The Zak Transform(s)

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1 Introduction

We introduce the operator Z that is often called the *Zak transform*. Our definition is a bit different from the one that usually appears in the literature. We will discuss this difference and will also give a historical account that the reader may find particularly interesting. In order to do this, however, we need to present our treatment of the operator Z (and $\tilde{Z}$) which shows that the Fourier transform and its inverse are unitary as an immediate consequence of the basic properties of Fourier series.

The operator Z maps each $f \in L^2(\mathbf{R})$ into the function

$$(Zf)(x,\xi) = \sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i} k\xi} \equiv \varphi(x,\xi)\;, \tag{1}$$

$x, \xi \in \mathbf{R}$. Let us explain the meaning of this equality. Since $f \in L^2(\mathbf{R})$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k\in\mathbf{Z}} |f(x+k)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x < \infty\;.$$

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Thus,

$$\sum_{k\in\mathbf{Z}} |f(x+k)|^2 < \infty \tag{2}$$

for a.e. $x \in \mathbf{R}$. This means that for a.e. $x \in \mathbf{R}$ the series in (1) is the Fourier series of a function in $L^2([-\frac{1}{2},\frac{1}{2}\rangle)$ (considered to be 1-periodic in ξ) we denote by $\varphi(x,\xi)$. Moreover, for a.e. $x \in \mathbf{R}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x,\xi)|^2 \mathrm{d}\xi = \sum_{k\in\mathbf{Z}} |f(x+k)|^2 .$$

It follows, therefore, that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x,\xi)|^2 \mathrm{d}\xi \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k\in\mathbf{Z}} |f(x+k)|^2 \mathrm{d}x = \|f\|_2^2 . \tag{3}$$

Z, therefore, maps $L^2(\mathbf{R})$ isometrically into a space of functions

$$\varphi \in L^2(\mathbf{T}^2) = L^2([-\frac{1}{2},\frac{1}{2}\rangle \times [-\frac{1}{2},\frac{1}{2}\rangle) .$$

Let us examine the space M of these images $\varphi(x,\xi) = (Zf)(x,\xi)$ of $L^2(\mathbf{R})$ under the transformation Z. We have seen that these images are functions of two real variables. Equality (3) asserts that the "norm" of φ involves only the variables $(x,\xi) \in \mathbf{T}^2$. The definition (1) indicates that $\varphi(x,\xi)$ should be 1-periodic in ξ. With respect to the variable x we have the easily established property

$$\varphi(x+j,\xi) = \mathrm{e}^{2\pi \mathrm{i} j\xi} \varphi(x,\xi) \tag{4}$$

for each $j \in \mathbf{Z}$. The property (4) tells us how $\varphi(x,\xi)$, for $x \in [-\frac{1}{2},\frac{1}{2}\rangle$ and $\xi \in \mathbf{R}$, extends to all $x \in \mathbf{R}$. This shows that $|\varphi(x,\xi)|$ is 1-periodic in each of the variables $x,\xi \in \mathbf{R}$.

It is also easy to show that Z maps $L^2(\mathbf{R})$ onto $L^2(\mathbf{T}^2)$. Let $\varphi \in L^2(\mathbf{T}^2)$. Then $\int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x,\xi)|^2 d\xi < \infty$ for a.e. $x \in [-\frac{1}{2},\frac{1}{2}\rangle$. For each such x, φ, as a function of ξ, is a member of $L^2(\mathbf{T})$; thus, for a.e. $x \in [-\frac{1}{2},\frac{1}{2}\rangle$, $\varphi(x,\xi)$ has a Fourier series

$$\varphi(x,\xi) \sim \sum_{k\in\mathbf{Z}} c_k(x) \mathrm{e}^{-2\pi ik\xi}$$

such that $\sum_{k\in\mathbf{Z}} |c_k(x)|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x,\xi)|^2 \mathrm{d}\xi < \infty$. We then define a function f on $\mathbf{R}$ by letting $f(x+k) = c_k(x)$ for each $k \in \mathbf{Z}$ and these $x \in [-\frac{1}{2},\frac{1}{2}\rangle$. This a.e. defined function on $\mathbf{R}$ belongs to $L^2(\mathbf{R})$ since

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \sum_{k\in\mathbf{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |c_k(x)|^2 \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x,\xi)|^2 \mathrm{d}\xi \mathrm{d}x < \infty.$$

This shows that $\varphi(x,\xi) = (Zf)(x,\xi)$ for $(x,\xi) \in \mathbf{T}^2$. We then extend $\varphi(x,\xi)$ for all $x,\xi \in \mathbf{R}$ by making it 1-periodic in ξ and using (4) when $x \in [-\frac{1}{2},\frac{1}{2}\rangle$ and $j \in \mathbf{Z}$.

We have, in fact, obtained the space M of all those measurable functions $\varphi(x,\xi)$, $(x,\xi) \in \mathbf{R}^2$, that are the general $\varphi \in L^2(\mathbf{T}^2)$ when (x,ξ) is restricted to $\mathbf{T}^2$, are 1-periodic in ξ and satisfy (4). Moreover, if we introduce the norm

$$\|\varphi\|_M \equiv (\int_{\mathbf{T}^2} |\varphi(x,\xi)|^2 \mathrm{d}\xi \mathrm{d}x)^{\frac{1}{2}}$$

on M we have obtained

Theorem 1. *The linear operator Z maps $L^2(\mathbf{R})$ isometrically onto M.*

We now introduce the space $\tilde{M}$, a "companion" space to M. Essentially $\tilde{M}$ is obtained from M by reversing the roles of the variables x and ξ. More precisely, $\tilde{M}$ consists of all functions $\tilde{\varphi}(x,\xi)$ on $\mathbf{R}^2$ that belong to $L^2(\mathbf{T}^2)$ when (x,ξ) is restricted to $\mathbf{T}^2$, $\tilde{\varphi}$ is 1-periodic in x and its value for the general $\xi \in \mathbf{R}$ is given by equality

$$\tilde{\varphi}(x,\xi+\ell) = \mathrm{e}^{-2\pi \mathrm{i} \ell x}\tilde{\varphi}(x,\xi) \ . \tag{5}$$

The norm of $\tilde{\varphi} \in \tilde{M}$ is, again, obtained from $L^2(\mathbf{T}^2)$:

$$\|\tilde{\varphi}\|_{\tilde{M}} = (\int_{\mathbf{T}^2} |\tilde{\varphi}(x,\xi)|^2 \mathrm{d}\xi \mathrm{d}x)^{\frac{1}{2}} \ .$$

There is a simple unitary map U from M onto $\tilde{M}$ (these two spaces are, clearly, Hilbert spaces):

$$(U\varphi)(x,\xi) = \mathrm{e}^{-2\pi \mathrm{i} x\xi}\varphi(x,\xi) \equiv \tilde{\varphi}(x,\xi) \ . \tag{6}$$

Let us explain why U maps M onto $\tilde{M}$. The general element, φ, of M, we have seen, is the Zak transform of an $f \in L^2(\mathbf{R}) : \varphi(x,\xi) = (Zf)(x,\xi) = \sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i} k\xi}$. Thus,

$$(U\varphi)(x,\xi) = \mathrm{e}^{-2\pi \mathrm{i} x\xi} \sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i} k\xi} = \sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i}(x+k)\xi} \ . \tag{7}$$

We see, therefore, that $\tilde{\varphi} = U\varphi$ is 1-periodic in x. Since $\varphi(x,\xi), (x,\xi) \in \mathbf{T}^2$, is the general function of $L^2(\mathbf{T}^2)$, it is clear that $\tilde{\varphi}(x,\xi) = \mathrm{e}^{-2\pi \mathrm{i} x\xi}\varphi(x,\xi)$ is, also, the general function of $L^2(\mathbf{T}^2)$. To see that $\tilde{\varphi}$ satisfies (5) we can use some of the ideas that showed Z was onto. Since $\tilde{\varphi} \in L^2(\mathbf{T}^2)$ we see that $\tilde{\varphi}(x,\xi) \cong \sum_{\ell\in\mathbf{Z}} c_\ell(\xi)\mathrm{e}^{2\pi \mathrm{i}\ell x}$ for a.e. $\xi \in [-\frac{1}{2},\frac{1}{2}\rangle$ and, for such a ξ, $\sum_{\ell\in\mathbf{Z}} |c_\ell(\xi)|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\tilde{\varphi}(x,\xi)|^2 \mathrm{d}x < \infty$. Define $g \in L^2(\mathbf{R})$ by letting $g(\xi+\ell) = c_\ell(\xi)$ for $\xi \in [-\frac{1}{2},\frac{1}{2}\rangle$ obtaining

$$\tilde{\varphi}(x,\xi) \cong \sum_{\ell\in\mathbf{Z}} g(\xi+\ell)\mathrm{e}^{2\pi \mathrm{i}\ell x} \ , \tag{8}$$

where "$\cong$" denotes the fact that the series on the right of (8) is the Fourier series of $\tilde{\varphi}(x,\xi)$ as a function of x for each of the $\xi \in [-\frac{1}{2},\frac{1}{2}\rangle$ described above. This enables us to define $\tilde{\varphi}(x,\xi)$ for all $\xi \in \mathbf{R}$ by using (8):

$$\tilde{\varphi}(x,\xi+j) = \sum_{\ell\in\mathbf{Z}} g(\xi+j+\ell)\mathrm{e}^{2\pi\mathrm{i}\ell x} = \\ = \mathrm{e}^{-2\pi\mathrm{i}jx}\sum_{\ell\in\mathbf{Z}} g(\xi+j+\ell)\mathrm{e}^{2\pi\mathrm{i}x(j+\ell)} = \mathrm{e}^{-2\pi\mathrm{i}jx}\tilde{\varphi}(x,\xi)\ .$$

This shows (5) and our definition of the space $\tilde{M}$ is complete once we define $\|\tilde{\varphi}\|_{\tilde{M}}$ to be

$$\Big(\int_{\mathbf{T}^2}|\tilde{\varphi}(x,\xi)|^2\mathrm{d}x\mathrm{d}\xi\Big)^{\frac{1}{2}}.$$

Observe that we have also shown that $\tilde{M}$ is the image of an isometric Zak-like transform $\tilde{Z}$:

$$(\tilde{Z}g)(x,\xi) = \sum_{\ell\in\mathbf{Z}} g(\xi+\ell)\mathrm{e}^{2\pi\mathrm{i}\ell x} \equiv \tilde{\varphi}(x,\xi) \tag{9}$$

for $g\in L^2(\mathbf{R})$.

It is also natural to ask what is the relation between the functions f and g that satisfy

$$UZf = \tilde{Z}g. \tag{10}$$

We can determine this relation easily once we determine the inverse operators Z^{-1} and $\tilde{Z}^{-1}$. In fact, we have

$$(a)\ if\ \varphi(x,\xi)\in M,\ then\ (Z^{-1}\varphi)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(x,\xi)\mathrm{d}\xi\ ;$$
$$(b)\ if\ \tilde{\varphi}(x,\xi)\in \tilde{M},\ then\ (\tilde{Z}^{-1}\tilde{\varphi})(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{\varphi}(x,\xi)\mathrm{d}x\ . \tag{11}$$

To see this, recall that if $\varphi\in M$, Theorem 1 tells us that $\varphi(x,\xi)=\sum_{k\in\mathbf{Z}} f(x+\xi)\mathrm{e}^{-2\pi\mathrm{i}k\xi}$ for a unique $f\in L^2(\mathbf{R})$ and, as we have seen in (2), the sequence $\{f(x+k)\}, k\in\mathbf{Z}$, is, for a.e. $x\in\mathbf{Z}$, the sequence of Fourier coefficients of the $L^2(\mathbf{T})$ function $\varphi(x,\xi)$ that is 1-periodic in ξ for each such x. Hence, $\int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(x,\xi)\mathrm{d}\xi = f(x)$, the zero coefficient of the function $\xi\to\varphi(x,\xi)$. This establishes (a) and the same argument, adapted to $\tilde{M}$, gives us (b).

The relation between f and g is then easily seen to be: $g=\hat{f}$ and $f=\check{g}$:

Theorem 2. *(a)* $\tilde{Z}^{-1}UZf=\hat{f}\equiv\mathscr{F}f$, *the Fourier transform of* f*;*
(b) $Z^{-1}U^*\tilde{Z}g=\check{g}\equiv\mathscr{F}^{-1}g$, *the inverse Fourier transform of* g.

Proof. Assume $f\in L^1(\mathbf{R})\cap L^2(\mathbf{R})$. From (7) we have $\tilde{\varphi}(x,\xi)=(U\varphi)(x,\xi)=\sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}(x+k)\xi}$. Thus, applying $\tilde{Z}^{-1}$ to the first and third expression in this equality we obtain (by $(11a)$):

$$(\tilde{Z}^{-1}UZf)(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}(x+k)\xi}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{d}x = \hat{f}(\xi)\ .$$

This shows that $\tilde{Z}^{-1}UZ$ maps the space $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ onto the space of Fourier transforms of these functions. Since $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, $\tilde{Z}^{-1}UZ$ is the unique extension of the map $f \to \hat{f}$ to a unitary operator, $\mathscr{F}$, on $L^2(\mathbf{R})$. Of course, $\mathscr{F}$ is the Fourier transform on $L^2(\mathbf{R})$.

We also have that the inverse Fourier transform, $\mathscr{F}^{-1}$, satisfies $\mathscr{F}^{-1} = Z^{-1}U^*\tilde{Z}$. □

This rather simple derivation of these important properties of the Fourier transform (in particular, that $\mathscr{F}$ is unitary) has surprised some of our colleagues. That the Zak transform can be used to obtain these properties, however, is not new. Let us discuss the relevant history of this matter. Zak introduced the transform we denoted by $\tilde{Z}$ in 1967 [7]. This operator, however, was also introduced by Gelfand [1] seventeen years earlier; he included in this article an argument very similar to the one we gave above that showed the unitary property of the Fourier transform. Gelfand, actually, gives credit to A. Weil for this proof in a book [6] that came out essentially at the same time (this book is a "reprint" of an earlier one). In fact some authors call the operator $\tilde{Z}$ the "Weil-Brezin" mapping; however, this occurs quite a bit later.

The two transforms we introduced, Z and $\tilde{Z}$, we believe, are rather natural and explain, perhaps, more clearly, the relationship between the Fourier transform (and its inverse) and the operators we introduced. The relations that were derived by the authors we quoted appear to us to involve a "Deus ex machina" in which the integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}x$ is applied to the expression $\sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}(x+k)\xi}$ (see (7)) to obtain the Fourier transform. Our Theorem 2 does explain why this is done; the works we cite do not motivate the use of this integral for this purpose. If $h \in L^2(\mathbf{T})$ and $\sum_{n\in\mathbf{Z}} \hat{h}(n)\mathrm{e}^{2\pi\mathrm{i}n\xi}$ is the Fourier series of h we show how the well known equality $\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(\xi)|^2 \mathrm{d}\xi = \sum_{n\in\mathbf{Z}} |\hat{h}(n)|^2$ easily implies the Plancherel property of the Fourier transform $\mathscr{F}$. Our motivation for using the expression $\mathrm{e}^{-2\pi\mathrm{i}k\xi}$ (instead of $\mathrm{e}^{2\pi\mathrm{i}k\xi}$) in the definition of Z (see (1)) is that it leads us to the Fourier transform identities, see Theorem 2, more directly.

The spaces M and $\tilde{M}$ are really quite interesting and, as we will show in another paper, are very much worth studying.

The operators $Z, \tilde{Z}, U$ and their inverses play an important role in various different settings. In the next section we shall present some observations that explain why this is the case.

2 More uses of the Zak transforms and their extensions.

We first make some simple observations that follow immediately from the equalities in Theorem 2. Suppose f is, say, a Schwartz function (though we do not need so much smoothness and rapid vanishing at ∞); this allows us to formulate the result in Theorem 2 as a pointwise result valid for all $(x, \xi) \in \mathbf{R}^2$:

$$\sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}(x+k)\xi} = \sum_{k\in\mathbf{Z}} \hat{f}(\xi+k)\mathrm{e}^{2\pi\mathrm{i}kx} . \tag{12}$$

If we set $x = 0 = \xi$ we obtain the *Poisson summation formula*

$$\sum_{k\in\mathbf{Z}} f(k) = \sum_{k\in\mathbf{Z}} \hat{f}(k) . \tag{13}$$

Suppose we only set $x = 0$. We then have the equality

$$\sum_{k\in\mathbf{Z}} f(k)\mathrm{e}^{-2\pi\mathrm{i}k\xi} = \sum_{k\in\mathbf{Z}} \hat{f}(\xi+k) . \tag{14}$$

If we make the additional band-limited assumption that $\mathrm{Supp}\hat{f} \subset [-\frac{1}{2}, \frac{1}{2}\rangle$ we then obtain

$$\hat{f}(\xi) = \sum_{k\in\mathbf{Z}} f(k)\mathrm{e}^{-2\pi\mathrm{i}k\xi}$$

for $\xi \in [-\frac{1}{2}, \frac{1}{2}\rangle$. Let us multiply each side of this equality by $\mathrm{e}^{2\pi\mathrm{i}x\xi}$ and integrate both sides with respect to ξ. We obtain

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}x\xi}\mathrm{d}\xi = \sum_{k\in\mathbf{Z}} f(k)\int_{-\infty}^{\infty} \chi_{[-\frac{1}{2},\frac{1}{2}\rangle}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi(x-k)}\mathrm{d}\xi$$

(we inserted $\chi_{[-\frac{1}{2},\frac{1}{2}\rangle}$ since $\mathrm{Supp}\hat{f} \subset [-\frac{1}{2}, \frac{1}{2}\rangle$). Since $\check{\chi}_{[-\frac{1}{2},\frac{1}{2}\rangle}(x) = sinc(x) \equiv \frac{\sin \pi x}{\pi x}$, we have

$$f(x) = \sum_{k\in\mathbf{Z}} f(k) sinc(x-k) . \tag{15}$$

It is easy to check that the convergence of this series is absolute and in the norm of all the $L^p(\mathbf{R})$ spaces, $2 \leqq p \leqq \infty$. The reader will recognize this result as the *Shannon Sampling Theorem*. It tells us that if f is band limited, with $\hat{f}(\xi) = 0$ if $|\xi| > \frac{1}{2}$, then it is completely determined by its values on the integers (that is, by "sampling" f on the set $\mathbf{Z}$). There is a considerable literature devoted to extending this sampling result. The Zak transforms play an important role for obtaining these extensions; we intend to explain these results in future publications. We presented these facts as an illustration of the many uses of these operators in various areas of analysis.

It is clear that the two Zak transforms can be introduced in the setting of $L^2(\mathbf{R}^n), n \geqq 1$, and the results we derived in §1 can be extended in this more general case. These operators can also be considered to act on other $L^p(\mathbf{R}^n)$ spaces, $1 \leqq p \leqq 2$. In fact, the Zak transforms can be extended to much more general settings in which the groups $\mathbf{Z}^n$ and $\mathbf{T}$ are replaced by a locally compact abelian (LCA) group G and its dual $\hat{G}$. The space $L^2(\mathbf{R}^n)$ corresponds to a separable Hilbert space $\mathbf{H}$ on which acts a unitary representation of G (see [3] and [4]). In these two articles just cited we study "Principal Shift Invariant Subspaces" of $L^2(\mathbf{R}^n)$, "Gabor Spaces" and their extensions in the LCA group setting. The extensions of the Zak transforms to these situations are also most useful in this context. Some of these extensions are also obtained by Gröchenig [2].

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