DEMOCRATIC SYSTEMS OF TRANSLATES

BY
E. HERNÁNDEZ, M. NIELSEN, H. ŠIKIĆ AND F. SORIA

§1. Introduction

Democratic systems arise in the context of greedy approximations in Banach spaces. The greedy algorithm is an important method in numerical mathematics; for more details see the early work of V. N. Temlyakov [16], [17]. At the same time it also points to some interesting issues in functional analysis. In particular, we would like to emphasize the basic theorem of S. V. Konyagin and V. N. Temlyakov (see [8]) which states that a basis in a Banach space is greedy if and only if it is unconditional and democratic. Hence, one could expect (and this is indeed so) that there are democratic bases which are not unconditional. It is important to mention that it is often difficult to produce valuable examples of conditional bases (it is good to consult books by I. Singer [15], J. Lindenstrauss and L. Tzafriri [9], R. Young [18]; numerous notions and fundamental results that we use here one can find there). We find an interesting direction in an article by P. Wojtaszczyk [19] where even more general systems than bases are studied in a similar context. The author studies quasi-greedy biorthogonal systems and develops a method to construct a wide range of quasi-greedy conditional bases. This partially motivates our work, since we shall also study that democratic property for general systems and we shall also try to offer deeper understanding of a particular, but fairly rich class of conditional democratic bases.

Let us comment on the second notion in our title. Systems of translates are recently again studied thoroughly, from the point of view of various reproducing function systems, like wavelets, Gabor systems, etc. Very often are the most basic properties of such systems characterized in terms of some associated system of (integer or other) translates of a function (or functions); see Ch. Heil and A. Powell [3], as well as [4], [14] and the references therein. In particular, there is a hierarchy of simple and usable conditions that characterize various basis-like properties of a system of integer translates of a square integrable function (see [5] and, for some far reaching generalizations, [6]).

Date: December 3, 2010.

The simplicity of such characterizations allows systematic construction of (perhaps surprisingly) rich families of examples (including also the case of a conditional basis); see [13] for one such method in the case of wavelets.

There is a natural question to be answered. Consider a system of integer translations of a single square integrable function on the real line. What are necessary and sufficient conditions for such a system to be democratic? Let us emphasize immediately two features of this question. First of all, the question is not a simple one; even in this, most elementary example among various systems of translates. As we shall see, in one important subcase, when our system forms a Parseval frame, the question is most closely related to the L^2-norm concentration (see [1] and [2] for recent developments and for basic literature on the subject). As a consequence, at this point we are not able to fully answer our main question, i.e., to offer a characterization of a system of translates that is given in simple terms which are not difficult to check. Still, we offer several necessary criteria and several sufficient criteria that meet such standards, and in some subclasses, like in the continuous case, we provide very elegant characterizations.

The second feature we would like to emphasize, connects nicely to the issue of conditional Schauder bases, mentioned earlier. We already know that such systems of translates form a Schauder basis if and only if the periodization of the Fourier transform of the generating function satisfies the celebrated Muckenhoupt A_p condition (see [11] and [10] for details). We also know (recall the result of P. Wojtaszczyk) that such systems of translates can not form conditional, quasi-greedy, Schauder bases; see [12]. Hence, the only chance to have conditional Schauder basis of translates with some properties akin to "greediness" is within the realm of democratic, conditional Schauder bases. As we shall see, even in the case of system of translates this class is very rich and exhibits some interesting properties.

We explain some basic details about democratic systems in Section 2. In Section 3 we offer a quick overview of various known properties of systems of translates. In Section 4 we develop our theory of democratic systems of integer translates of a single square integrable function on the real line.

§2. Democratic families. Basic properties

We shall work eventually within a particular Hilbert space. However, most properties of democratic families can be formulated in a much more general space. We take the middle ground here, which is easily understandable to a mathematician of any background.

Consider a normed space $(X, \| \|)$. For a finite and non-empty family $\mathcal{G} \subseteq X \backslash \{\mathbf{0}\}$ we introduce the notation

$$\sum\nolimits_{\mathcal{G}} := \Big\| \sum_{x \in \mathcal{G}} \frac{x}{\|x\|} \Big\| \tag{2.1}$$

It is straightforward to check the following properties. If $card(\mathcal{G}) = 1$, then $\sum_{\mathcal{G}} = 1$. In general, $0 \leqslant \sum_{\mathcal{G}} \leqslant card(\mathcal{G})$. Already with two vectors one can combine $\sum_{\mathcal{G}}$ to achieve any value in $[0, card(\mathcal{G})]$. For example, $\mathcal{G} = \{-x, x\}$ gives $\sum_{\mathcal{G}} = 0$, while $\mathcal{G} = \{x, 2x\}$ gives $\sum_{\mathcal{G}} = 2$. Obviously, if $\sum_{\mathcal{G}} = 0$, then $\mathcal{G}$ is linearly dependent. The reverse, however, is not necessarily true. If $\{\mathcal{G}_1, \mathcal{G}_2\}$ is a non-trivial partition of $\mathcal{G}$, then the triangle inequality gives

$$\sum\nolimits_{\mathcal{G}_1} - \sum\nolimits_{\mathcal{G}_2} \leqslant \sum\nolimits_{\mathcal{G}} \leqslant \sum\nolimits_{\mathcal{G}_1} + \sum\nolimits_{\mathcal{G}_2} ; \tag{2.2}$$

in particular, if $g \in \mathcal{G}$ and $card(\mathcal{G}) \geqslant 2$, then

$$\sum\nolimits_{\mathcal{G}\backslash\{g\}} -1 \leqslant \sum\nolimits_{\mathcal{G}} \leqslant \sum\nolimits_{\mathcal{G}\backslash\{g\}} +1 \tag{2.3}$$

Given $\emptyset \neq \mathcal{F} \subseteq X \backslash \{\mathbf{0}\}$ and $n \in \mathbb{N}$ such that $n \leqslant \operatorname{card} \mathcal{F}$, we define

$$\begin{aligned} D^+(n\,;\mathcal{F}) &:= \sup_{\substack{\mathcal{G} \subseteq \mathcal{F} \\ \operatorname{card}(\mathcal{G}) = n}} \sum\nolimits_{\mathcal{G}} \\ D_-(n\,;\mathcal{F}) &:= \inf_{\substack{\mathcal{G} \subseteq \mathcal{F} \\ \operatorname{card}(\mathcal{G}) = n}} \sum\nolimits_{\mathcal{G}} \end{aligned} \tag{2.4}$$

It follows from previous observations that $D_-(1\,;\mathcal{F}) = D^+(1\,;\mathcal{F}) = 1$ and

$$0 \leqslant D_-(n\,;\mathcal{F}) \leqslant D^+(n\,;\mathcal{F}) \leqslant n\ .$$

Furthermore, if $k, \ell \in \mathbb{N}$ are such that $k + \ell = n$, then

$$\begin{aligned} D_-(k\,;\mathcal{F}) - D^+(\ell\,;\mathcal{F}) \leqslant D_-(n\,;\mathcal{F}) \leqslant \\ \leqslant D^+(n\,;\mathcal{F}) \leqslant D^+(k\,;\mathcal{F}) + D^+(\ell\,;\mathcal{F}) \end{aligned} \tag{2.5}$$

Let us explore some extremal cases first. We have seen already that $D_-(n\,;\mathcal{F}) = 0$ is possible. What about $D^+(n\,;\mathcal{F}) = 0$? The following examples and the following lemma describe such a possibility more or less completely.

Example 2.6. (**a**) *Let* $\mathcal{F} = \{-x, x\}$ *for some* $x \in X \backslash \{\mathbf{0}\}$. *Then* $D_-(1\,;\mathcal{F}) = D^+(1\,;\mathcal{F}) = 1$ *and* $D_-(2\,;\mathcal{F}) = D^+(2\,;\mathcal{F}) = 0$.

(**b**) *Let* $X = \mathbb{R}^3$. *Take* $x_1 = (1, 0, 0)$ *and find* x_2 *so that* $\|x_2\| = 1$ *and* $\|x_1 + x_2\| = 1$; *for example* $x_2 = (\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$ *satisfies these properties. Take* $\mathcal{F} = \{x_1, x_2, -(x_1 + x_2)\}$. *Then* $D_-(1\,;\mathcal{F}) = D^+(1\,;\mathcal{F}) = D_-(2\,;\mathcal{F}) = D^+(2\,;\mathcal{F}) = 1$ *and* $D_-(3\,;\mathcal{F}) = D^+(3\,;\mathcal{F}) = 0$

(**c**) *Observe that in general for a finite $\mathcal{F}$ we have*

$$D_{-}(\mathrm{card}(\mathcal{F});\mathcal{F}) = D^{+}(\mathrm{card}(\mathcal{F});\mathcal{F}) = \sum\nolimits_{\mathcal{F}} \tag{2.7}$$

Observe also that for $n < \mathrm{card}(\mathcal{F})$ it is possible to have $D_{-}(n\,;\mathcal{F}) = 0$; take, for example, $\mathcal{F} = \{x, -x, 2x\}$. Compare this with the following result.

♦

Lemma 2.8. *If $n < \mathrm{card}(\mathcal{F})$, then $D^{+}(n\,;\mathcal{F}) > 0$.*

Proof. For $n = 1$ the statement is always true. Consider $n \in \mathbb{N} \setminus \{1\}$. Since $n < \mathrm{card}(\mathcal{F})$ we can find $n-1$ mutually distinct vectors in $\mathcal{F}$. Let us fix some choice of such vectors $x_1, \dots, x_{n-1} \in \mathcal{F}$. Since $n < \mathrm{card}(\mathcal{F})$, we can find two vectors $y_1, y_2 \in \mathcal{F} \setminus \{x_1, \dots, x_{n-1}\}$ such that $y_1 \neq y_2$.

Suppose now, to the contrary, that $D^{+}(n\,;\mathcal{F}) = 0$. Then it would follow

$$\sum\nolimits_{\{x_1,\dots,x_{n-1},y_1\}} = \sum\nolimits_{\{x_1,\dots,x_{n-1},y_2\}} = 0\ ,$$

which implies $\frac{y_1}{\|y_2\|} = \frac{y_2}{\|y_2\|}$. Take any $k \in \{1, \dots, n-1\}$, replace x_k with y_1 and keep y_2. Then, using $0 = \sum_{\{x_1,\dots,y_1,\dots,x_{n-1},y_2\}}$, we obtain $\frac{y_1}{\|y_1\|} = \frac{x_k}{\|x_k\|}$. Since k was arbitrary, we proved that $\frac{y_1}{\|y_1\|} = \frac{x_1}{\|x_1\|} = \cdots = \frac{x_{n-1}}{\|x_{n-1}\|}$. Hence, using $0 = \sum_{\{x_1,\dots,x_{n-1},y_1\}}$, we obtain $n \cdot \frac{y_1}{\|y_1\|} = \mathbf{0}$; which is not possible, since $y_1 \in \mathcal{F} \subseteq X \setminus \{\mathbf{0}\}$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Let us now define the main notion of this article.

Definition 2.9. A non-empty family $\mathcal{F} \subseteq X \setminus \{\mathbf{0}\}$ is **democratic** if there exists $D > 0$ such that

$$\big(\mathcal{G},\ \mathcal{H} \subseteq \mathcal{F},\ \mathcal{G} \text{ and } \mathcal{H} \text{ finite},\ \mathrm{card}(\mathcal{G}) = \mathrm{card}(\mathcal{H}) \Rightarrow \sum\nolimits_{\mathcal{G}} \leqslant D \sum\nolimits_{\mathcal{H}}\big)\ .$$

Using Lm. 2.8 we immediately obtain the following:

$$\big(n < \mathrm{card}(\mathcal{F}),\ D_{-}(n\,;\mathcal{F}) = 0\ \Rightarrow\ \mathcal{F} \text{ is not democratic}\big). \tag{2.10}$$

Obviously, if $n = \mathrm{card}(\mathcal{G}) = \mathrm{card}(\mathcal{H})$ in Def. 2.9, then $D_{-}(n\,;\mathcal{F}) \leqslant \min\{\sum_{\mathcal{F}}, \sum_{\mathcal{H}}\} \leqslant \max\{\sum_{\mathcal{G}}, \sum_{\mathcal{H}}\} \leqslant D^{+}(n\,;\mathcal{F})$ and both bounds can be approximated with $\min\{\sum_{\mathcal{G}}, \sum_{\mathcal{H}}\}$ and $\max\{\sum_{\mathcal{G}}, \sum_{\mathcal{H}}\}$ arbitrarily close. Using (2.7) and $\sum_{\{x\}} = 1$ we obtain the following straightforward equivalence:

$$\begin{aligned} &\emptyset \neq \mathcal{F} \subseteq X \setminus \{\mathbf{0}\} \text{ is democratic} \\ \Leftrightarrow \quad &\sup\left\{\frac{D^{+}(n\,;\mathcal{F})}{D_{-}(n\,;\mathcal{F})}\ :\ 2 \leqslant n < \mathrm{card}(\mathcal{F})\right\} < +\infty\,. \end{aligned} \tag{2.11}$$

In particular, if $\operatorname{card}(\mathcal{F}) \in \{1, 2\}$, then $\mathcal{F}$ is always democratic. Furthermore, for democratic systems the smallest constant that satisfies the defining relationship is exactly the supremum taken in (2.11). In particular, if D satisfies Def. 2.9 for some system $\mathcal{F}$, then $D \geqslant 1$.

In this article we shall be interested only in countable families. Let us first describe the case of finite $\mathcal{F}$, which, as we shall see, will not be otherwise of a particular interest to us. In this case we only have finitely many subfamilies, so Lm. 2.8 and (2.11) easily imply the following characterization.

Proposition 2.12. *A non-empty and finite family $\mathcal{F} \subseteq X \setminus \{\mathbf{0}\}$ is democratic if and only if*

$$D_-(n\,;\mathcal{F}) > 0\,,$$

for every $n \in \{1, \ldots, \operatorname{card}(\mathcal{F}) - 1\}$. If this is the case then we can take D in Def. 2.9 to be

$$D = \max_{1 \leqslant n < \operatorname{card}(\mathcal{F})} \frac{D^+(n\,;\mathcal{F})}{D_-(n\,;\mathcal{F})}$$

(with an obvious interpretation for $\operatorname{card}(\mathcal{F}) = 1$*).*

Since for a finite $\mathcal{F}$ we can have min instead of inf in the definition of D_- (see (2.4)), we obtain the following result.

Corollary 2.13. *If $\emptyset \neq \mathcal{F} \subseteq X$ is finite and linearly independent, then $\mathcal{F}$ is democratic.*

Observe that in Ex. 2.6 (a) and (b) we have examples of finite democratic families which are not linearly independent.

Let us now turn our attention to the most interesting case for us, the case of infinite and countable family $\mathcal{F}$, i.e., the case when $\mathcal{F} \subseteq X \setminus \{\mathbf{0}\}$ and $\operatorname{card}(\mathcal{F}) = \aleph_0$. Obviously (see (2.11)) such a family is democratic if and only if

$$D(\mathcal{F}) := \sup \left\{ \frac{D^+(n\,;\mathcal{F})}{D_-(n\,;\mathcal{F})} \;:\; n \in \mathbb{N} \right\} \tag{2.14}$$

is finite (observe that in general $D(\mathcal{F}) \in [1, +\infty]$).

This tells us that for democratic families $\mathcal{F}$ we can take any sequence $(\mathcal{G}_n \colon n \in \mathbb{N})$ of subfamilies $\mathcal{G}_n \subseteq \mathcal{F}$, with $\operatorname{card}(\mathcal{G}) = n$, for every $n \in \mathbb{N}$, and we will have

$$\big(\mathcal{H} \subseteq \mathcal{F},\ \operatorname{card}(\mathcal{H}) = n \ \Rightarrow\ \frac{1}{D(\mathcal{F})} \sum\nolimits_{\mathcal{G}_n} \leqslant \sum\nolimits_{\mathcal{H}} \leqslant D(\mathcal{F}) \sum\nolimits_{\mathcal{G}_n} \big). \tag{2.15}$$

Hence, in analyzing infinite and countable democratic families we are primarily interested in "the order of growth" of $\sum_{\mathcal{F}_n}$, with respect to n, where $\mathcal{F}_n \subseteq \mathcal{F}$, $\operatorname{card}(\mathcal{F}_n) = n$.

Example 2.16. *Consider $X = \mathbb{H}$, where $\mathbb{H}$ is a Hilbert space.*
(**a**) *If $\{e_k \colon k \in \mathbb{N}\} \subseteq \mathbb{H}$ is a Riesz basis with constants $0 < A \leqslant B < +\infty$ (see [7] or [18] for definitions and basic properties), then*

$$\frac{A}{B}\,\mathrm{card}(\mathcal{G}) = A \sum_{e_k \in \mathcal{G}} \frac{1}{B} \leqslant \Big\| \sum_{e_k \in \mathcal{G}} \frac{e_k}{\|e_k\|} \Big\|^2 \leqslant \leqslant B \sum_{e_k \in \mathcal{G}} \frac{1}{A} = \frac{B}{A}\,\mathrm{card}(\mathcal{G}),$$

for every finite $\mathcal{G} \subseteq \{e_k \colon k \in \mathbb{N}\}$. Hence, $\mathcal{F} = \{e_k \colon k \in \mathbb{N}\}$ is democratic, and

$$\sqrt{\frac{A}{B}} \cdot \sqrt{n} \leqslant D_-(n\,; \mathcal{F}) \leqslant D^+(n\,; \mathcal{F}) \leqslant \sqrt{\frac{B}{A}} \cdot \sqrt{n}\,. \tag{2.17}$$

In particular, if $\mathcal{F}$ is an orthonormal basis, then $A = B = 1$ and

$$D_-(n\,; \mathcal{F}) = D^+(n\,; \mathcal{F}) = \sqrt{n} \tag{2.18}$$

Hence, the most typical behavior that we can expect is that "the order of growth" of $\sum_{\mathcal{F}_n}$ is $\sqrt{n}$.

♦

As we shall see, our translation systems are going to exhibit this same "rate of growth". Therefore, we shall not go deeper into the study of abstract democratic systems, despite the fact that there are some interesting results there. As an illustration, let us at the end of this section provide some academic examples which show that neither linear independence nor "the rate of growth" of $\sqrt{n}$ are necessary features of democratic systems in general.

Example 2.19. *Take $x \neq \mathbf{0}$ and consider $\mathcal{F} = \{n \cdot x \colon n \in \mathbb{N}\}$. Then, $D_-(n\,; \mathcal{F}) = D^+(n\,; \mathcal{F}) = n$, $\mathcal{F}$ is democratic and $D(\mathcal{F}) = 1$. Observe that $\mathcal{F}$ is "highly" linearly dependent.*

♦

Example 2.20. *Take an orthonormal basis $\{e_k \colon k \in \mathbb{N}\}$ in a Hilbert space $\mathbb{H}$. Take a vector $x \neq \mathbf{0}$ with the property that $(-n)\frac{x}{\|x\|}$ is not equal to any $\sum\limits_{e_k \in \mathcal{G}} e_k$, with $\mathcal{G} \subseteq \{e_k\}$ being finite. For example, $x = e_1 - e_2$ has such a property. Let*

$$\mathcal{F} := \{n\,x \colon n \in \mathbb{N}\} \cup \{e_k \colon k \in \mathbb{N}\}\,.$$

Then $\mathcal{F}$ contains a democratic system (with "growth" $\sqrt{n}$) and

$$D_-(n\,; \mathcal{F}) > 0\,,\ \forall n \in \mathbb{N}\,.$$

However, the system $\mathcal{F}$ is not democratic, since

$$D_-(n\,; \mathcal{F}) \leqslant \sqrt{n} = \sum\nolimits_{\mathcal{G}_n} \leqslant \sum\nolimits_{\mathcal{H}_n} = n = D^+(n\,; \mathcal{F})\,;$$

where $\mathcal{G}_n = \{e_1, \dots, e_n\}$ *and* $\mathcal{H}_n = \{x, 2x, \dots, nx\}$.

♦

§3. Systems of Translates. Basic Properties

In this section we offer a quick review of basic results about systems of translates we intend to study here. For definitions, proofs and details we refer to [14], [11], [5].

Let $\psi \in L^2(\mathbb{R})$ and consider $\mathcal{F}_\psi := \{T_k\psi \colon k \in \mathbb{Z}\}$, where $T_k\psi(x) = \psi(x-k)$, $x \in \mathbb{R}$. It is known that, for every $\psi \neq \mathbf{0}$, $\mathcal{F}_\psi$ is linearly independent. The system $\mathcal{F}_\psi$ has interesting properties within the *principal shift invariant space* generated by ψ, i.e.,

$$\langle\psi\rangle := \overline{\operatorname{span}}(\mathcal{F}_\psi)\,.$$

It turns out that these properties are given in terms of the periodization function

$$p_\psi(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2\,,\ \xi \in \mathbb{R}\,, \tag{3.1}$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ. The following picture provides the set of all $\psi \neq \mathbf{0}$, depending on various properties of $\mathcal{F}_\psi$ within $\langle\psi\rangle$.

Let us provide a legend for the figure above. It represents a set of all $\psi \neq \mathbf{0}$ and its partition into various subclasses based on the properties of $\mathcal{F}_\psi$ within $\langle\psi\rangle$.

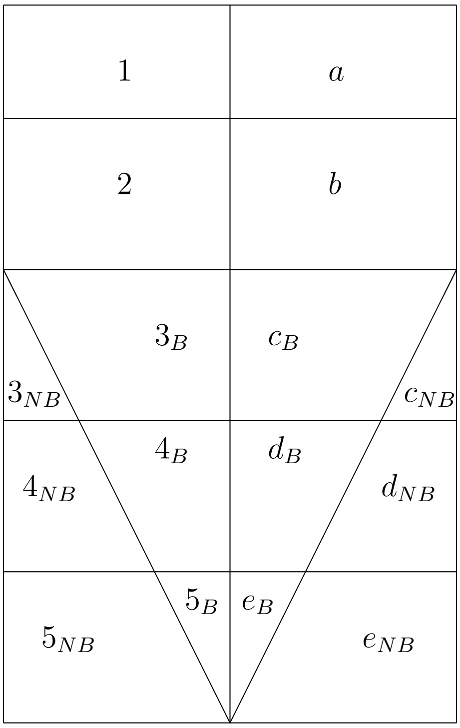
All classes on the left, named by numbers, have the property that

$$|\{\xi \colon p_\psi(\xi) = 0\}| > 0\,, \tag{3.2}$$

where $|A|$ denotes the Lebesgue measure of A. All classes on the right, named by letters, have the property

$$p_\psi > 0 \quad \text{a.e.} \tag{3.3}$$

In particular, for all these classes, $\mathcal{F}_\psi$ is ℓ^2-linearly independent. See [15] for many results on various levels of linear independence. Observe that convergence for such systems is not necessarily unconditional and for some properties (ℓ^2-linear independence, Schauder basis) the order of vectors matters. In all such situations we assume that $\mathbb{Z}$ is ordered as $\{0, 1, -1, 2, -2, \dots\}$. It is still an open question if (3.3) gives the characterization of ℓ^2-linear independence. However, candidates that may disprove such a statement would have to be within the NB (non-Besselian) classes. As we shall see in the next section, there are no democratic systems within NB. Hence, for Besselian systems (3.3) is



$$B := 3_B \cup 4_B \cup 5_B \cup c_B \cup d_B \cup e_B$$
$$x := x_B \cup x_{NB} \quad x \in \{3, 4, 5, c, d, e\}$$

the characterization of ℓ^2-linear independence. Recall that $\mathcal{F}_\psi$ is a Besselian system if and only if there exists $0 < B' < +\infty$ such that

$$p_\psi \leqslant B' \quad \text{a.e.} \tag{3.4}$$

In our figure we have that $\mathcal{F}_\psi$ is a Besselian system if and only if

$$\psi \in B \cup 1 \cup 2 \cup a \cup b\,. \tag{3.5}$$

After describing "vertical" and "diagonal" division in our figure, let us turn to the "horizontal" one. We start from the bottom.

The lowest part, i.e., the class $5 \cup e$ consists of these $\psi \neq \mathbf{0}$ for which

$$\frac{1}{p_\psi} \cdot \chi_{\{\xi :\, p_\psi(\xi) > 0\}}$$

is not integrable on the torus $\mathbb{T} := {}^{\mathbb{R}}/_{\mathbb{Z}}$ (observe that p_ψ is 1-periodic on $\mathbb{R}$, i.e., can be considered as a function on $\mathbb{T}$). This class is still not well understood. We know that e_B consists of ψ-s such that $\mathcal{F}_\psi$ is Besselian, ℓ^2-linearly independent, but not minimal (see [15], [18] for various results on minimal systems), while in the e_{NB} case we have same properties except Besselian one. In the class 5_B we have Besselian systems which are not ℓ^2-linearly independent and do not allow dual system in the way we shall explain shortly after. In the case of 5_{NB} the issue of ℓ^2-linear independence has not been settled completely, as we mentioned earlier.

Obviously then, $\psi \notin 5 \cup e$ if and only if

$$\frac{1}{p_\psi} \cdot \chi_{\{\xi\colon p_\psi(\xi)>0\}} \in L^1(\mathbb{T})\,. \tag{3.6}$$

In this case we define the *dual function* $\widetilde{\psi}$ in $L^2(\mathbb{R})$ using

$$\widehat{\widetilde{\psi}}(\xi) := \frac{\widehat{\psi}(\xi)}{p_\psi(\xi)} \cdot \chi_{\{\xi\colon p_\psi(\xi)>0\}}\,. \tag{3.7}$$

It turns out that $\widetilde{\psi} \in \langle \psi \rangle$ (actually $\langle \widetilde{\psi} \rangle = \langle \psi \rangle$),

$$p_{\widetilde{\psi}} = \frac{1}{p_\psi} \chi_{\{p_\psi>0\}}\,, \tag{3.8}$$

(observe $\{p_{\widetilde{\psi}} > 0\} = \{p_\psi > 0\}$) and

$$\langle T_k \psi, T_\ell \widetilde{\psi} \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{T}} e^{2\pi i \xi(\ell-k)} \chi_{\{\xi\colon p_\psi(\xi)>0\}}\, d\,\xi\,, \tag{3.9}$$

where $\ell, k \in \mathbb{Z}$. In particular,

$$((3.3) \text{ and } (3.6)) \Leftrightarrow \frac{1}{p_\psi} \in L^1(\mathbb{T})\,, \tag{3.10}$$

and, in this case, $\mathcal{F}_\psi$ and $\mathcal{F}_{\widetilde{\psi}}$ form a biorthogonal system (observe that in (3.9) we then have $\delta_{\ell,k}$ on the right hand side); which is equivalent to $\mathcal{F}_\psi$ being minimal (see [15] and [5] for details).

The distinction on the next level is provided by the following criterium. We define that $\psi \notin 5 \cup e \cup 4 \cup d$ if there exists a constant $0 < C' < +\infty$ such that

$$\left[\frac{1}{|I|} \int_I p_\psi(\xi)\, d\,\xi\right] \left[\frac{1}{|I|} \int_I \frac{1}{p_\psi(\xi)} \cdot \chi_{\{\xi\colon p_\psi(\xi)>0\}}\, d\,\xi\right] \leqslant C'\,, \tag{3.11}$$

for all intervals $I \subseteq \mathbb{T}$. At this point it is still open what exactly is the consequence of the distinction between classes 4 and 3. However, it is well-known what happens between d and c. More precisely, under (3.3) the condition (3.11) is equivalent to the celebrated Muckenhoupt A_2 condition. As shown in [11], we have

$$\psi \in a \cup b \cup c \;\Leftrightarrow\; \mathcal{F}_\psi \text{ is a Schauder basis for } \langle \psi \rangle\,. \tag{3.12}$$

Furthermore, if this is the case, then

$$\mathcal{F}_{\widetilde{\psi}} \text{ is the dual basis.} \tag{3.13}$$

The distinction on the next level is provided by the following criterium. We define that $\psi \in 1 \cup a \cup 2 \cup b$ if there exists constant $0 < A' \leqslant B' < +\infty$ such that

$$A' \cdot \chi_{\{\xi:\, p_\psi > 0\}} \leqslant p_\psi \leqslant B' \cdot \chi_{\{\xi:\, p_\psi > 0\}} \,. \tag{3.14}$$

Recall that

$$(3.14) \;\Leftrightarrow\; \mathcal{F}_\psi \text{ is a frame for } \langle \psi \rangle \,. \tag{3.15}$$

Furthermore, the best constants A', B' are exactly equal to frame bounds. Since

$$B' \chi_{\{p_\psi > 0\}} \leqslant B' \,,$$

it is obvious that such systems are always Besselian. As before, when we add (3.3) we obtain basis type properties. In particular,

$$\psi \in a \cup b \quad \Leftrightarrow \quad \mathcal{F}_\psi \text{ is a Riesz basis for } \langle \psi \rangle \,. \tag{3.16}$$

Observe that class c consists of such ψ for which $\mathcal{F}_\psi$ is a Schauder basis, but it is not a Riesz basis, i.e., c consists of ψ-s for which $\mathcal{F}_\psi$ is a conditional Schauder basis. In particular, the class c_B consists of conditional Hilbertian Schauder bases (see [15] or [5] for terminology).

The final distinction comes from the request that $A' = B' = 1$. Hence,

$$\begin{aligned} \big(\psi \in 1 \cup a \quad &\Leftrightarrow \quad p_\psi = \chi_{\{p_\psi > 0\}} \\ &\Leftrightarrow \quad \mathcal{F}_\psi \text{ is a Parseval frame for } \langle \psi \rangle\big) \,. \end{aligned} \tag{3.17}$$

In particular,

$$\begin{aligned} \big(\psi \in a \quad &\Leftrightarrow \quad p_\psi \equiv 1 \;\text{ a.e.} \\ &\Leftrightarrow \quad \mathcal{F}_\psi \text{ is an orthonormal basis for } \langle \psi \rangle\big) \,. \end{aligned} \tag{3.18}$$

This provides the detailed description of our figure, which describes the various properties of $\mathcal{F}_\psi$ within $\langle \psi \rangle$. This is also the stage for our paper and we can now turn our attention to the main question.

§4. Democratic Systems of Integer Translates

Given $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$, what are necessary and sufficient conditions for a system $\mathcal{F}_\psi$ to be democratic? Observe that $\langle \psi \rangle$ is a closed linear space, so we are working within the Hilbert space $\langle \psi \rangle$ and our system $\mathcal{F}_\psi$ is always linearly independent. Since $\|T_k \psi\|_2 = \|\psi\|_2$, for all $k \in \mathbb{Z}$, it is enough to study the behavior of

$$\Big\| \sum_{k \in \Gamma} T_k \psi \Big\|_2 \,, \quad \Gamma \subseteq \mathbb{Z} \,, \quad \Gamma \text{ finite.} \tag{4.1}$$

It is well known (see, for example, [5]) that $\langle \psi \rangle$ is isometrically isomorphic to $L^2(\mathbb{T}; p_\psi)$, the space of square integrable functions on the torus, with respect to the measure $p_\psi(\xi)\,d\xi$. The role of functions $T_k\psi$ is played by the exponentials, so we have

$$\|\sum_{k\in\Gamma} T_k\psi\|_2^2 = \int_{-1/2}^{1/2} |\sum_{k\in\Gamma} e^{-2\pi i k\xi}|^2\, p_\psi(\xi)\,d\xi\,. \tag{4.2}$$

It will be convenient for us to abuse somewhat the notation from our Section 2. For $\Gamma \subseteq \mathbb{Z}$, Γ finite, we introduce the notation

$$\sum\nolimits_\Gamma = \sum\nolimits_\Gamma(\xi) := \sum_{k\in\Gamma} e^{-2\pi i k\xi} \quad , \xi \in \mathbb{R}\,.$$

Obviously, $\sum_\Gamma$ is a 1-periodic function, so it can be (and will be most often) considered as a function on $\mathbb{T}$.

Let us first prove that the democratic property for systems $\mathcal{F}_\psi$ is equivalent to having "the rate of growth" of $\sqrt{n}$.

Theorem 4.3. *Let* $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$*. Then* $\mathcal{F}_\psi$ *is democratic if and only if*

$$\|\sum_{k\in\Gamma} T_k\psi\|_2^2 \asymp \operatorname{card}(\Gamma)\,,\ \Gamma\subseteq\mathbb{Z}\,,\ \Gamma \textit{ finite.}$$

Proof. Suppose first that the given condition is valid, i.e., there exist constants $0 < D_- \leqslant D^+ < +\infty$, independent of Γ, such that

$$D_-\sqrt{\operatorname{card}(\Gamma)} \leqslant \|\sum_{k\in\Gamma} T_k\psi\|_2 \leqslant D^+\sqrt{\operatorname{card}(\Gamma)}\,,$$

for every $\Gamma \subseteq \mathbb{Z}$, Γ finite. In particular, we obtain

$$\|\psi\|_2\cdot\sqrt{n}\,D_- \leqslant D_-(n\,;\mathcal{F}_\psi) \leqslant D^+(n\,;\mathcal{F}_\psi) \leqslant \|\psi\|_2\cdot\sqrt{n}\,D^+\,.$$

It follows that

$$\sup\left\{\frac{D^+(n\,;\mathcal{F}_\psi)}{D_-(n\,;\mathcal{F}_\psi)} : n\in\mathbb{N}\right\} \leqslant \frac{D^+}{D_-} < +\infty\,,$$

and we conclude that $\mathcal{F}_\psi$ is democratic.

Suppose now that $\mathcal{F}_\psi$ is democratic, i.e., $0 < D := D(\mathcal{F}_\psi) < +\infty$. For any $\Gamma\subseteq\mathbb{Z}$, Γ finite, we have

$$\frac{D^+\big(\operatorname{card}(\Gamma)\,;\mathcal{F}_\psi\big)}{D} \leqslant D_-(\operatorname{card}(\Gamma)\,;\mathcal{F}_\psi) \leqslant \|\sum_{k\in\Gamma}\frac{T_k\psi}{\|\psi\|_2}\|_2 \leqslant$$
$$\leqslant D^+\big(\operatorname{card}(\Gamma)\,;\mathcal{F}_\psi\big) \leqslant D\cdot D_-\big(\operatorname{card}(\Gamma)\,;\mathcal{F}_\psi\big)\,.$$

Hence, it is enough to show that for every $n\in\mathbf{N}$ there exists $\Gamma = \Gamma(n)$ such that $\operatorname{card}(\Gamma) = n$ and

$$\|\sum_{k\in\Gamma} T_k\psi\|_2^2 = n\cdot\|\psi\|_2^2\,. \tag{4.4}$$

Indeed, if ψ has a compact support, say contained in $[-2^M, 2^M]$, for some $M \in \mathbb{N}$, then it is easy to find such Γ-s. For $n \in \mathbb{N}$ take $\Gamma = \{2^{M+1}, 2^{M+2}, \dots, 2^{M+n}\}$. Observe that

$$\operatorname{supp}(T_{2^{M+j}}\,\psi) \;\cap\; \operatorname{supp}(T_{2^{M+\ell}}\,\psi) = \emptyset\,,$$

whenever $\ell, j \in \{q, \dots, n\}$, $\ell \neq j$. Hence,

$$\|\sum_{k\in\Gamma} T_k\psi\|_2^2 = \sum_{j=1}^{n} \|T_{2^{M+j}}\psi\|_2^2 = n \cdot \|\psi\|_2^2\,.$$

For arbitrary $\psi \in L^2(\mathbb{R})$, we can approximate ψ with functions of compact support in an obvious way to get arbitrary close to (4.4), which is enough to prove this theorem.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Remark 4.5. *As we have seen in Ex. 2.16, $\sqrt{n}$ is precisely "the order of growth" in Riesz bases. If we consider our figure in Section 3, then $\mathcal{F}_\psi$ is democratic for every $\psi \in a \cup b$. Hence, we know exactly what happens within $a \cup b$ and we shall not explore these two subclasses any further.*

■

Obviously, we need much more operative criterium for the democracy property of $\mathcal{F}_\psi$ systems. This problem seems to be a difficult one. Let us first take care of the easy part; the upper bound.

Lemma 4.6. *If $\mathcal{F}_\psi$ is democratic, then there exists a constant $0 < B' < +\infty$ such that for every $\Gamma \subseteq \mathbb{Z}$, Γ finite, and for any sequence $\{\alpha_k \in \mathbb{C} \colon k \in \Gamma\}$, with $|\alpha_k| \leqslant 1$, we have*

$$\|\sum_{k\in\Gamma} \alpha_k T_k\psi\|_2 \leqslant B'\sqrt{\operatorname{card}(\Gamma)}\,.$$

Proof. It is standard to show that it is enough to prove the claim for $\alpha_k \in [-1, 1]$. By Tm. 4.3 there exists $0 < D^+ < +\infty$ such that

$$\|\sum_{k\in\Gamma} T_k\psi\|_2 \leqslant D^+\sqrt{\operatorname{card}(\Gamma)}\,.$$

For a given $\Gamma \subseteq \mathbb{Z}$, Γ finite, and a given choice of $\{\alpha_k \colon k \in \Gamma\} \subseteq [-1, 1]$, we define $\Gamma^+ := \{k \in \Gamma \colon \alpha_k \geqslant 0\}$, $\Gamma^- := \Gamma \setminus \Gamma^+$, and

$$\varepsilon_k := \begin{cases} +1 & ;\ k \in \Gamma^+ \\ -1 & ;\ k \in \Gamma^- \end{cases}\,.$$

We, then, have

$$\begin{aligned}
&\|\sum_{k\in\Gamma}\alpha_k T_k\psi\|_2 \leqslant \max_{\varepsilon_k=\pm1}\|\sum_{k\in\Gamma}\varepsilon_k T_k\psi\|_2 \leqslant\\
&\leqslant \max_{\varepsilon_k=\pm1}\Big[\|\sum_{k\in\Gamma^+}T_k\psi\|_2 + \|\sum_{k\in\Gamma^-}T_k\psi\|_2\Big] \leqslant\\
&\leqslant \max_{\varepsilon_k=\pm1}\Big[D^+\sqrt{\mathrm{card}(\Gamma^+)} + D^+\sqrt{\mathrm{card}(\Gamma^-)}\Big] \leqslant\\
&\leqslant \max_{\varepsilon_k=\pm1}\Big[D^+\sqrt{2}\cdot\sqrt{\mathrm{card}(\Gamma^+)+\mathrm{card}(\Gamma^-)}\Big] =\\
&= D^+\sqrt{2}\cdot\sqrt{\mathrm{card}(\Gamma)}\,.
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Theorem 4.7. *If $\mathcal{F}_\psi$ is democratic, then there exists a constant $0 < B'' < +\infty$ such that*

$$p_\psi(\xi) \leqslant B'' \quad , \textit{for a.e. } \xi \in \mathbb{R}\,.$$

Proof. Let us denote by D_N the "symmetric" Dirichlet kernel, i.e.,

$$D_N(\xi) := \sum_{|k|\leqslant N} e^{2\pi i k\xi} \quad , \ \xi \in \mathbb{R}\,.$$

Then, for $u \in [-1/2, 1/2]$ we obtain, using Lm. 4.6,

$$\|\sum_{|k|\leqslant N} e^{2\pi i k u} T_k\psi\|_2^2 \leqslant (B')^2 \cdot \mathrm{card}\{k\in\mathbb{Z}\colon |k|\leqslant N\}\,.$$

Hence, there exists $0 < B'' < +\infty$, independent of Γ, such that

$$\begin{aligned}
(2N+1)\cdot B'' &\geqslant \|\sum_{|k|\leqslant N} e^{2\pi i k u} T_k\psi\|_2^2 =\\
&= \int_{-1/2}^{1/2} |\sum_{|k|\leqslant N} e^{2\pi i k u} e^{-2\pi i k\xi}|^2\, p_\psi(\xi)\, d\xi =\\
&= \int_{-1/2}^{1/2} |D_N(u-\xi)|^2\, p_\psi(\xi)\, d\xi\,.
\end{aligned}$$

Observe that $\|D_N\|_2^2 = 2N+1$ and that

$$\left\{\frac{|D_N(u-\xi)|^2}{\|D_N\|_2^2}\right\}_{N=1}^{\infty}$$

is a summability kernel at u. Since we proved that for every $u \in [-1/2, 1/2]$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|D_N(u-\xi)|^2}{\|D_N\|_2^2}\, p_\psi(\xi)\, d\xi \leqslant B'' ,$$

we deduce that $p_\psi(u) \leqslant B''$, for almost every u.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Remark 4.8. (**a**) *Observe that a necessary condition given in Thm. 4.7 is also a sufficient one for the upper bound, since, by* (4.2), $p_\psi \leqslant B''$ *a.e. implies*

$$\| \sum_{k\in\Gamma} T_k\psi\|_2^2 \leqslant B'' \int_{-1/2}^{1/2} |\sum\nolimits_\Gamma|^2 = B'' \cdot \operatorname{card}(\Gamma)\, .$$

Hence, it is the lower bound which creates the real problem.
(**b**) *Considering Section 3, observe that Thm. 4.7 proves that if* $\mathcal{F}_\psi$ *is democratic, then* $\mathcal{F}_\psi$ *is Besselian, i.e.,*

$$\psi \in B \cup 1 \cup 2 \cup a \cup b\, .$$

Since (see Rm. 4.5) $a \cup b$ *consists of systems which are democratic, the real question is what happens with*

$$\psi \in B \cup 1 \cup 2\, . \tag{4.9}$$

■

Let us illustrate that the problem of the lower bound is closely connected with the result of Wiener and Shapiro on concentration inequalities (see [1] and [2], and references therein). The important role there is played by functions with positive Fourier coefficients. In particular, such functions must be symmetric with respect to the origin. Let us first make a calculation which is very similar to the original Wiener-Shapiro method.

Consider a real-valued function $\varphi \in L^1(\mathbb{T})$ (for measure-theoretic purposes we identify $\mathbb{T}$ with symmetric interval $[-1/2, 1/2\rangle$). If $\varphi(\xi) = \varphi(-\xi)$, for a.e. ξ, then $\widehat{\varphi}(-k) = \widehat{\varphi}(k) \in \mathbb{R}$, for every $k \in \mathbb{N}$. Hence, for such a function we obtain, for every $\Gamma \subseteq \mathbb{Z}$, Γ finite,

$$\begin{aligned} \int_{-1/2}^{1/2} |\sum\nolimits_\Gamma|^2 \varphi(\xi)\, d\xi &= \sum_{j,k\in\Gamma} \int_{-1/2}^{1/2} e^{2\pi i(k-j)\xi} \varphi(\xi)\, d\xi = \\ &= \widehat{\varphi}(0)\cdot \operatorname{card}(\Gamma) + \sum_{\substack{j,k\in\Gamma\\ j<k}} 2\widehat{\varphi}(k-j)\, . \end{aligned} \tag{4.10}$$

Let us now symmetrize our periodization function, i.e., for $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ we define

$$s_\psi(\xi) := \frac{p_\psi(\xi) + p_\psi(-\xi)}{2} \quad , \xi \in \mathbb{R}\,. \tag{4.11}$$

Clearly, $s_\psi \in L^1(\mathbb{T})$, s_ψ is 1-periodic, $s_\psi \geqslant 0$, $s_\psi(\xi) = s_\psi(-\xi)$, for every $\xi \in \mathbb{R}$, and

$$\begin{aligned} \|s_\psi\|_1 := \|s_\psi\|_{L^1(\mathbb{T})} = \|p_\psi\|_{L^1(\mathbb{T})} = \|\psi\|_2^2 = \\ \widehat{p}_\psi(0) = \widehat{s}_\psi(0)\,. \end{aligned} \tag{4.12}$$

Furthermore, observe (see (4.2)) that

$$\|\sum_{k\in\Gamma} T_k\psi\|_2^2 = \int_{-1/2}^{1/2} |\sum\nolimits_\Gamma|^2\, s_\psi(\xi)\, d\xi\,, \tag{4.13}$$

and that

$$p_\psi \text{ is bounded above } 0 \;\Leftrightarrow\; s_\psi \text{ is bounded above}\,. \tag{4.14}$$

Using properties of s_ψ with (4.10) and (4.13), one directly obtains the following result.

Proposition 4.15. *Let $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ such that s_ψ is bounded above. Then $\mathcal{F}_\psi$ is democratic, i.e., there exists a constant $0 < D_0$ such that, for every $\Gamma \subseteq \mathbb{Z}$, Γ finite,*

$$\|\sum_{k\in\Gamma} \frac{T_k\psi}{\|\psi\|_2}\|_2^2 \geqslant D_0 \cdot \operatorname{card}(\Gamma)\,,$$

if and only if

$$\sum_{\substack{j.k\in\Gamma \\ j<k}} 2\,\frac{\widehat{s}_\psi(k-j)}{\widehat{s}_\psi(0)} \geqslant (D_0 - 1)\operatorname{card}(\Gamma)\,. \tag{4.16}$$

Obviously, in most cases one can not consider (4.16) as a "simple and operative" condition, so we have to explore the lower bound further. Let us first observe that the case $D_0 = 1$ is somewhat simpler, more accessible, and very much in the spirit of the Wiener-Shapiro result.

Theorem 4.17. *Let $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$. Then,*

$$\|\sum_{k\in\Gamma} \frac{T_k\psi}{\|\psi\|_2}\|_2^2 \geqslant \operatorname{card}(\Gamma)\,,$$

for every $\Gamma \subseteq \mathbb{Z}$, Γ finite, if and only if

$$\widehat{s}_\psi(n) \geqslant 0 \;, \textit{ for every } n \in \mathbb{N}\,.$$

Proof. Since $\psi \neq \mathbf{0}$ and $s_\psi \geqslant 0$, we obtain $\widehat{s}_\psi(0) > 0$. Observe also that $\widehat{s}_\psi(-n) = \widehat{s}_\psi(n) \in \mathbb{R}$, for every $n \in \mathbb{N}$. Hence, using (4.10) with $\|\psi\|_2^2 = \widehat{s}_\psi(0)$ we obtain

$$\Big\| \sum_{k\in\Gamma} \frac{T_k\psi}{\|\psi\|_2} \Big\|_2^2 = \operatorname{card}(\Gamma) + \sum_{\substack{j,k\in\Gamma \\ j<k}} 2\, \frac{\widehat{s}_\psi(k-j)}{\widehat{s}_\psi(0)} .$$

Obviously then, if $\widehat{s}_\psi(n) \geq 0$, for all $n \in \mathbb{N}$, we obtain the desired lower bound.

If, on the other hand, we have (see (4.13)),

$$\Big\| \sum_{k\in\Gamma} \frac{T_k\psi}{\|\psi\|_2} \Big\|_2^2 = \frac{1}{\widehat{s}_\psi(0)} \int_{-1/2}^{1/2} \Big| \sum\nolimits_\Gamma \Big|^2 s_\psi(\xi)\, d\xi \geqslant \operatorname{card}(\Gamma) ,$$

then consider $\Gamma_n = \{0, n\}$ to obtain

$$\frac{1}{\widehat{s}_\psi(0)} \int_{-1/2}^{1/2} \Big| \sum\nolimits_{\Gamma_n} \Big|^2 s_\psi(\xi)\, d\xi = 2 + 2\, \frac{\widehat{s}_\psi(n)}{\widehat{s}_\psi(0)} \geqslant 2 = \operatorname{card}(\Gamma_n) .$$

Hence, $\widehat{s}_\psi(n) \geqslant 0$, and this holds for all $n \in \mathbb{N}$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Even if s_ψ does not have positive Fourier coefficients, we can still obtain democratic property. For example it is enough "to insert" a function with positive Fourier coefficients between zero and s_ψ.

Corollary 4.18. *Let $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ such that s_ψ is bounded above. If there exists a function φ on $\mathbb{T}$ with the following properties:*

(i) $0 \leqslant \varphi(\xi) \leqslant s_\psi(\xi)$, *for a.e.* ξ ;
(ii) $\widehat{\varphi}(0) > 0$;
(iii) $\widehat{\varphi}(-n) = \widehat{\varphi}(n) \geqslant 0$, *for every* $n \in \mathbb{N}$;

then $\mathcal{F}_\psi$ is democratic.

Proof. By (i) we obtain

$$\Big\| \sum_{k\in\Gamma} T_k\psi \Big\|_2^2 = \int_{-1/2}^{1/2} \Big| \sum\nolimits_\Gamma \Big|^2 s_\psi(\xi)\, d\xi \geqslant \int_{-1/2}^{1/2} \Big| \sum\nolimits_\Gamma \Big|^2 \varphi(\xi)\, d\xi ,$$

and then we apply (4.10). Since $\widehat{\varphi}(k-j) \geqslant 0$, for $k \neq j$, we obtain

$$\Big\| \sum_{k\in\Gamma} T_k\psi \Big\|_2^2 \geqslant \widehat{\varphi}(0) \cdot \operatorname{card}(\Gamma) .$$

Since (ii) is valid, we obtain the democratic property.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Exactly as in the basic Wiener-Shapiro concentration inequality, one can "insert" a function φ as above in any interval around zero. The following result is well-known; we comment on its proof via (4.10) just for our reader's convenience.

Corollary 4.19. *Let* $0 < \delta \leqslant 1/2$ *and* $\Gamma \subseteq \mathbb{Z}$ *finite. Then*

$$\delta \cdot \operatorname{card}(\Gamma) \leqslant \int_{-\delta}^{\delta} \big| \sum\nolimits_\Gamma(\xi) \big|^2 \, d\xi \leqslant \operatorname{card}(\Gamma) \,.$$

Proof. Consider a function

$$\varphi_\delta(x) := \big(1 - \frac{|x|}{\delta}\big)_+$$

as a 1-periodic function. Observe that $\widehat{\varphi}_\delta(-n) = \widehat{\varphi}_\delta(n) \geqslant 0$, $\widehat{\varphi}_\delta(0) = \delta$ and

$$\operatorname{card}(\Gamma) = \int_{-1/2}^{1/2} \big| \sum\nolimits_\Gamma \big|^2 \, d\xi \geqslant \int_{-1/2}^{1/2} 1_{[-\delta,\delta]}(\xi) \, \big| \sum\nolimits_\Gamma(\xi) \big|^2 \, d\xi \geqslant$$

$$\geqslant \int_{-1/2}^{1/2} \varphi_\delta(\xi) \, \big| \sum\nolimits_\Gamma(\xi) \big|^2 \, d\xi \,.$$

Apply (4.10) to obtain the result.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

The following conditions are only sufficient, but they qualify as "simple and operative".

Corollary 4.20. *If there exist positive constants* $0 < D_0,\ D_1 < +\infty$ *and* $0 < \delta \leqslant 1/2$ *such that*

(i) $s_\psi(\xi) \leqslant D_1$, *for a.e.* ξ *;*
(ii) $s_\psi(\xi) \geqslant D_0$, *for a.e.* $\xi \in [-\delta, \delta]$ *;*

then $\mathcal{F}_\psi$ *is democratic.*

Proof. Directly from Cor. 4.19, since

$$D_1 \cdot \operatorname{card}(\Gamma) \geqslant \int_{-1/2}^{1/2} \big| \sum\nolimits_\Gamma \big|^2 s_\psi \, d\xi \geqslant \int_{-\delta}^{\delta} \big| \sum\nolimits_\Gamma \big|^2 s_\psi \, d\xi \geqslant$$

$$\geqslant D_0 \int_{-\delta}^{\delta} \big| \sum\nolimits_\Gamma \big|^2 \, d\xi \geqslant D_0 \cdot \delta \cdot \operatorname{card}(\Gamma) \,.$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Let us turn our attention to necessary conditions. As we already observed it is the lower bound that presents the problem and we need to study the estimates of the form

$$\int_{-1/2}^{1/2} \left| \sum\nolimits_{\Gamma} \right|^2 s_\psi(\xi)\, d\xi \geqslant \text{const.} \cdot \operatorname{card}(\Gamma)\,.$$

Obviously, we are facing a large variety of sums $|\sum_\Gamma|$. However, we can emphasize at least some significant ones. Since all functions involved in $|\sum_\Gamma|$ are unimodular, without loss of generality we can always assume that

$$\min(\Gamma) = 0\,, \tag{4.21}$$

and we shall do so in the following discussion.

One typical sum $|\sum_\Gamma|$ occurs when there are "no gaps" between terms, i.e., when

$$\sum\nolimits_{\Gamma}(\xi) = \widetilde{D_N}(\xi) := \sum_{k=0}^{N-1} e^{2\pi i k \xi}\,; \tag{4.22}$$

the case of "one-sided" Dirichlet kernel.

Theorem 4.23. *Let* $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ *such that* s_ψ *is bounded above. There exists* $0 < C_0 < +\infty$ *such that*

$$\int_{-1/2}^{1/2} |\widetilde{D_N}(\xi)|^2\, s_\psi(\xi)\, d\xi \geqslant C_0 N\ , \quad \text{for any } N \in \mathbb{N}\,,$$

if and only if

$$\inf_{0<\varepsilon<1/2} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} s_\psi(\xi)\, d\xi > 0\,. \tag{4.24}$$

In particular, condition (4.24) *is necessary for a system* $\mathcal{F}_\psi$ *to be democratic.*

Proof. Suppose first that (4.24) is valid. Then

$$\int_{-1/2}^{1/2} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi \;\geqslant\; \int_{-1/2N}^{1/2N} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi \geqslant$$

$$\geqslant \int_{-1/2N}^{1/2N} \big|\widetilde{D_N}(\tfrac{1}{2N})\big|^2 s_\psi(\xi)\,d\xi \;\geqslant\; \text{const. } N^2 \int_{-1/2N}^{1/2N} s_\psi(\xi)\,d\xi =$$

$$= N\left(\text{const. } \frac{1}{2\frac{1}{2N}} \int_{-1/2N}^{1/2N} s_\psi(\xi)\,d\xi\right) \geqslant \text{ const. } N\,.$$

Suppose now that there exists C_0, a positive constant, such that, for every $N \in \mathbb{N}$,

$$\int_{-1/2}^{1/2} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi \geqslant C_0 N\,.$$

Since s_ψ is bounded above, we have $\|s_\psi\|_\infty < +\infty$. For $L, N \in \mathbb{N}$ such that $N \geqslant 2L$ we have

$$\int_{-1/2}^{1/2} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi = \int_{|\xi|\leqslant \frac{L}{N}} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi +$$

$$+ \int_{\frac{L}{N}<|\xi|\leqslant\frac{1}{2}} \big|\widetilde{D_N}(\xi)\big|^2 s_\psi(\xi)\,d\xi \leqslant \int_{|\xi|\leqslant\frac{L}{N}} N^2 s_\psi(\xi)\,d\xi +$$

$$+ \|s_\psi\|_\infty \int_{\frac{L}{N}<|\xi|\leqslant\frac{1}{2}} \frac{|\sin \pi N\xi|^2}{|\sin \pi\xi|^2}\,d\xi \leqslant$$

$$\leqslant N^2 \int_{-L/N}^{L/N} s_\psi(\xi)\,d\xi + \|s_\psi\|_\infty \int_{|\xi|>\frac{L}{N}} \frac{1}{4|\xi|^2}\,d\xi =$$

$$= N^2 \int_{-L/N}^{L/N} s_\psi(\xi)\,d\xi + \|s_\psi\|_\infty \frac{N}{2L}\,.$$

Consider, in addition, L big enough so that $L > \dfrac{\|s_\psi\|_\infty}{C_0}$. For such an L we obtain

$$C_0 N \leqslant N^2 \int_{-L/N}^{L/N} s_\psi(\xi)\,d\xi + C_0\frac{N}{2}\,,$$

which gives

$$0 < \frac{1}{L} \cdot \frac{C_0}{4} \leqslant \frac{N}{2L} \int_{-L/N}^{L/N} s_\psi(\xi)\, d\xi \,. \tag{4.25}$$

We claim that (4.25) implies (4.24). Indeed, given $0 < \varepsilon < 1/2$, we can choose $L, N \in \mathbb{N}$ large enough with $N \geqslant 2L$ so that

$$\frac{\varepsilon}{2} < \frac{L}{N} < \varepsilon \,. \tag{4.26}$$

Since $s_\psi \geqslant 0$, we obtain

$$\begin{aligned} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} s_\psi \, d\xi \geqslant \frac{1}{2\varepsilon} \int_{-L/N}^{L/N} s_\psi \, d\xi = \frac{1}{2\varepsilon} \cdot \frac{2L}{N} \cdot \frac{N}{2L} \int_{-L/N}^{L/N} s_\psi \, d\xi \geqslant \\ \geqslant \frac{1}{\varepsilon} \cdot \frac{L}{N} \cdot \left(\frac{1}{L} \cdot \frac{C_0}{4} \right) \geqslant \frac{C_0}{8} \cdot \frac{1}{L} > 0 \,. \end{aligned} \tag{4.27}$$

Observe that in our construction we can first choose L large enough and keep it fixed afterwards. On the other hand, $N \geqslant 2L$ we can change freely to adjust for (4.26), with the original L kept in place. Therefore, (4.27) proves (4.24).

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Of course, one may hope that boundedness of s_ψ with (4.24) is also sufficient for democracy. However, this is not so. We thank Professor Aline Bonami for the following counter-example (it is given in [2], Remark 3). We shall briefly recall the example for reader's convenience and then comment on it.

Example 4.28. *For $\ell = 2, 3, 4, \dots$ defines*

$$E_\ell := \bigcup_{j=1}^{\ell-1} \langle \frac{j}{\ell} - \frac{1}{2\ell^3}, \frac{j}{\ell} + \frac{1}{2\ell^3} \rangle \,.$$

Take L large enough so that $A_L := \bigcup_{\ell=L}^{\infty} E_L \subseteq \langle 0, 1 \rangle$ and define $E_L := \langle 0, 1 \rangle \setminus A_L$. Take ψ such that $\widehat{\psi} = \chi_{E_L}$. Hence, p_ψ is equal to χ_{E_L} on $[0, 1\rangle$. It can be shown that

$$\lim_{k \to +\infty} \Big(\lim_{N \to +\infty} \frac{1}{N} \int_{E_L} \big| \widetilde{D_N}(k\xi) \big|^2 \, d\xi \Big) = 0 \,,$$

which shows that $\mathcal{F}_\psi$ is not democratic. On the other hand ψ satisfies (4.24) since, for $\varepsilon > 0$,

$$\frac{1}{2\varepsilon} \big| E_L \cap \langle -\varepsilon, \varepsilon \rangle \big| \geqslant 1 - \frac{1}{L} \,.$$

♦

Remark 4.29. (**a**) *Various subintervals within E_ℓ in Ex. 4.28 overlap in a way somewhat difficult to follow. For our purposes we can adjust this example into its "dyadic version".*
(**b**) *Condition* (4.24) *is the density type condition. One may feel that positive density is not good enough, but that one need to require certain level of density to reach the democratic property. Our adjusted example will show that this is not so. Related to this observe that* (4.24) *may achieve its infimum for ξ not close to zero, while we want "to measure" precisely density at zero. Hence, we shall consider ($\psi \neq \mathbf{0}$ and $\|s_\psi\|_\infty < \infty$)*

$$\Delta_\psi(0) := \liminf_{\varepsilon \searrow 0+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{s_\psi(\xi)}{\|s_\psi\|_\infty} \, d\xi \, .$$

The values of $\Delta_\psi(0)$ and the infimum in (4.24) *may differ. However,* (4.24) *is valid if and only if $\Delta_\psi(0) > 0$. Furthermore, $\Delta_\psi(0) \in [0,1]$ and, if $\Delta_\psi(0) = 1$, then the* lim inf *in the definition of $\Delta_\psi(0)$ becomes the limit. As we shall see, we can have a maximal density $\Delta_\psi(0) = 1$ and $\mathcal{F}_\psi$ may still not be democratic. In other words, "the density at zero" type condition is closely related to sums $\sum_\Gamma$ of the form $\widetilde{D_N}(\xi)$, where almost all mass is concentrated at zero.*
(**c**) *The sums that disprove democracy in Ex. 4.28 are of the form $\widetilde{D_N}(k\xi)$. Observe that they have "k peaks" evenly spread with each having roughly $^1/_k$ of the total mass.*
(**d**) *Observe that Ex. 4.28 provides ψ such that p_ψ is the characteristic function of a set, i.e., $\mathcal{F}_\psi$ forms a Parseval frame for $\langle\psi\rangle$. In other words $\psi \in 1$; according to our Figure in Section 3.*

■

Example 4.30. *For $k \in \mathbb{N}$ we define*

$$\widetilde{E_k} := \bigcup_{j=1}^{2^k-1} \langle \frac{j}{2^k} - \frac{1}{2 \cdot 2^{3k}}, \frac{j}{2^k} + \frac{1}{2 \cdot 2^{3k}} \rangle \, ,$$

and take $\widetilde{A} := \bigcup_{k=1}^{\infty} \widetilde{E_k} \subseteq \langle 0,1\rangle$. Let $\widetilde{B} := \langle 0,1\rangle \setminus \widetilde{A}$ and $\psi \in L^2(\mathbb{R})$ be such that $\widehat{\psi} = \chi_{\widetilde{B}}$. Hence, p_ψ is the periodization of $\chi_{\widetilde{B}}$ and, since $\widetilde{B}$ is symmetric, $p_\psi = s_\psi$.

Since

$$\int_{\widetilde{A}} \left|\widetilde{D_N}(2^k\xi)\right|^2 d\xi \geqslant \int_{\widetilde{E}_k} \left|\widetilde{D_N}(2^k\xi)\right|^2 d\xi =$$

$$= \sum_{j=1}^{2^k-1} \int_{\frac{j}{2^k}-\frac{1}{2\cdot 2^{3k}}}^{\frac{j}{k}+\frac{1}{2\cdot 2^{3k}}} \left|\widetilde{D_N}(2^k\xi)\right|^2 d\xi = \frac{1}{2^k}(2^k-1) \int_{-\frac{1}{2\cdot 2^{2k}}}^{\frac{1}{2\cdot 2^{2k}}} \left|\widetilde{D_N}(\xi)\right|^2 d\xi\,,$$

we obtain

$$1 \geqslant \liminf_{N\to\infty} \frac{1}{N} \int_{\widetilde{A}} |\widetilde{D_N}(2^k\xi)|^2\, d\xi \geqslant \frac{2^k-1}{2^k}\,,$$

which shows that

$$1 = \lim_{k\to\infty} \Big(\liminf_{N\to\infty} \frac{1}{N} \int_{\widetilde{A}} \left|\widetilde{D_N}(2^k\xi)\right|^2 d\xi \Big)\,.$$

Therefore,

$$\lim_{k\to\infty} \Big(\limsup_{N\to\infty} \frac{1}{N} \int_{\widetilde{B}} \left|\widetilde{D_N}(2^k\xi)\right|^2 d\xi \Big) = 0\,,$$

which shows that $\mathcal{F}_\psi$ is not democratic.

Observe that

$$\frac{\left|\widetilde{A} \cap \langle 0,\varepsilon\rangle\right|}{\varepsilon} \leqslant \frac{1}{\varepsilon}\Big[\frac{2}{2^{3(k_\circ+1)}} + \frac{2}{2^{3(k_\circ+2)}} + \frac{4}{2^{3(k_\circ+3)}} + \cdots\Big]\,,$$

for some $k_\circ \in \mathbb{N}$ large enough. Hence,

$$\frac{|\widetilde{A} \cap \langle 0,\varepsilon\rangle|}{\varepsilon} \leqslant \frac{1}{\varepsilon}\Big(\frac{1}{2^{3(k_\circ+1)}} + \frac{\frac{1}{2^{3(k_\circ+1)}}}{1-\frac{2}{2^3}}\Big) \leqslant \frac{7}{128}\varepsilon^2\,,$$

since $2^{\frac{1}{k_\circ+1}} < \varepsilon \leqslant \frac{1}{2^{k_\circ}}$. It follows that

$$1 \geqslant \lim_{\varepsilon\to 0} \frac{\left|\widetilde{B} \cap \langle -\varepsilon,\varepsilon\rangle\right|}{2\varepsilon} \geqslant \lim_{\varepsilon\to 0} \big(1 - \frac{7}{128}\varepsilon^2\big) = 1\,,$$

which shows $\Delta_\psi(0) = 1$; observe $s_\psi \leqslant 1$.

♦

As we have seen, Thm. 4.23 takes care of the sums of the form $\widetilde{D_N}(\xi)$, but it is not sufficient for the democratic property. We can use roughly a fairly similar technique on sums of the form $\widetilde{D_N}(k\xi)$; we leave the details of the proof as an exercise to our readers.

Proposition 4.31. *If $\mathcal{F}_\psi$ is democratic, then for every $L \in \mathbb{N}$*

$$\inf_{0<\varepsilon<1/2} \frac{1}{2\varepsilon} \int_{\bigcup_{j=0}^{L-1} \langle \frac{j}{L}-\frac{\varepsilon}{L}, \frac{j}{L}+\frac{\varepsilon}{L}\rangle} s_\psi(\xi)\, d\xi > 0\,. \tag{4.32}$$

Remark 4.33. *Despite the fact that* $\sum_\Gamma$ *can take many different forms; not just* $\widetilde{D_N}(\xi)$ *or* $\widetilde{D_N}(k\xi)$*, there are some hints to suggest that taking care of these two type of sums may be sufficient for the democratic property. Unfortunately, at this point we are not able neither to prove nor to disprove this conjecture. Hence, the question of general and "usable" necessary and sufficient conditions for democracy remains open. We devote the rest of the paper to the analysis of some special subclasses of functions.*

■

If we have some additional analytic property of either p_ψ or s_ψ around 0, we may be able to obtain very elegant characterizations. Let us illustrate this on a particular class and we leave other similar versions as an exercise to our readers.

Let us define the following class

$$\mathcal{L}im := \left\{\psi \in L^2(\mathbb{R}) \setminus \{\mathbf{0}\} : \quad p_\psi(0\pm) := \lim_{\xi \to 0\pm} p_\psi(\xi) \text{ exist}\right\}.$$

Corollary 4.34. *If* $\psi \in \mathcal{L}im$*, then* $\mathcal{F}_\psi$ *is democratic if and only if* $\mathcal{F}_\psi$ *is Besselian and*

$$p_\psi(0-) + p_\psi(0+) > 0\,. \tag{4.35}$$

Proof. Let $\psi \in \mathcal{L}im$ be such that $\mathcal{F}_\psi$ is Besselian. If (4.35) is fulfilled, then there exists $\delta > 0$ such that s_ψ is bounded from below on $[-\delta, \delta]$ a.e. By Cor. 4.20 $\mathcal{F}_\psi$ is democratic.

If $p_\psi(0-) + p_\psi(0+) = 0$, then $p_\psi(0-) = p_\psi(0+) = 0 = \lim_{\xi \to 0} s_\psi(\xi)\,.$

Then $\mathcal{F}_\psi$ can not be democratic by Thm. 4.23. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

There is an obvious, but sometimes useful, special case.

Corollary 4.36. *Let* $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ *be such that* $\mathcal{F}_\psi$ *is Besselian and* s_ψ *is continuous at zero. Then,* $\mathcal{F}_\psi$ *is democratic if and only if* $s_\psi(0) > 0$.

Consider now our Figure from Section 3. We know that if $\mathcal{F}_\psi$ is democratic, then $\psi \in 1 \cup 2 \cup a \cup b \cup B$. Furthermore, if $\psi \in a \cup b$, then $\mathcal{F}_\psi$ is democratic. We shall explore $\psi \in 1 \cup 2$ first. Since then p_ψ is bounded from below and above on $\{p_\psi > 0\}$, the analysis of the democratic property depends only on the property of the set $\{p_\psi > 0\}$. Hence, without loss of generality, we can consider the case of Parseval frames, i.e., $\psi \in 1$. This is the case characterized by $p_\psi = \chi_{\{p_\psi > 0\}}$. In particular, these systems are always Besselian and the dual function $\widetilde{\psi} = \psi$.

We introduce the following notion

$$\mathcal{D} := \Big\{ A \subseteq [-1/2, 1/2] : \ \big(\exists \psi \in L^2(\mathbb{R})\big) \\ (\mathcal{F}_\psi \text{ democratic }) \quad p_\psi\big|_{[-1/2,1/2]} = \chi_A \Big\} . \tag{4.37}$$

For $A \subseteq [-1/2, 1/2]$ we denote $[-1/2, 1/2] \setminus A$ by A^c. Observe that every set A in $\mathcal{D}$ has to be measurable and that "$A \in \mathcal{D}$" is really the property of equivalence class (like with functions $\psi \in L^2(\mathbb{R})$); if $A \in \mathcal{D}$ and $B \subseteq [-1/2, 1/2]$ is measurable and $|A \triangle B| = 0$, then $B \in \mathcal{D}$. Furthermore, it is obvious that $\mathcal{D}$ has the hereditary property in the sense

$$\big(A \in \mathcal{D} \text{ and } A \subseteq B \subseteq [-1/2, 1/2] \text{ measurable } \Rightarrow \ B \in \mathcal{D}\big) . \tag{4.38}$$

In particular, if $A, B \in \mathcal{D}$, then $A \cup B \in \mathcal{D}$. The following results and examples are easy consequences of our previous theorems.

Example 4.39. (**a**) *If* $0 \in \mathrm{Int}(A)$*, then* $A \in \mathcal{D}$ *and* $A^c \notin \mathcal{D}$*. Hence, whenever there is* $\delta > 0$ *such that* $\langle -\delta, \delta\rangle \subseteq \{p_\psi > 0\}$ *a.e. and* $\psi \in 1$*, then* $\mathcal{F}_\psi$ *is democratic. On the other hand, whenever there is* $\delta > 0$ *such that* $\langle -\delta, \delta\rangle \subseteq \{p_\psi = 0\}$ *a.e. and* $\psi \in 1$*, then* $\mathcal{F}_\psi$ *is not democratic.*

Obviously, then, if $0 \notin \mathrm{Cl}(A)$*, then* $A \notin \mathcal{D}$ *and* $A^c \in \mathcal{D}$*. The difficult cases are when* $0 \in \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$ *in "an essential way" (in the sense that there are no* B *such that* $|A \triangle B| = 0$ *and either* $0 \in \mathrm{Int}(B)$ *or* $0 \notin \mathrm{Cl}(B)$*).*
(**b**) *If* $0 \in \mathrm{Int}(A \cup -A)$*, then* $A \in \mathcal{D}$*. For example* $A = [-1/2, 0] \in \mathcal{D}$*, bur also* $A^c = \langle 0, 1/2] \in \mathcal{D}$*. Therefore, the intersection of two sets in* $\mathcal{D}$ *does not have to be in* $\mathcal{D}$*.*

As for an illustration let us emphasize that any set A *which contains a.e. any of the following sets must be in* $\mathcal{D}$*:*

$$[0, a] , \quad 0 < a \leqslant 1/2$$
$$[-b, 0] , \quad 0 < b \leqslant 1/2$$
$$\Big(\bigcup_{n\in\mathbb{N}} \langle b_n, a_n\rangle\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} \langle -b_n, -a_{n+1}\rangle\Big) , \, 1/2 \geqslant a_1 > b_1 > a_2 > b_2 \ldots \searrow 0 .$$

On the other hand, even for very small $\delta > 0$ *and* $\varepsilon > 0$*, set* $[1/2, -\delta] \cup [\varepsilon, 1/2]$ *is not in* $\mathcal{D}$*.*

♦

Observe that the density results apply easily to this situation and we obtain

$$\Big(A \in \mathcal{D} \Rightarrow \liminf_{\varepsilon \searrow 0} \frac{\big|A \cap \langle -\varepsilon, \varepsilon\rangle\big|}{2\varepsilon} > 0\Big) . \tag{4.40}$$

Using (4.40) and Ex. 4.30 we obtain set A with the property $A \notin \mathcal{D}$ and $A^c \notin \mathcal{D}$. One can describe these various options using that, for every $\Gamma \subseteq \mathbb{Z}$, Γ finite, we have

$$\operatorname{card}(\Gamma) = \int_{-1/2}^{1/2} |\textstyle\sum_\Gamma|^2\, d\xi = \int_A |\textstyle\sum_\Gamma|^2\, d\xi + \int_{A^c} |\textstyle\sum_\Gamma|^2\, d\xi\,.$$

Hence, we obtain directly the following result.

Proposition 4.41. *If $A \subseteq [-1/2, 1/2]$ is measurable, then the following are equivalent:*

(*a*) $A \in \mathcal{D}$;

(*b*) $\inf_\Gamma \dfrac{\int_A |\sum_\Gamma|^2\, d\xi}{\operatorname{card}(\Gamma)} > 0$;

(*c*) $\sup_\Gamma \dfrac{\int_{A^c} |\sum_\Gamma|^2\, d\xi}{\operatorname{card}(\Gamma)} < 1$.

It is also not difficult to see that

$$\big(A \in \mathcal{D} \Leftrightarrow A \cup (-A) \in \mathcal{D}\big)\,. \tag{4.42}$$

Hence, resolving the democratic property for $\psi \in 1$ is equivalent to answering the following question. Given a measurable, symmetric (i.e., $-H = H$) subset $H \subseteq [-1/2, 1/2]$, what are necessary and sufficient conditions on H to satisfy that here exists a constant $0 < C_0 < +\infty$ such that

$$\int_H \big|\textstyle\sum_\Gamma(\xi)\big|^2\, d\xi \geqslant C_0 \int_{-1/2}^{1/2} \big|\textstyle\sum_\Gamma(\xi)\big|^2\, d\xi\,, \tag{4.43}$$

for every $\Gamma \subseteq \mathbb{Z}$, Γ finite.

Remark 4.44. *Inequality* (4.43) *is the well-known concentration inequality for $p = 2$ (for a sample of rich literature on the subject see* [1], [2], *and the references therein). The p-concentration problem has been resolved in the case $p = 2$. However, our question remains open. There is a different approach in* (4.43)*; we fix a set and try to satisfy* (4.43) *for* ***all*** Γ *(with a constant which does not depend on* Γ*). For the p-concentration problem one usually seeks for some idempotent function (in terms of convolutions) to satisfy the concentration inequality, i.e., the function may depend on the set.*

We have seen several sufficient conditions and some necessary ones, but not a complete answer.

■

Consider now the case $\psi \in B$. Roughly speaking the analysis of the democratic property for a class x_B, $x = 3, 4, 5$, will be a combination of conditions for class 1 and y_B, $y = c, d, e$; respectively. Therefore, we shall focus our attention to $\psi \in c_B \cup d_B \cup e_B$.

Consider $\psi \in c_B$ first. This means that $\mathcal{F}_\psi$ is a Schauder basis for $\langle\psi\rangle$, it is a conditional basis which satisfies property (H) and does not satisfy property (B) (see [15] for terminology). The dual function $\widetilde{\psi}$ generates the dual basis $\mathcal{F}_{\widetilde{\psi}}$ which is not Besselian; in particular it is never democratic. What about $\mathcal{F}_\psi$? Can it be democratic? Must it be democratic? The following example is typical; at least for functions with the limit at zero.

Example 4.45. *Let $0 < \alpha < 1$ and let ψ be such that $(p_\psi\big|_{[-\frac{1}{2},\frac{1}{2}]})(\xi) = |\xi|^\alpha$; recall that p_ψ is 1-periodic. It is not difficult to check that p_ψ satisfies the Muckenhoupt A_2 condition, which shows that $\mathcal{F}_\psi$ is a Schauder basis. Since p_ψ is bounded above, but is not bounded below (away from zero), we have $\psi \in c_B$ (this is actually famous Babenko's example). If we take $a \in \langle -1/2, 1/2\rangle$ and define $p_{\psi,a}(\xi) := p_\psi(\xi - a)$, it follows that there exists $\varphi_a \in L^2(\mathbb{R})$ with the property $p_{\varphi_a} = p_{\psi,a}$ (for $a = 0$, $\varphi_a = \psi$) and that $\mathcal{F}_{\varphi_a} \in c_B$. Since $p_{\psi,a}$ is continuous at zero in all cases, but it is equal to zero only for $a = 0$, we obtain:*

$$\begin{aligned}&\mathcal{F}_{\varphi_a} \textit{ is democratic for } a \neq 0\\ &\mathcal{F}_\psi \textit{ is not democratic}.\end{aligned}$$

Observe a "catastrophic" behavior where we have democratic systems for all small $a \neq 0$, but not for $a = 0$.

♦

One can adjust numerous examples of the same type, by taking some other continuous function to begin with; instead of $|\xi|^\alpha$. There are many (even polynomial) examples which are going to satisfy the A_2 condition. They are all going to exhibit the same property; if the value at zero is not zero, then the system is democratic, otherwise it is not. Completely analogous types of examples can be adjusted for classes d_B and e_B. Therefore we can find democratic and non-democratic systems in all these cases. Unfortunately, we were not able to improve on full characterization of democracy in neither of these classes. It appears that none of the conditions (3.3), (3.10) and (3.11), is particularly related to the democratic property.

Let us complete this article with observations that in classes c_B and d_B we have dual functions $\widetilde{\psi}$, but systems $\mathcal{F}_{\widetilde{\psi}}$ are not Besselian (hence not democratic). Therefore we have the following result.

Corollary 4.46. *Let* $\mathbf{0} \neq \psi \in L^2(\mathbb{R})$ *be such that* $\mathcal{F}_\psi$ *is democratic. If* $\mathcal{F}_\psi$ *is a minimal system (in particular, if* $\mathcal{F}_\psi$ *is a Schauder basis), then* $\mathcal{F}_{\tilde{\psi}}$ *is democratic if and only if* $\mathcal{F}_\psi$ *is a Reisz basis.*

Observe that $\mathcal{F}_\psi$ from Ex. 4.45 is an example of a conditional Schauder basis where neither $\mathcal{F}_\psi$ nor $\mathcal{F}_{\widetilde{\psi}}$ are democratic.

References

[1] A. Bonami, Sz. Gy. Révész *"Failure of Wieuer's property for positive definite periodic functions"*, C. R. Math. Acad. Sci. Paris 346 (2008), no. 1-2, 39-44.

[2] A. Bonami, Sz. Gy. Révész *"Integral concentration of idempotent trigonometric polynomials with gaps"*, Amer. J. Math. 131 (2009), no.4, 1065-1108.

[3] Ch. Heil and A. Powell *"Gabor Schauder bases and the Balian-Low theorem"*, J. Math. Phys. 47 (11) (2006), 113506, 21pp.

[4] E. Hernández, D. Labate, G. Weiss, E. N. Wilson *"Oversampling quasi-affine frames, and wave packets"*, Appl. Comput. Harmon. Anal. 16 (2004), no.2, 111-147.

[5] E. Hernández, H. Šikić, G. Weiss, E. N. Wilson *"Cyclic subspaces for unitary representations of LCA groups; generalized Zak transform"*, Collog. Math. 118 (2010), no.1, 313-332.

[6] E. Hernández, H. Šikić, G. Weiss, E. N. Wilson *"On the properties of the integer translates of a square integrable function"*, Contemp. Math. 505 (2010), 233-249.

[7] E. Hernández and G. Weiss, *"A First Course on Wavelets"*, Stud. Adv. Math., CRC Press, Boca Raton, FL, (1996).

[8] S.V. Konyagin, V. N. Temlyakov *"A remark on greedy approximation in Banach spaces"*, East. J. Approx. 5 (3) (1999), 365-379.

[9] J. Lindenstrauss, L. Tzafriri *"Classical Banach Spaces I: Sequence spaces"*, Ergeb. Math. Grenzgeb., vol. 92, Springer, Berlin, 1977.

[10] M. Nielsen *"Trigonometric bases for matrix weighted Lp-spaces"*, J. Math. Anal. Appl. 371 (2010), no. 2, 784-792.

[11] M. Nielsen, H. Šikić *"Schauder bases of integer translates"*, Appl. Comput. Harmon. Anal. 23 (2007), no. 2, 259-262.

[12] M. Nielsen, H. Šikić *"Quasi-greedy systems of integer translates"*, J. Approx. Theory 155 (2008), no. 1, 43-51.

[13] H. Šikić, D. Speegle *"Dyadic PFW's and* $W_\circ$*-bases"*, Functional Analysis IX, 85-90, Various Publ. Ser. (Aarhus), (48), Univ. Aarhus, Aarhus, 2007.

[14] H. Šikić, D. Speegle and G. Weiss, *"Structure of the set of dyadic PFW's"*, Contemp. Math. 451 (2008), 263-291.

[15] I. Singer *"Bases in Banach Spaces I"*, Die Grundlehren der mathematischen Wissenscheften, vol. 154, Springer-Verlag, New-York, 1970.

[16] V. N. Temlyakov *"The best m-term approximation and greedy algorithms"*, Adv. Comput. Math. 8 (3) (1998), 249-265.

[17] V. N. Temlyakov *"Greedy algorithm and m-term trigonometric approximation"*, Constr. Approx. 14 (4) (1998), 569-587.

[18] R. Young *"An Introduction to Nonharmonic Fourier Series"*, revised 1st ed., Academic, San Diego, 2001.

[19] P. Wojtaszczyk *"Greedy algorithm for general biorthogonal systems"*, J. Approx. Theory 107 (2) (2000), 293-314.

Eugenio Hernández, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain. e-mail address: eugenio.hernandez@uam.es

Hrvoje Šikić, Dept. of Mathematics, Univ. of Zagreb, Bijenička 30, HR-10 000, Zagreb, Croatia. e-mail address: hsikic@math.hr