu(M)$

Let $\mathcal{F}_g(M)_i$ be the space of tuples $(W, a, d_0, \dots, d_i)$ where

1. $(W, u) \in \mathcal{E}_g^\nu(M)$
2. $d_0, \dots, d_i \colon D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all j.

These spaces form a semi-simplicial space $\mathcal{F}_g(M)_\bullet$ where the jth face map forgets the jth disc, and there is an augmentation to $\mathcal{E}_g^\nu(M)$ that forgets all the discs.

Proposition

$\mathcal{F}_g(M)$ is a resolution of $\mathcal{E}_g^\nu(M)$.

Proof: Resolution of $\mathcal{E}_g^\nu(M)$

Let $\mathcal{F}_g(M)_i$ be the space of tuples $(W, a, d_0, \ldots, d_i)$ where

1. $(W, u) \in \mathcal{E}_g^\nu(M)$
2. $d_0, \ldots, d_i \colon D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all j.

These spaces form a semi-simplicial space $\mathcal{F}_g(M)_\bullet$ where the jth face map forgets the jth disc, and there is an augmentation to $\mathcal{E}_g^\nu(M)$ that forgets all the discs.

Proposition

$\mathcal{F}_g(M)$ is a resolution of $\mathcal{E}_g^\nu(M)$.

Proof.

Let $\mathcal{F}'_g(M)_\bullet$ the semi-simplicial space defined as $\mathcal{F}_g(M)_\bullet$, except that the embeddings are only required to be disjoint at the centers of the discs. Then

- the inclusion $\mathcal{F}_g(M)_\bullet \subset \mathcal{F}'_g(M)_\bullet$ is a levelwise equivalence.
- $\mathcal{F}'_g(M)_\bullet$ is a topological flag complex augmented over $\mathcal{E}_g^\nu(M)$.
- $\mathcal{F}'_g(M)_\bullet$ satisfies the conditions of our lemma on topological flag complexes, hence is a resolution. □

Proof: Resolution of $\Gamma_c(\mathcal{S}(TM) \to M)_g$

Let $\mathcal{G}_g(M)_i$ be the space of tuples $(f, d_0, \ldots, d_i, h_0, \ldots, h_i)$ where

1. $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$;
2. $d_0, \ldots, d_i \colon D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all j.
3. $h_0, \ldots, h_i$ are smooth homotopies of sections of $d_j^*(\mathcal{S}(TM))$, constant near the boundary, and such that

$$h_j(x, 0) = f \circ d_j, \qquad h_j(0, 1) = \infty.$$

The jth face map forgets d_j and h_j, and there is an augmentation to $\Gamma_c(\mathcal{S}(TM) \to M)_g$ by forgetting all discs and homotopies.

Proof: Resolution of $\Gamma_c(\mathcal{S}(TM) \to M)_g$

Let $\mathcal{G}_g(M)_i$ be the space of tuples $(f, d_0, \ldots, d_i, h_0, \ldots, h_i)$ where

1. $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$;
2. $d_0, \ldots, d_i \colon D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all j.
3. $h_0, \ldots, h_i$ are smooth homotopies of sections of $d_j^*(\mathcal{S}(TM))$, constant near the boundary, and such that

$$h_j(x, 0) = f \circ d_j, \qquad h_j(0, 1) = \infty.$$

The jth face map forgets d_j and h_j, and there is an augmentation to $\Gamma_c(\mathcal{S}(TM) \to M)_g$ by forgetting all discs and homotopies.

Proposition

$\mathcal{G}_g(M)_\bullet$ is a resolution of $\Gamma_c(\mathcal{S}(TM) \to M)_g$.

Proof.

□

Proof: Resolution of the scanning map

We can extend the scanning map to a map of resolutions:

$$\begin{array}{ccc} \mathcal{F}_g(M)_\bullet & \longrightarrow & \mathcal{G}_g(M)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}_g^\nu(M) & \longrightarrow & \Gamma_c(\mathcal{S}(TM) \to M)_g \end{array}$$

by sending a tuple $(W, u, d_0, \dots, d_i)$ to $(\mathscr{S}(W, u), d_0, \dots, d_i, h_0, \dots, h_i)$, where h_j are constant homotopies.

Proof: First step accomplished

1. construct **resolutions** of the source and target of the scanning map

$$\mathcal{F}_g(M)_\bullet \longrightarrow \mathcal{E}_g^\nu(M), \qquad \mathcal{G}_g(M)_\bullet \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

and a resolution of the scanning map

$$\begin{array}{ccc} \mathcal{F}_g(M)_\bullet & \longrightarrow & \mathcal{G}_g(M)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}_g^\nu(M) & \longrightarrow & \Gamma_c(\mathcal{S}(TM) \to M)_g . \end{array}$$

2. Construct vertical maps (called **approximations**)

$$\begin{array}{ccc} \mathcal{E}_g^\nu(M \setminus \{p_1, \dots, p_i\}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \{p_1, \dots, p_i\}) \to M \setminus \{p_1, \dots, p_i\})_g \\ \downarrow & & \downarrow \\ \mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \end{array}$$

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically $\frac{2}{3}(g-1)$-connected.

Proof: The approximation maps

Forgeting the surface + tubular neighbourhood or the section defines a pair of maps

$$\begin{array}{ccc} \mathcal{F}_g(M)_i & \qquad & \mathcal{G}_g(M)_i \\ \big\downarrow & & \big\downarrow \\ C_i(M) & & C_i(M), \end{array}$$

to the space $C_i(M) := \mathrm{Emb}([i] \times D^d, M)$.

Proof: The approximation maps

Forgeting the surface W + tubular neighbourhood or the section gives homotopy fibre sequences

$$\begin{array}{ccc} \mathcal{E}^{\nu}_g(M \setminus \mathbf{p}) & & \Gamma_c(\mathcal{S}(TM \setminus \mathbf{p}) \to M \setminus \mathbf{p})_g \\ \downarrow & & \downarrow \\ \mathcal{F}_g(M)_i & & \mathcal{G}_g(M)_i \\ \downarrow & & \downarrow \\ C_i(M) & & C_i(M), \end{array}$$

to the space $C_i(M) := \mathrm{Emb}([i] \times D^d, M)$. The fibre is taken over the point $(d_0, \dots, d_j)$ and $\mathbf{p} = \{d_0(0), \dots, d_i(0)\}$.

Proof: The approximation maps

Forgeting the surface + tubular neighbourhood or the section defines a pair of maps

$$\begin{array}{ccc}
\mathcal{E}^{\nu}_{g}(M \setminus \mathbf{p}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \mathbf{p}) \to M \setminus \mathbf{p})_g \\
\downarrow & & \downarrow \\
\mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \\
\downarrow & & \downarrow \\
C_i(M) & = & C_i(M),
\end{array}$$

to the space $C_i(M) := \mathrm{Emb}([i] \times D^d, M)$. The fibre is taken over the point $(d_0, \dots, d_j)$ and $\mathbf{p} = \{d_0(0), \dots, d_i(0)\}$.

Proof: The approximation maps

Forgeting the surface + tubular neighbourhood or the section defines a pair of maps

$$\begin{array}{ccc}
\mathcal{E}^{\nu}_g(M \setminus \mathbf{p}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \mathbf{p}) \to M \setminus \mathbf{p})_g \\
\downarrow & & \downarrow \\
\mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \\
\downarrow & & \downarrow \\
C_i(M) & = & C_i(M),
\end{array}$$

to the space $C_i(M) := \mathrm{Emb}([i] \times D^d, M)$. The fibre is taken over the point $(d_0, \ldots, d_j)$ and $\mathbf{p} = \{d_0(0), \ldots, d_i(0)\}$.

The scanning map commutes with the map between spaces of i-simplices.

Proof: The approximation maps

Forgeting the surface + tubular neighbourhood or the section defines a pair of maps

$$\begin{array}{ccc}
\mathcal{E}^{\nu}_g(M \setminus \mathbf{p}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \mathbf{p}) \to M \setminus \mathbf{p})_g \\
\downarrow & & \downarrow \\
\mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \\
\downarrow & & \downarrow \\
C_i(M) & = & C_i(M),
\end{array}$$

to the space $C_i(M) := \mathrm{Emb}([i] \times D^d, M)$. The fibre is taken over the point $(d_0, \dots, d_j)$ and $\mathbf{p} = \{d_0(0), \dots, d_i(0)\}$.
The scanning map commutes with the map between spaces of i-simplices.

Corollary

Since the scanning map on the fibres is a homology isomorphism in degrees $* \leq \frac{2}{3}(g-1)$, it follows from a previous lemma that the map between total spaces is a homology isomorphism in those degrees.

Proof: Second step accomplished

1. construct resolutions of the source and target of the scanning map

$$\mathcal{F}_g(M)_\bullet \longrightarrow \mathcal{E}^\nu_g(M), \qquad \mathcal{G}_g(M)_\bullet \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

and a resolution of the scanning map

$$\begin{array}{ccc} \mathcal{F}_g(M)_\bullet & \longrightarrow & \mathcal{G}_g(M)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E}^\nu_g(M) & \longrightarrow & \Gamma_c(\mathcal{S}(TM) \to M)_g. \end{array}$$

2. Construct a map of pairs (called *approximation*)

$$\begin{array}{ccc} \mathcal{E}^\nu_g(M \setminus \{p_1, \ldots, p_i\}) & \longrightarrow & \Gamma_c(\mathcal{S}(TM \setminus \{p_1, \ldots, p_i\}) \to M \setminus \{p_1, \ldots, p_i\})_g \\ \downarrow & & \downarrow \\ \mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \end{array}$$

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically $\frac{2}{3}(g-1)$-connected.