# On the Properties of the Integer Translates of a Square Integrable Function in $L^2(\mathbb{R})$

by
Eugenio Hernández, Hrvoje Šikić, Guido Weiss, Edward Wilson

## §1 Introduction

Those who work in the area of wavelets are very much aware of the importance of the shift invariant spaces generated by a function $\psi$ in $L^2(\mathbb{R})$. These are those closed subspaces of $L^2(\mathbb{R})$, denoted by $\langle\psi\rangle$, that are defined by

$$\langle\psi\rangle = \overline{span\{\psi(\cdot - k) \equiv \psi_k : k \in \mathbb{Z}\}} \tag{1.1}$$

That is, $\langle\psi\rangle$ is the $L^2(\mathbb{R})$ closure of the space generated by the finite linear combinations of the integer translates, $(T_k\psi)(x) \equiv \psi(x-k) \equiv \psi_k(x)$, of $\psi \in L^2(\mathbb{R})$. Such a space is often called the *principal shift invariant space generated by* $\psi$. In general, a closed subspace $V \subset L^2(\mathbb{R})$ is *shift invariant* if and only if $T_k V \subset V$ for all translates $T_k, k \in \mathbb{Z}$. The theory of wavelets is based on the properties of the principal shift invariant spaces. The scaling space $V_0$ in a *multiresolution analysis* (MRA) is an example of such a space; the *zero resolution wavelet space* $W_0$ is another such space. These two spaces enjoy very different features. The basic properties of $\langle\psi\rangle$ are determined by those of the generating system $\mathcal{B} = \{\psi_k = T_k\psi : k \in \mathbb{Z}\}$. As we will show, those properties are reflected by simple properties of the *periodization function*

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2\,. \tag{1.2}$$

Of course, $\hat{\psi}$ is the Fourier transform of $\psi$:

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-2\pi i \xi x} dx \tag{1.3}$$

[0]The research was supported by the US-Croatian grant NSF-INT-0245238. The research of the first named author was supported by grants MTM2007-60952 of Spain and SIMUMAT S-0505/ESP-0158 of the Madrid Community Region. The research of the second named author was also supported by the MZOS grant 037-0372790-2799 of the Republic of Croatia.

[0]*Keywords and Phrases.* Principal shift invariant spaces, bases, frames, periodization function.

(well defined when $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and with the well known extension for the general $\psi \in L^2(\mathbb{R})$).

Let us begin by showing how $p_\psi$ leads us to a natural correspondence between the space $\langle\psi\rangle$ and the space $\mathcal{M}_\psi = L^2([0,1), p_\psi)$ of all 1-periodic functions $m$ on $\mathbb{R}$ satisfying

$$\int_0^1 |m(\xi)|^2 p_\psi(\xi) d\xi < \infty \,.$$

We observe that it follows from the definition of $p_\psi$ that this function is the general non-negative 1-periodic function in $L^1([0,1), d\xi)$. We introduce the *bracket* of two functions $\hat{f}$ and $\hat{g} \in L^2(\hat{\mathbb{R}})$ defined by

$$[\hat{f}, \hat{g}](\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + k)\overline{\hat{g}(\xi + k)} \,. \tag{1.4}$$

Note that $p_\psi = [\hat{\psi}, \hat{\psi}]$. For $f, g \in L^2(\mathbb{R})$, $|[\hat{f}, \hat{g}]| \leqq [\hat{f}, \hat{f}]^{1/2}[\hat{g}, \hat{g}]^{1/2}$. Moreover,

$$\begin{aligned}\langle T_k f, g\rangle = \langle (T_k f)^\wedge, \hat{g}\rangle &= \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\xi)e^{-2\pi i\xi k} d\xi \\ &= \int_0^1 [\hat{f}, \hat{g}](\xi)e^{-2\pi ik\xi} d\xi = k^{th} \text{ Fourier coefficient of } [\hat{f}, \hat{g}] \,,\end{aligned}$$

which is a function in $L^1([0,1])$. Hence, we have:

**Lemma 1.5.** *For $f, g \in L^2(\mathbb{R})$, $\langle f\rangle \perp \langle g\rangle$ if and only if $[\hat{f}, \hat{g}](\xi) = 0$ a.e.*

This is clear since $\langle f\rangle \perp \langle g\rangle$ iff $\langle T_j f, T_l g\rangle = 0$ for all $j, l \in \mathbb{Z}$ iff $\langle T_k f, g\rangle = 0$ for all $k \in \mathbb{Z}$ iff $[\hat{f}, \hat{g}](\xi) = 0$ a.e.)

This perpendicularity criterion provides an easy proof of a basic result that will be used by us many times. In order to state this theorem we introduce the map $J_\psi : \mathcal{M}_\psi \to L^2(\mathbb{R})$ defined by letting $J_\psi m = (m\hat{\psi})^\vee$ for each $m \in \mathcal{M}_\psi$. We then have

**Theorem 1.6.** *$J_\psi$ is an isometry between $\mathcal{M}_\psi$ and $\langle\psi\rangle$.*

*Proof.* That $J_\psi$ is an isometry is a consequence of the following "periodization" argument:

$$\begin{aligned}\|J_\psi m\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |m(\xi)|^2|\hat{\psi}(\xi)|^2 d\xi = \int_0^1 |m(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2 d\xi \\ &= \int_0^1 |m(\xi)|^2 p_\psi(\xi) d\xi \equiv \|m\|^2_{\mathcal{M}_\psi} \,.\end{aligned}$$

If $h \in \langle\psi\rangle^\perp$, then $\langle h\rangle \perp \langle\psi\rangle$ and, thus, by Lemma 1.5,

$$[\hat{h}, (J_\psi m)^\wedge](\xi) = \overline{m(\xi)}[\hat{h}, \hat{\psi}](\xi) = 0 \text{ a.e.} \quad \text{for all } m \in \mathcal{M}_\psi\,.$$

It follows that $J_\psi m \in (\langle\psi\rangle^\perp)^\perp = \langle\psi\rangle$. Hence $J_\psi$ maps $\mathcal{M}_\psi$ into $\langle\psi\rangle$.

If $J_\psi$ were not onto $\langle\psi\rangle$, then $J_\psi\mathcal{M}_\psi$ is a proper subspace of $\langle\psi\rangle$ (since $J_\psi$ is an isometry). Hence, there exists a non zero $g \in \langle\psi\rangle$ such that

$$0 = \langle g, J_\psi m\rangle = \langle \hat{g}, m\hat{\psi}\rangle = \int_0^1 \overline{m(\xi)}[\hat{g}, \hat{\psi}](\xi)d\xi$$

for all $m \in \mathcal{M}_\psi$. We can clearly select $m \in \mathcal{M}_\psi$, of absolute value 1, such that $\overline{m}[\hat{g}, \hat{\psi}] = |[\hat{g}, \hat{\psi}]|$. Thus,

$$0 = \int_0^1 |[\hat{g}, \hat{\psi}](\xi)|d\xi$$

and it follows that $[\hat{g}, \hat{\psi}](\xi) = 0$ a.e. Using Lemma 1.5 again, we see that $g \perp \langle\psi\rangle$. Since $g \in \langle\psi\rangle$ we have the contradiction that the non zero function $g$ satisfies $g(\xi) = 0$ a.e. ■

Under the operator $J_\psi$ the functions $\psi_k, k \in \mathbb{Z}$, are the images of the exponential functions $e_k(\xi) = e^{-2\pi i k\xi}$. Thus, the various properties of the system $\mathcal{B}$ that generates $\langle\psi\rangle$ correspond to the various properties of the exponential system $\{e_k\}$ on $\mathcal{M}_\psi$. Here are some examples which also show the role played by the periodization function $p_\psi$:

**Theorem 1.7.** *The system $\mathcal{B}$ is linearly independent.*

This is an immediate consequence of the linear independence of the exponential system $\{e_k\}$.

**Theorem 1.8.** *$\mathcal{B}$ is an orthonormal system if and only if $p_\psi(\xi) = 1$ a.e..*

**Theorem 1.9.** *If $p_\psi(\xi) > 0$ a.e. and $\varphi = (\frac{1}{\sqrt{p_\psi}}\hat{\psi})^\vee$ then $\{\varphi_k = T_k\varphi : k \in \mathbb{Z}\}$ is an orthonormal basis of $\langle\psi\rangle$.*

Since, in this case, $\{\frac{e_k}{\sqrt{p_\psi}} : k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{M}_\psi$, Theorem 1.9 follows by applying $J_\psi$ to $\frac{e_k}{\sqrt{p_\psi}}$.

These examples illustrate the main theme of this paper: how the properties of $p_\psi$ correspond to those of $\mathcal{B}$ (and, thus, those of $\psi$). We will examine various generating properties of $\mathcal{B}$, more general than that of being an orthonormal basis, and show how $p_\psi$ reflects these characteristics. By doing this we will obtain a rather complete study of the various principal shift invariant spaces. In addition we will obtain many important concrete examples of various concepts that arise in functional analysis. We also are very much aware that one can extend this study in several different directions: to higher dimensions, to spaces that are defined by applying not only translations but other operators (like dilations and modulations) to not only one function $\psi$, but a set of functions, as is the case for Gabor systems, affine systems and wave packet systems. Moreover, we do not need to restrict ourselves to $L^2$-spaces, but can consider the various function spaces that arise in analysis and group representation theory.

Although many of the results we present are in the literature, some of the proofs we give are simpler than the original ones; in some cases we present more general theorems then those we cite. We also present new results. One of our purposes is to have a unified approach to this material. It is much more appropriate to present most of the relevant references after we have developed our material. The last section of this monograph will be devoted to these references and explanations of these matters.

## §2 Various ways in which the system $\mathcal{B}$ generates the space $\langle\psi\rangle$

In the last section we have seen that $\mathcal{B}$ is always linearly independent. We also found out precisely when it is an orthonormal basis of $\langle\psi\rangle$; moreover, we showed how to modify $\mathcal{B}$ to an orthonormal basis (Theorem 1.9). This was done when $\Omega_\psi = \{\xi \in [0,1) : p_\psi(\xi) \neq 0\}$ has measure $|\Omega_\psi| = 1$. If $0 < |\Omega_\psi| < 1$ we cannot consider the function $1/\sqrt{p_\psi}$; however, if we let $\varphi = [\frac{\tilde{\chi}_{\Omega_\psi}}{\sqrt{p_\psi}}\hat{\psi}]^\vee$ and understand $\tilde{\chi}_{\Omega_\psi}$ to be the 1-periodic extension of $\chi_{\Omega_\psi}$ to the entire real line $\mathbb{R}$ (so that $\tilde{\chi}_{\Omega_\psi}(\xi)/\sqrt{p_\psi(\xi)} = 0$ when $p_\psi(\xi) = 0$) we do have a well defined $\varphi \in \langle\psi\rangle$ (by Theorem 1.6). Let $m \in \mathcal{M}_\psi$ so that $f = (m\hat{\psi})^\vee$ is the general function in $\langle\psi\rangle$. Then

$$\langle f, \varphi_k\rangle = \langle f, T_k\varphi\rangle = \langle \hat{f}, \hat{\varphi}_k\rangle = \int_{\mathbb{R}} m(\xi)\hat{\psi}(\xi)\frac{\tilde{\chi}_{\Omega_\psi}(\xi)}{\sqrt{p_\psi(\xi)}}e^{2\pi ik\xi}\overline{\hat{\psi}}(\xi)d\xi$$
$$= \int_{\Omega_\psi} m(\xi)\sqrt{p_\psi(\xi)}e^{2\pi ik\xi}d\xi$$

is the $-k^{th}$ Fourier coefficient of the function $m(\xi)\sqrt{p_\psi(\xi)}$ which is in $L^2\big([0,1)\big)$. Thus,

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_k\rangle|^2 = \int_0^1 |m(\xi)|^2 p_\psi(\xi)d\xi = \|f\|^2_{L^2(\mathbb{R})}.$$

This extends Theorem 1.9 to

**Theorem 2.1.** *If* $\varphi = [\frac{\tilde{\chi}_{\Omega_\psi}}{\sqrt{p_\psi}}\hat{\psi}]^\vee$, *then the system* $\{\varphi_k : k \in \mathbb{Z}\} = \{T_k\varphi : k \in \mathbb{Z}\}$ *is a Parseval frame for* $\langle\psi\rangle$. *This means that for each* $f \in \langle\psi\rangle$

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_k\rangle|^2 = \|f\|^2_{L^2(\mathbb{R})} \,. \tag{2.2}$$

*As is well known, this is equivalent to*

$$f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_k\rangle \varphi_k \,, \tag{2.2'}$$

*where this sum is the* $L^2(\mathbb{R})$*-norm limit of the partial sums* $\sum_{|k|\leqq n} \langle f, \varphi_k\rangle\varphi_k$ .

This is typical of the fact that the case when $|\Omega_\psi| = 1$ allows us to obtain results that need an appropriate extension when $0 < |\Omega_\psi| < 1$.

This situation occurs when we want to extend Theorem 1.8 to a characterization of those $\mathcal{B}$ that are *Riesz bases.* A definition of this notion is that $\mathcal{B}$ satisfies the following conditions: There exist constants $A$ and $B$, $0 < A \leqq B < \infty$ , such that for each $\{a_j\} \in \ell^2(\mathbb{Z})$ we have

$$A\sum_{j\in\mathbb{Z}} |a_j|^2 \leqq \|\sum_{j\in\mathbb{Z}} a_j\psi_j\|^2_{L^2(\mathbb{R})} \leqq B\sum_{j\in\mathbb{Z}} |a_j|^2 \,. \tag{2.3}$$

In the Hilbert space setting these bases can be shown to be the image of an orthonormal basis under a bounded invertible linear operator. Our setting gives us a very satisfactory example of this fact:

Let us suppose $|\Omega_\psi| = 1$. Then it is clear that the operator $L$ that maps $f \in \langle\psi\rangle\,(= \langle\varphi\rangle)$ into the function $(\sqrt{p_\psi}\hat{f})^\vee$ sends the orthonormal basis $\{\varphi_k : k \in \mathbb{Z}\}$ onto $\mathcal{B}$. $L$ is clearly bounded iff $\|p_\psi\|_\infty < \infty$ and so is $L^{-1}$ bounded iff $\|\frac{1}{p_\psi}\|_\infty < \infty$. In fact, $\|L\| = \|p_\psi^{1/2}\|_\infty$ and $\|L^{-1}\| = \|\frac{1}{\sqrt{p_\psi}}\|_\infty$. If $\{a_j\} \in \ell^2(\mathbb{Z})$ and $f = \sum_{j\in\mathbb{Z}} a_j\varphi_j$, then, assuming $p_\psi$ and $\frac{1}{p_\psi}$ are bounded, we have

$$\sum_{j\in\mathbb{Z}} a_j\widehat{\psi}_j = \sqrt{p_\psi}\sum_{j\in\mathbb{Z}} a_j\widehat{\varphi}_j \qquad \text{and, thus,}$$

$$\|Lf\|^2_{L^2(\mathbb{R})} = \|\sum_{j\in\mathbb{Z}} a_j\widehat{\psi}_j\|^2_{L^2(\mathbb{R})} \leqq \|L\|^2\sum_{j\in\mathbb{Z}} |a_j|^2 \,.$$

Similarly,

$$\sum_{j\in\mathbb{Z}} |a_j|^2 = \|f\|^2_{L^2(\mathbb{R})} = \|L^{-1}Lf\|^2_{L^2(\mathbb{R})} \leqq \|L^{-1}\|^2\,\|\sum a_j\psi_j\|^2_{L^2(\mathbb{R})}$$

We thus obtain the inequalities (2.3) with $A = \|L^{-1}\|^{-2}$ and $B = \|L\|^2$. In fact we easily obtain:

**Theorem 2.4.** *The system $\mathcal{B}$ is a Riesz basis for the space $\langle\psi\rangle$ if and only if*

$$0 < A \leqq p_\psi(\xi) \leqq B < \infty \qquad \text{for a.e. } \xi. \tag{2.5}$$

Before continuing the discussion of Riesz bases, let us introduce the notion of *biorthogonal dual systems.* Suppose $\mathbb{H}$ is a separable Hilbert space and $\mathcal{B} = \{\psi_k : k \in \mathbb{Z}\}$ is a countable set in $\mathbb{H}$ such that $\overline{span\{\psi_k : k \in \mathbb{Z}\}} = \mathbb{H}$. A collection $\tilde{\mathcal{B}} = \{\tilde{\psi}_k : k \in \mathbb{Z}\} \subset \mathbb{H}$ is *biorthogonal to* $\mathcal{B}$ if and only if

$$\langle \psi_j, \tilde{\psi}_k \rangle = \delta_{jk} \quad \text{for all} \quad j, k \in \mathbb{Z}\,. \tag{2.6}$$

Suppose $\{\tilde{\varphi}_k : k \in \mathbb{Z}\}$ is another system biorthogonal to $\mathcal{B}$. Then

$$\langle \psi_j, \tilde{\psi}_k - \tilde{\varphi}_k \rangle = 0 \quad \text{for all } j \text{ and } k\,;$$

hence, $\tilde{\psi}_k - \tilde{\varphi}_k = 0$ for all $k \in \mathbb{Z}$ since $\overline{span\{\psi_k : k \in \mathbb{Z}\}} = \mathbb{H}$. This means that if $\tilde{\mathcal{B}}$ is a biorthogonal dual to $\mathcal{B}$, it is unique. We then call $\tilde{\mathcal{B}}$ the *biorthogonal dual* to $\mathcal{B}$.

Suppose $\mathbb{H} = \langle\psi\rangle$ and $\mathcal{B} = \{\psi_k = T_k\psi : k \in \mathbb{Z}\}$ (as we have been considering). If there exists $\tilde{\psi} \in \langle\psi\rangle$ such that $\langle \psi_k, \tilde{\psi} \rangle = \delta_{k0}$ for all $k \in \mathbb{Z}$ and $\tilde{\psi}_j \equiv T_j\tilde{\psi}$, $j \in \mathbb{Z}$, a change of variables gives us $\langle \psi_k, \tilde{\psi}_j \rangle = \delta_{jk}$ for all $k$ and $j$ in $\mathbb{Z}$. That is, if there exists $\tilde{\psi} \in \langle\psi\rangle$ with the property $\langle \psi_k, \tilde{\psi} \rangle = \delta_{k0}$, the system $\tilde{\mathcal{B}} = \{\tilde{\psi}_j : j \in \mathbb{Z}\}$ is, then, a biorthogonal dual to $\mathcal{B}$. The function $\tilde{\psi}$, if it exists, will be called the *canonical dual function to* $\psi$. The argument above shows that if this canonical dual exists it is unique (justifying the term "canonical").

The following theorem tells us precisly when the canonical dual to $\psi$ exists:

**Theorem 2.7.** *There exists a canonical dual, $\tilde{\psi}$, of $\psi$ that belongs to $\langle\psi\rangle$ if and only if $\frac{1}{p_\psi}$ belongs to $L^1\big([0,1)\big)$. In this case, $\tilde{\psi} = (\frac{1}{p_\psi}\hat{\psi})^\vee$.*

*Proof.* For $\tilde{\psi}$ to be in $\langle\psi\rangle$ it follows from Theorem 1.6 that there exists a unique $m \in \mathcal{M}_\psi$ such that $\tilde{\psi}^\wedge = m\hat{\psi}$. Moreover, $\tilde{\psi}$ must satisfy $\langle \psi_k, \tilde{\psi} \rangle = \delta_{k0}$ for all $k \in \mathbb{Z}$. Moreover, (see Lemma 1.5 and the equalities preceding it), $\langle \psi_k, \tilde{\psi} \rangle$ is the $k^{th}$ Fourier coefficient of $[\hat{\psi}, \hat{\tilde{\psi}}] = \bar{m}p_\psi$ which, by the inequality following (1.4) is a function in $L^1\big([0,1)\big)$. This can only be the case if and only if $\overline{m(\xi)}p_\psi(\xi) = 1$ a.e.. Hence, $m = \overline{m} = \frac{1}{p_\psi}$. Finally, since $\frac{1}{p_\psi} = \frac{p_\psi}{p_\psi^2}$, we see that $\frac{1}{p_\psi} \in \mathcal{M}_\psi$ if and only if $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$. ∎

Let us return to the case where $\mathcal{B}$ is a Riesz basis (Theorem 2.4). The condition $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$ is automatic since $0 < A \leqq p_\psi(\xi)$ a.e.. In fact, both $\sqrt{p_\psi}$ and $\frac{1}{\sqrt{p_\psi}}$ are bounded functions in this case. Theorem 1.9 stated that $\{\varphi_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $\langle\psi\rangle$ if $\varphi = (\frac{1}{\sqrt{p_\psi}}\hat{\psi})^\vee$ and observe that $\sqrt{p_\psi}\widehat{\tilde{\psi}}_k = \widehat{\varphi}_k$ and $\hat{\psi} = \sqrt{p_\psi}\,\widehat{\varphi}$. Thus, when $\mathcal{B}$ is a Riesz basis, we obtain a reproducing formula for $f \in \langle\psi\rangle$:

$$\hat{f} = \frac{\hat{f}}{\sqrt{p_\psi}}\sqrt{p_\psi} = \sum_{k\in\mathbb{Z}} \langle \frac{\hat{f}}{\sqrt{p_\psi}}, \hat{\varphi}_k\rangle \hat{\varphi}_k \sqrt{p_\psi} = \sum_{k\in\mathbb{Z}} \langle \hat{f}, \hat{\tilde{\psi}}_k\rangle \hat{\psi}_k \,.$$

Taking inverse Fourier transforms, we obtain

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_k\rangle \psi_k = \sum_{k\in\mathbb{Z}} \langle f, \psi_k\rangle \tilde{\psi}_k \tag{2.8}$$

for all $f \in \langle\psi\rangle$ (the second equality is a consequence of the argument we just presented and the fact that $\mathcal{B} = \{\psi_k : k \in \mathbb{Z}\}$ is the canonical dual system to $\tilde{\mathcal{B}} = \{\tilde{\psi}_k : k \in \mathbb{Z}\}$). When we drop the assumption that $\mathcal{B}$ is a Riesz basis and only assume that $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$ (which is equivalent to the existence of the biorthogonal dual system $\tilde{\mathcal{B}}$ – see Theorem 2.7) the reproducing formulae (2.8) may fail since we can find $f = (m\hat{\psi})^\vee$ in $\langle\psi\rangle$ (for $m \in \mathcal{M}_\psi$) with $\frac{m}{\sqrt{p_\psi}} \notin \mathcal{M}_\psi$.

We will discuss more delicate issues involving reproducing formulae for $\mathcal{B}$ and $\tilde{\mathcal{B}}$ in §3. At this point, however, we examine these reproducing properties when $0 < |\Omega_\psi| < 1$. As we have seen at the beginning of this section, where Parseval frames corresponded to orthonormal bases when $|\Omega_\psi| = 1$ (Theorem 2.1), we have an extension of Riesz bases in case $0 < |\Omega_\psi| < 1$. Let us begin by observing that the inequalities (2.3), that characterized Riesz bases, are associated with the inequalities

$$A\|f\|_2^2 \leqq \sum_{k\in\mathbb{Z}} |\langle f, \psi_k\rangle|^2 \leqq B\|f\|_2^2 \tag{2.9}$$

for all $f \in \langle\psi\rangle$. To see this we use the operator $L$ we introduced just after (2.3): $Lf = (\sqrt{p_\psi}\hat{f})^\vee$. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\langle f, \psi_k\rangle|^2 &= \sum_{k\in\mathbb{Z}} |\langle (Lf)^\wedge, \frac{1}{\sqrt{p_\psi}}\hat{\psi}_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle (Lf)^\wedge, \hat{\varphi}_k\rangle|^2 \\ &= \|(Lf)^\wedge\|_2^2 = \|Lf\|_2^2\,.\end{aligned}$$

But, as we have seen above (immediately before Theorem 2.4),

$$A\|f\|_2^2 = \|L^{-1}\|^{-2}\,\|f\|_2^2 \leqq \|Lf\|^2 \leqq \|L\|^2\,\|f\|_2^2 \leqq B\|f\|_2^2$$

and we obtain (2.9). But these inequalities are the ones that characterize those systems $\mathcal{B}$ that are frames for $\langle\psi\rangle$. We can extend these facts to the case $0 < |\Omega_\psi| < 1$ and obtain the following characterization (and properties) of those systems $\mathcal{B}$ that are frames for $\langle\psi\rangle$. More precisely, we have

**Theorem 2.10.** *The system $\mathcal{B}$ is a frame for $\langle\psi\rangle$ (that is, it satisfies the inequalities* (2.9)*) if and only if there exist constants $0 < A \leqq B < \infty$ such that*

$$A\chi_{\Omega_\psi}(\xi) \leqq p_\psi(\xi) \leqq B\chi_{\Omega_\psi}(\xi) \qquad a.e.\,. \tag{2.11}$$

*When this is the case, there exists a unique dual system $\tilde{\mathcal{B}} = \{T_k\tilde{\psi} \equiv \tilde{\psi}_k : k \in \mathbb{Z}\} \subset \langle\psi\rangle$, where $\tilde{\psi} = (\frac{1}{p_\psi}\tilde{\chi}_{\Omega_\psi}\hat{\psi})^\vee$ and the reproducing equalities* (2.8) *are satisfied for all $f \in \langle\psi\rangle$.*

*Proof.* We remind the reader that $\tilde{\chi}_{\Omega_\psi}$ is the 1-periodic extension of $\chi_{\Omega_\psi}$ introduced at the beginning of this section. Theorem 1.6 shows that $\tilde{\psi}$ is well defined and belongs to $\langle\psi\rangle$ provided $\chi_{\Omega_\psi}/p_\psi \in L^1\big([0,1)\big)$ (which is certainly the case here since we are assuming (2.11)). In fact, $\langle\tilde{\psi}\rangle = \langle\psi\rangle$ since, clearly, $\langle\tilde{\psi}\rangle \subset \langle\psi\rangle$ and $p_{\tilde{\psi}} = \frac{\chi_{\Omega_\psi}}{p_\psi}$, so that $(\tilde{\psi})^\sim = \psi$ and, thus, $\langle\psi\rangle \subset \langle\tilde{\psi}\rangle$. The operator $L$ that maps $f \in \langle\psi\rangle$ into $(\sqrt{p_\psi}\tilde{\chi}_{\Omega_\psi}\hat{f})^\vee$ is bounded and has a bounded inverse on $\langle\psi\rangle$. The argument that we used to prove (2.8) can be extended to this case (we use Theorem 2.1 in place of Theorem 1.9). ■

*Comments.* (a) When $0 < |\Omega_\psi| < 1$ the biorthogonality (2.6) is no longer valid.

(b) The "best" constants $A$ and $B$ are $\|L^{-1}\|^{-2}$ and $\|L\|^2$, as before; however, $L$ is not a bounded invertible operator on $L^2(\mathbb{R})$ (just on $\langle\psi\rangle$).

(c) Once again we see that a result that is true when $|\Omega_\psi| = 1$ has an extension to the case $0 < |\Omega_\psi| < 1$ if we use $\chi_{\Omega_\psi}$ appropriately (see Theorem 2.1). In fact Theorem 2.1 can be considered to be a special case of Theorem 2.10 (when $p_\psi(\xi) = \chi_{\Omega_\psi}(\xi)$ a.e.; that is, $A = B = 1$ — a Parseval frame is also called a *tight frame* when $A = 1 = B$).

Recall that $\mathcal{B}$ is always an "algebraic basis" in the sense that it is a linearly independent system (Theorem 1.7). We have obtained several results that show how $\mathcal{B}$ can be used to obtain explicit representations of elements of $\langle\psi\rangle$ (sometimes with the help of $\tilde{\mathcal{B}}$). It is natural to ask whether there are other notions of independence (or non-redundancy) that are associated with such representations. The next section is devoted to these concepts.

## §3 Non-redundancy and the Representation of the Elements of $\langle\psi\rangle$ by $\mathcal{B}$.

We begin by examining a notion of independence, or non-redundancy, that we will show is equivalent to the existance of a canonical dual, $\tilde{\psi}$, of $\psi$ (so that $\tilde{\mathcal{B}}$ and $\mathcal{B}$ are biorthogonal systems — see Theorem 2.7).

**Definition.** $\mathcal{B}$ is *minimal* if and only if there does not exist an integer $k_0$ such that

$$\langle\psi\rangle = \overline{span\{\psi_k : k \in \mathbb{Z},\, k \neq k_0\}} \tag{3.1}$$

It turns out that each of the integers $k \in \mathbb{Z}$ can replace $k_0$ in (3.1):

**Lemma 3.2.** *If $\mathcal{B}$ is not minimal, then*

$$\psi_j \in \overline{span\,\{\psi_k : k \in \mathbb{Z}\setminus\{j\}\}}$$

*for each $j \in \mathbb{Z}$.*

*Proof.* If $\mathcal{B}$ is not minimal, there exists $k_0 \in \mathbb{Z}$ such that $\lim\limits_{n\to\infty} \|v_n - \psi_{k_0}\|_2 = 0$ for a sequence $\{v_n : n \in \mathbb{N}\} \subset V_{k_0} \equiv span\{\psi_k : k \neq k_o\}$. For $j \in \mathbb{Z}$, we have $\psi_j = T^{j-k_0}\psi_{k_0}$ if $T$ is the translation by 1 operator. Moreover, $T^{j-k_0}V_{k_0} \equiv V_j$. Since $T^{j-k_0}$ is bounded on $L^2(\mathbb{R})$,

$$\psi_j = T^{j-k_0}\psi_{k_0} = \lim_{n\to\infty} T^{j-k_0}v_n \in \overline{V_j} = \overline{span\{\psi_l : l \neq j\}}\,.$$

■

We now examine the connection between minimality and the existence of $\tilde{\mathcal{B}}$. Assume $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$ so that $\tilde{\mathcal{B}}$ exists and $\mathcal{B}$, $\tilde{\mathcal{B}}$ is a biorthogonal dual system. Then,

$$\langle\tilde{\psi}, \psi_l\rangle = \langle\tilde{\psi}_l, \psi\rangle = 0 \qquad \text{if} \qquad l \neq 0\,.$$

Hence, the non-zero vector $\tilde{\psi} = \tilde{\psi}_0$ is orthogonal to

$$\overline{V} = \overline{V_0} = \overline{span\{\psi_l, l \neq 0\}}\,.$$

Since $\langle\tilde{\psi}_0, \psi_0\rangle = 1$ we have $\psi_0 = a\tilde{\psi}_0 + \theta$ where $a \neq 0$ and $\theta \perp \tilde{\psi}_0$. It follows that $\psi_0 \notin \overline{V}$ and, thus, $\mathcal{B}$ is minimal.

Now assume $\mathcal{B}$ is minimal. Then $\overline{V} \subsetneqq \langle\psi\rangle$; consequently, there exists $\tilde{\psi}_0 (\neq 0)$ in $\langle\psi\rangle \cap \overline{V}^{\perp}$. Clearly, $\langle\psi_0, \tilde{\psi}_0\rangle \neq 0$ and we might as well assume $\langle\psi_0, \tilde{\psi}_0\rangle = 1$. Since $\langle\psi_l, \tilde{\psi}_0\rangle = 0$ for $l \neq 0$ we then have $\langle\psi_j, \tilde{\psi}_k\rangle = \langle\psi_{j-k}, \tilde{\psi}_0\rangle = \delta_{jk}$ for all $j, k \in \mathbb{Z}$. In the proof of Theorem 2.7 we showed that this biorthogonality property implies $1/p_\psi \in L^1\big([0,1)\big)$.

We see, therefore, we have the following characterization of minimality:

**Theorem 3.3.** *$\mathcal{B}$ is minimal if and only if $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$.*

We have, therefore, shown that the condition $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$ is equivalent to the biorthogonality of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ and to the fact that $\mathcal{B}$ is minimal. The fact that biorthogonality and minimality are equivalent is well-known in a more general setting (see [19]). We presented a rather simple proof of this fact in the present setting.

Let us now consider another non-redundancy notion that arises in functional analysis: $\ell^2$-linear independence. In our case, $\mathcal{B}$ is said to be *$\ell^2$-linearly independent* provided there is no non-zero sequence $\{c_n\} \in \ell^2(\mathbb{Z})$ such that $\sum_{k\in\mathbb{Z}} c_k\psi_k = 0$. We must be careful about what we mean by this series. Consistent with the fact that we are dealing with Hilbert spaces (within $L^2(\mathbb{R})$) we want convergence to be with respect to the $L^2(\mathbb{R})$-norm. We will not always be dealing with unconditionally convergent series and, thus, we must decide on an appropriate order that determines the partial sums. Let us agree, therefore, that here (as well as later on) we order $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$ as is usually done when considering Fourier series. Thus, it can be shown that we can interpret $\sum_{k\in\mathbb{Z}} c_k\psi_k$ as the $L^2(\mathbb{R})$-limit of the symmetric partial sums $s_n = \sum_{|k|\leqq n} c_k\psi_k$ .

Suppose $\mathcal{B}$ is $\ell^2$-linearly dependent. Then there exists $\{c_k\}$, a non-zero sequence in $\ell^2(\mathbb{Z})$, such that

$$\lim_{n\to\infty} \| \sum_{|k|\leqq n} c_k\psi_k \|_{L^2(\mathbb{R})} = 0\,.$$

Let $m(\xi)$ be the function in $L^2\big([0,1)\big)$ with Fourier series $\sum\limits_{k\in\mathbb{Z}} c_k e^{2\pi i k\xi}$. Then, $m(\xi) \neq 0$ on a set $E \subset [0,1)$ of positive measure. Moreover, it follows from Theorem 1.6 that the partial sums $\sum\limits_{|k|\leqq n} c_k e^{2\pi i k\xi} = m_n(\xi)$ satisfy

$$\int_0^1 |m_n(\xi)|^2 p_\psi(\xi) d\xi = \| \sum_{|k|\leqq n} c_k\hat{\psi}_k \|^2_{L^2(\mathbb{R})} \longrightarrow 0$$

as $n \to \infty$. It follows that $m(\xi)\sqrt{p_\psi(\xi)} = 0$ a.e.. But $m(\xi) \neq 0$ for $\xi \in E$, a set of positive measure. It follows that $p_\psi(\xi) = 0$ for $\xi \in E$. In particular $0 < |\Omega_\psi| < 1$. We have shown

**Fact 3.4.** *If $p_\psi(\xi) > 0$ a.e., then $\mathcal{B}$ is $\ell^2$-linearly independent.*

It is natural to ask if the property $p_\psi(\xi) > 0$ a.e. is equivalent to $\ell^2$-linear independence. Unfortunately, we have not been able to determine whether this condition is a characterization of $\ell^2$-linear independence of $\mathcal{B}$. We therefore state that it is an *open question* that $\mathcal{B}$ *is $\ell^2$-linearly independent if and only if* $p_\psi(\xi) > 0$ a.e. A particular result in this direction is the following one. Let us say that $\mathcal{B}$ is a *Bessel system* with bound $B < \infty$ provided the second inequality in (2.9) is satisfied for all $f \in \langle\psi\rangle$. We then have

**Fact 3.5.** *If $\mathcal{B}$ is a Bessel system that is $\ell^2$-linearly independent, then $p_\psi(\xi) > 0$ a.e..*

The proof of this fact uses the following version of the fact that $L^\infty\big([0,1)\big)$ represents the dual of $L^1\big([0,1)\big)$

**Lemma 3.6.** *Suppose a measurable non-negative function $s$ on $[0,1)$ satisfies $sm \in L^1\big([0,1)\big)$ whenever $m \in L^1\big([0,1)\big)$, then $s \in L^\infty\big([0,1)\big)$.*

*Proof.* If $\|s\|_\infty = \infty$, there exists a sequence $\{j_k\}$, $k \geqq 1$, in $\mathbb{N}$ such that $1 < j_1 < j_2 < j_3 < \ldots$ and sets $E_k = \{\xi \in [0,1) : 2^{j_k} \leqq s(\xi) < 2^{j_{k+1}}\}$ of positive measure $\nu_k, k \in \mathbb{N}$. Let $m = \sum_{k\in\mathbb{N}} 2^{-j_k/2}\nu_k^{-1}\chi_{E_k}$. Then

$$\infty > \int_0^1 s(\xi)m(\xi)d\xi \geqq \sum_{k\in\mathbb{N}} 2^{-j_k/2}2^{j_k} = \sum_{k\in\mathbb{N}} 2^{j_k/2} = \infty\,.$$

■

**Theorem 3.7.** *$\mathcal{B}$ is a Bessel system if and only if $\|p_\psi\|_\infty < \infty$.*

*Proof.* The second inequality in (2.9) involves the quantity

$$\sum_{k\in\mathbb{Z}} |\langle f, \psi_k\rangle|^2$$

for the general $f \in \langle\psi\rangle$. From Theorem 1.6 we know that $\hat{f} = q\hat{\psi}$, where $q$ is the general element in $\mathcal{M}_\psi$ and

$$\|f\|_2^2 = \int_0^1 |q(\xi)|^2 p_\psi(\xi)d\xi\,.$$

On the other hand,

$$\sum_{k\in\mathbb{Z}} |\langle f, \psi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}}(\xi)e^{2\pi ik\xi}d\xi|^2 = \sum_{k\in\mathbb{Z}} |\int_0^1 q(\xi)p_\psi(\xi)e^{2\pi ik\xi}d\xi|^2$$

which is the sum of the squares of the absolute values of the Fourier coefficients of the function $qp_\psi = (q\sqrt{p_\psi})\sqrt{p_\psi}$. Thus, the second inequality in (2.9) asserts that

$$\int_0^1 \left[|q(\xi)|^2 p_\psi(\xi)\right] p_\psi(\xi)d\xi \leqq B\int_0^1 |q(\xi)|^2 p_\psi(\xi)d\xi\,.$$

In the notation of Lemma 3.6, where $|m(\xi)| = |q(\xi)|^2 p_\psi(\xi)$ and $s(\xi) = p_\psi(\xi)$, and observing that $|m(\xi)|$ is the absolute value of the general $m \in L^1\big([0,1)\big)$, we see that s satisfies the hypotheses of Lemma 3.6. It follows that $\|s\|_\infty = \|p_\psi\|_\infty < \infty$. The converse follows easily from the above arguments. Hence, Theorem 3.7 is true and we see that the "best" $B$ in inequality (2.9) is $\|p_\psi\|_\infty$. ■

*Proof.* Let us now turn to the proof of Fact 3.5. It suffices to show that $|\Omega_\psi| < 1$ implies $\mathcal{B}$ is $\ell^2$-linearly dependent. In this case let $m$ be the characteristic function of $[0,1) \setminus \Omega_\psi$. Then $m(\xi)\sqrt{p_\psi(\xi)} = 0$ on $[0,1)$. Let $\sum_{|k| \leqq n} c_k e^{2\pi i k\xi} = m_n(\xi)$ be the symmetric partial sums of the Fourier series of $m, n \in \mathbb{N}$. By Theorem 3.7 and our assumption that $\mathcal{B}$ is a Bessel system, we have $\|p_\psi\|_\infty < \infty$. Then

$$\|m_n\sqrt{p_\psi}\|_2^2 = \|(m_n - m)\sqrt{p_\psi}\|_2^2 \leqq \|p_\psi\|_\infty \|m_n - m\|_2^2 \to 0 \quad \text{as} \quad n \to \infty$$

Hence,

$$\|\sum_{|k| \leqq n} c_k \psi_k\|_{L^2(\mathbb{R})} = \|m_n \sqrt{p_\psi}\|_{L^2([0,1))} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $\mathcal{B}$ is $\ell^2$-linearly dependent. ■

Let us now pass to examining another notion of non-redundancy that allows us to represent the elements of $\langle\psi\rangle$ in terms of $\mathcal{B}$. There is a natural notion of *basis* on a general topological vector space: a collection $\mathcal{B} = \{\psi_k : k \in \mathbb{Z}\}$ in this vector space is such a (countable) basis if and only if for each vector $v$ there exists a unique set of coefficients in $\mathbb{C}$, $\{f_k(v) : k \in \mathbb{Z}\}$, such that

$$v = \lim_{n \to \infty} \sum_{|k| \leqq n} f_k(v)\psi_k \tag{3.8}$$

in the topology of the space. The maps $v \to f_k(v)$ are each linear; however, it may happen that they are not continuous linear functionals. In a Banach space, however, it can be shown (see Theorem (1.6) in Chapter 5 of [11]) that these linear functional are continous. Such a basis for a general topological vector space (for which the $f_k$'s are continous linear functionals) is called a *Schauder basis.* Thus, any basis of a Banach space is a Schauder basis.

We will now determine the property of $p_\psi$ that characterizes those systems $\mathcal{B}$ that are Schauder bases. Let us consider the space $\mathcal{M}_\psi$ in which the exponential functions $e_k, k \in \mathbb{Z}$, correspond, via the isometry $J_\psi$ (see Theorem 1.6), with the elements $\psi_k \in \mathcal{B}$ . Hunt, Muckenhoupt and Wheeden (see

[12]) introduced a class of non-negative weights $\{w(\xi)\}$ on $[0,1)$ associated with various properties of Fourier series and related operators. A particular class of these weights is the class $\mathcal{A}_2 = \mathcal{A}_2\big([0,1)\big)$ : $w \in \mathcal{A}_2$ if and only if there exists a positive $C \in \mathbb{R}$ such that

$$\left[\frac{1}{|I|}\int_I w(\xi)d\xi\right]\left[\frac{1}{|I|}\int_I \frac{d\xi}{w(\xi)}\right] \leqq C \tag{3.9}$$

for all intervals $I \subset [0,1)$.

Suppose $p_\psi \in \mathcal{A}_2$. Then, in particular, $\frac{1}{p_\psi} \in L^1\big([0,1)\big)$. If $m \in \mathcal{M}_\psi$, then $m \in L^1\big([0,1)\big)$ since

$$\int_0^1 |m(\xi)|d\xi = \int_0^1 \{|m(\xi)|\sqrt{p_\psi(\xi)}\}\{\frac{1}{\sqrt{p_\psi(\xi)}}\}d\xi \leqq$$
$$\leqq \left(\int_0^1 |m(\xi)|^2 p_\psi(\xi)d\xi\right)^{1/2}\left(\int_0^1 \frac{1}{p_\psi(\xi)}d\xi\right)^{1/2} < \infty\,.$$

Hence, the Fourier coefficients

$$c_k = \int_0^1 m(\xi)e_k(\xi)d\xi\,, \quad k \in \mathbb{Z}\,,$$

are well defined, and so are the symmetric partial sums

$$s_n(\xi) = s_n(m;\xi) = \sum_{|k|\leqq n} c_k e^{2\pi i k\xi}\,, \quad n \in \mathbb{N}\,,$$

of the Fourier series of $m$. Theorem 8 in [12] asserts that the following three statements are equivalent:

$$\begin{cases} (a) & p_\psi \in \mathcal{A}_2\,; \\ (b) & \text{there exists a constant } C \in (0,\infty),\ \text{independent of} \\ & m \in \mathcal{M}_\psi,\ \text{such that} \\ & \int_0^1 |s_n(m;\xi)|^2 p_\psi(\xi)d\xi \leqq C\int_0^1 |m(\xi)|^2 p_\psi(\xi)d\xi; \\ (c) & \text{if } m \in \mathcal{M}_\psi,\ \lim_{n\to\infty}\int_0^1 |m(\xi) - s_n(m;\xi)|^2 p_\psi(\xi)d\xi = 0\,. \end{cases} \tag{3.10}$$

It is shown in [14] that it follows from these equivalences that $\{e_k : k \in \mathbb{Z}\}$ is a Schauder basis for $\mathcal{M}_\psi$ if and only if $p_\psi \in \mathcal{A}_2$. Thus, by Theorem 1.6 we can conclude that

**Theorem 3.11.** *$\mathcal{B}$ is a Schauder basis for $\langle\psi\rangle$ if and only if $p_\psi \in \mathcal{A}_2$.*

*Comments:* (1) Though, in our case, the convergence

$$\sum_{|k| \leqq n} c_k \psi_k \to f \, (= J_\psi m)$$

is in the $L^2(\mathbb{R})$ norm, the coefficients $c_k$, $k \in \mathbb{Z}$, do not necessarily belong to $\ell^2(\mathbb{Z})$. To see this we need to produce, for a $p_\psi \in \mathcal{A}_2$, an $m \in \mathcal{M}_\psi$ such that $m \notin L^2\big([0,1)\big)$ (recall that we showed that $\mathcal{M}_\psi \subset L^1\big([0,1)\big)$) This is easily done.

(2) The condition $p_\psi \in \mathcal{A}_2$ is stronger than the integrability of $1/p_\psi$. Thus, by Theorem 3.3, if $\mathcal{B}$ is a Schauder basis, then $\mathcal{B}$ is minimal; however, the converse is not always true. It is easy to verify that $\psi$ such that

$$\tilde{\psi}(\xi) = \begin{cases} 1/\sqrt{\xi}(1 - \ln \xi) & \text{for } 0 < \xi < 1\,, \\ 0 & \text{for } \xi \notin (0,1) \end{cases}$$

provides us an example of a $\psi \in L^2(\mathbb{R})$ for which $\frac{1}{p_\psi(\xi)} \in L^1\big([0,1)\big)$; however, $p_\psi \notin \mathcal{A}_2$. We are grateful to Profesor J. M. Martell for providing us this example.

Let us now examine more closely the situation when $\mathcal{B}$ has some of these non-redundant properties such as $\ell^2$-linear independence, and some redundant property, such as non-minimality, in order to see how the space $\langle\psi\rangle$ is generated. Let us also assume $\mathcal{B}$ is a Bessel system. In this specific situation we have $|\Omega_\psi| = 1$ (by Fact 3.5) and $\frac{1}{p_\psi} \notin L^1\big([0,1)\big)$ (by Theorem 3.3). Since $p_\psi(\xi) \leqq B < \infty$ a.e. (by Theorem 3.7) the operator $I_\psi : L^2\big([0,1)\big) \to \mathcal{M}_\psi$ defined by $I_\psi m = m$ is bounded. We then have

**Theorem 3.12.** *If $\mathcal{B}$ is $\ell^2$-linearly independent, is non-minimal and Bessel, then $\sum_{k\in\mathbb{Z}} c_k\psi_k$ converges to an element $f \in \langle\psi\rangle$ (that is,*

$$\| \sum_{|k|\leqq n} c_k \psi_k - f \|_{L^2(\mathbb{R})} \to 0 \quad \textit{as} \quad n \to \infty\,),$$

*whenever $\{c_k\} \in \ell^2(\mathbb{Z})$.*

*Proof.* Suppose $\{c_k\} \in \ell^2(\mathbb{Z})$, then

$$m = \sum_{k\in\mathbb{Z}} c_k e_k = \lim_{n\to\infty} \sum_{|k|\leqq n} c_k e_k \in L^2\big([0,1)\big).$$

Moreover, $J_\psi \circ I_\psi$ is a bounded linear transformation of $L^2\big([0,1)\big)$ into $\langle\psi\rangle$ so that

$$\sum_{k\in\mathbb{Z}} c_k \psi_k = \sum_{k\in\mathbb{Z}} c_k (J_\psi \circ I_\psi)(e_k) = (J_\psi \circ I_\psi)(m) \equiv f\,.$$

■

The $\ell^2$-linear independence of $\mathcal{B}$ shows that the above representation (in terms of $\ell^2(\mathbb{Z})$ sequences) is unique. Furthermore, the elements $f \in \langle\psi\rangle$ that have the representation $f = \sum_{k\in\mathbb{Z}} c_k\psi_k$ are precisely those in the range of $J_\psi \circ I_\psi \equiv Image(J_\psi \circ I_\psi)$. Unfortunately, this range is not all of $\langle\psi\rangle$. If $m = \frac{1}{\sqrt{p_\psi}}$, then $m$ is a well defined measurable function on $[0,1)$ and it belongs to $\mathcal{M}_\psi$; however, $m \notin L^2\big([0,1)\big)$ since $\frac{1}{p_\psi}(= |m|^2)$ is not integrable.

We see, therefore, that even when $\mathcal{B}$ is not minimal but is a Bessel system there are some positive results about the $\ell^2$-representations of elements of $\langle\psi\rangle$. There are, however, some negative facts, such as we cannot represent every $f \in \langle\psi\rangle$ in such a way.

Let us now turn our attention, again, to the case when $\mathcal{B}$ is minimal. We have seen, in this case, that the dual system, $\tilde{\mathcal{B}}$, is well defined. We want to explore further when $f \in \langle\psi\rangle$ can be expressed as a sum $f = \sum_{k\in\mathbb{Z}} c_k\psi_k$. In order to understand this question better, as well as the properties of the coefficients $c_k$ we consider two notions that arise in functional analysis (see [19] pages 337-8):

Let $\mathcal{B}$ be a minimal system, then

(a) $\mathcal{B}$ is it Besselian if and only if

$$\sum_{k\in\mathbb{Z}} c_k\psi_k \text{ is convergent implies } \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty\,;$$

(b) $\mathcal{B}$ is *Hilbertian* if and only if

$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty \text{ implies } \sum_{k\in\mathbb{Z}} c_k\psi_k \text{ is convergent.}$$

We shall say $\mathcal{B}$ *satisfies property* (B) in case (a) is true, and $\mathcal{B}$ *satisfies property* (H) in case (b) is true.

**Theorem 3.13.** *Suppose* $\frac{1}{p_\psi} \in L^1\left([0,1)\right)$ *(or, equivalently, by Theorem 3.7,* $\mathcal{B}$ *is minimal). Then the following are equivalent:*

*(a)* $\mathcal{B}$ *satisfies property (H);*

*(b)* $\|p_\psi\|_\infty < \infty$*;*

*(c)* $\mathcal{B}$ *is a Bessel system.*

*If any of these equivalent statements is true, then* $\tilde{\mathcal{B}}$ *satisfies property (B).*

*If $\mathcal{B}$ satisfies property (B) and, for each $\varphi \in \langle \psi \rangle$ there exists a sequence $\{c_n\}$ such that $\varphi = \sum_{k \in \mathbb{Z}} c_k \psi_k$, then $\| \frac{1}{p_\psi} \|_\infty < \infty$.*

*Proof.* Assume (a). Since $\mathcal{B}$ satisfies property (H), if $\{c_k\} \in \ell^2(\mathbb{Z})$ then the partial sums $S_n = \sum_{|k| \leqq n} c_k \psi_k$ converge to an $f \in \langle \psi \rangle$ in the $L^2(\mathbb{R})$ norm. By Theorem 1.6 we have a corresponding situation in $\mathcal{M}_\psi$ involving $J_\psi^{-1} f = m \in \mathcal{M}_\psi$ and the exponential system $\{e_k : k \in \mathbb{Z}\}$: the symmetric partial sums

$$s_n(m; \xi) = \sum_{|k| \leqq n} c_k e_k(\xi) = J_\psi^{-1} S_n$$

converge to $m$ in the $\mathcal{M}_\psi$ norm. Since $\{c_k\} \in \ell^2(\mathbb{Z})$, the partial sums $s_n(m; \xi)$ also converge in the $L^2\big([0,1)\big)$ norm to a function $\mu \in L^2\big([0,1)\big)$. Thus, a subsequence $\{s_{n_j}(m; \xi)\}$ converges a.e. to $\mu(\xi)$ in $[0,1)$. But

$$\int_0^1 |s_{n_j}(m; \xi) \sqrt{p_\psi(\xi)} - m(\xi) \sqrt{p_\psi(\xi)}|^2 d\xi =$$
$$= \int_0^1 |s_{n_j}(m; \xi) - m(\xi)|^2 p_\psi(\xi) d\xi \longrightarrow 0$$

as $j \to \infty$. A subsequence of $\{s_{n_j}(m; \xi)\}$ converges to $m(\xi)$ a.e. (since $p_\psi(\xi) > 0$ a.e.). It follows that $\mu(\xi) = m(\xi)$ a.e.. Since $\{c_k\}$ is an arbitrary element of $\ell^2(\mathbb{Z})$, $q(z) = |m(\xi)|^2$ is the general non-negative function in $L^1\big([0,1)\big)$. But the last convergence result shows $q p_\psi \in L^1\big([0,1)\big)$. We can then apply Lemma 3.6 to $s = p_\psi$ to conclude that $\|p_\psi\|_\infty = \|s\|_\infty < \infty$; thus, (b) is true.

Theorem 3.7 shows that (b) implies (c). In fact, this result is the equivalence between (b) and (c). That (b) implies (a) follows immediately from the boundedness of $J_\psi \circ I_\psi$ as an operator from $L^2([0,1))$ into $\langle \psi \rangle$ (see the proof of Theorem 3.12).

We now pass to the proof that $\tilde{\mathcal{B}}$ satisfies property (B) if any one of the equivalent statements (a), (b) and (c) is true. We are given $\frac{1}{p_\psi(\xi)} \in L^1\big([0,1)\big)$ and let us assume $\|p_\psi\| = B < \infty$. By Theorem 2.7 we know $\tilde{\mathcal{B}}$ is well defined and belongs to $\langle \psi \rangle = \langle \tilde{\psi} \rangle$. We want to show that $\tilde{\mathcal{B}}$ satisfies property (B): if $\sum_{k \in \mathbb{Z}} c_k \tilde{\psi}_k$ converges then $\{c_k\} \in \ell^2(\mathbb{Z})$.

By Theorem 1.6

$$\| \sum_{\mu \leqq |k| \leqq \nu} c_k \tilde{\psi}_k \|_{L^2(\mathbb{R})} = \| ( \sum_{\mu \leqq |k| \leqq \nu} c_k e_k ) \sqrt{p_{\tilde{\psi}}} \|_{L^2\left([0,1)\right)}$$

Hence,

$$\text{if } \sum_{k \in \mathbb{Z}} c_k \psi_k \text{ converges in } \langle \tilde{\psi} \rangle, \text{ then } \sum_{k \in \mathbb{Z}} c_k e_k \sqrt{p_{\tilde{\psi}}} \text{ converges in } L^2\big([0,1)\big).$$

Since $\frac{1}{p_{\tilde{\psi}}(\xi)} = p_\psi(\xi) \leqq B$ a.e.

$$\begin{aligned}\|\sum_{\mu\leqq|k|\leqq\nu} c_k e_k\|^2_{L^2([0,1))} &= \int_0^1 |\sum_{\mu\leqq|k|\leqq\nu} c_k e_k(\xi)|^2\, p_{\tilde{\psi}}(\xi)\frac{1}{p_{\tilde{\psi}}(\xi)} d\xi \\ &\leqq B\int_0^1 |\sum_{\mu\leqq|k|\leqq\nu} c_k e_k(\xi)|^2\, p_{\tilde{\psi}}(\xi) d\xi \to 0\end{aligned}$$

as $\mu \to \infty$. This means $\sum_{k\in\mathbb{Z}} c_k e_k$ converges in $L^2\big([0,1)\big)$. But this implies $\{c_k\} \in \ell^2(\mathbb{Z})$ as desired.

Let us, finaly, consider the last statement, which is a partial converse of what we just proved. Let $m \in L^2\big([0,1)\big)$. Then $m/\sqrt{p_\psi} \in L^2([0,1), p_\psi)$ (in our case $p_\psi(\xi) > 0$ a.e.; thus, $m/\sqrt{p_\psi}$ is well defined). Since each $\varphi \in \langle\psi\rangle$ has an associated $\{c_k\}$ such that

$$\varphi = \sum_{k\in\mathbb{Z}} c_k\psi_k\,, \qquad \frac{m}{\sqrt{p_\psi}} = \sum_{k\in\mathbb{Z}} c_k e_k \quad \text{in } L^2([0,1), p_\psi)\,.$$

Since $\mathcal{B}$ satisfies property (B), $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$. Thus,

$$h = \sum_{k\in\mathbb{Z}} c_k e_k \in L^2\big([0,1)\big)\,.$$

By the same subsequence argument we used in the first part of this proof, we can show $h = m/\sqrt{p_\psi}$. We showed $\frac{m}{\sqrt{p_\psi}} \in L^1([0,1), p_\psi)$ when $m \in L^1\big([0,1)\big)$. In particular, if $q \geqq 0$ and $q \in L^1\big([0,1)\big)$ then $q/\sqrt{p_\psi} \in L^1\big([0,1)\big)$. We can apply Lemma 3.6 to show that $\|\frac{1}{p_\psi}\|_\infty < \infty$. ■

The reader might find the terminology "satisfying property (B)" a bit confusing when considering the notion "$\mathcal{B}$ is a Bessel system." We will not try to confuse the reader further by continuing to discuss this matter.

It is clear that one can say much more about how $\mathcal{B}$ (and $\tilde{\mathcal{B}}$) can be used to represent the elements of $\langle\psi\rangle$. We have seen that, though we are using $L^2$-norms to approximate these elements, series of the type $\sum_{k\in\mathbb{Z}} c_k\psi_k$ do not always involve sequences $\{c_k\}$ in $\ell^2(\mathbb{Z})$. We have obtained a host of results that explain these matters further; however, we will publish them elsewhere. As mentioned in the first section, one of our purposes was to show how simple properties of $p_\psi$ we associated with various properties of $\mathcal{B}$. We believe that we have presented a sufficient number of examples. They can also be used to illustrate many examples of notions in functional analysis.

## §4 References and comments

Perhaps the most important aspect of this paper is to provide an introduction to a much more general theory. Let us begin by saying a few words about Gabor systems. These are systems of the form

$$g_{mn}(\xi) = e^{2\pi in\xi}g(\xi - m) = (M_nT_mg)(\xi)\,, \quad m, n \in \mathbb{Z}$$

($M_n$ denotes the *modulation operator* that maps $f(\xi)$ into $e^{2\pi in\xi}f(\xi)$). An important operator used for the study of these systems is the *Zak transform* that maps $\varphi \in L^2(\mathbb{R})$ into the function

$$(Z\varphi)(s,t) = \sum_{m\in\mathbb{Z}} \varphi(s+m)e^{2\pi imt}\,,$$

$(s,t) \in [0,1) \times [0,1)$. One can show that $[\varphi,\psi](s,t) \equiv (Z\varphi)(s,t)\overline{Z\psi(s,t)}$ corresponds to the bracket we introduced in the first section of this paper. In particular, $p_\psi(s,t) = [\psi,\psi](s,t)$ plays the role that the weight $p_\psi(\xi)$ on $[0,1)$ played for the study of the systems $\mathcal{B}$, when applied to the Gabor systems $\{\psi_{m,n}\}$, $m,n \in \mathbb{Z}$, as a weight on $[0,1) \times [0,1)$. One can obtain a result that corresponds to Theorem 1.6 and use it to obtain extensions of the various theorems we developed here to the study of Gabor systems. The authors of this paper are involved in a project that, not only extends to higher dimensions, but applies to locally compact abelian groups (playing the role of the integers) and associated to unitary representations of these groups on appropriate Hilbert spaces.

The bracket product was introduced by Rong-Qing Jia and Charles A. Micchelli in [13]. Our notation of the bracket product was introduced in the pioneering work of C. de Boor, R. A. DeVore and A. Ron (see [2] and [3]) as one of the main tools for studying shift invariant spaces. Lemma 1.5 and other basic results are proved there from a somewhat different point of view. There are several statements and proofs of Theorem 1.6 in the literature (see [16], prop 2.18 and [21]); the proof we offer here is, perhaps, the most elementary one and is consistent with the basic method of studying the periodization function we use throughout this article. For more details about recent developments in the theory of shift invariant spaces, an excellent article is the one by M. Bownik [4].

Theorems 1.7, 1.8 and 1.9 are "folklore" by now (see [1] and [11] for Theorem 1.8). [1] also contains the results Theorem 2.1, 2.4 and 2.10 (although there are some unnecessary assumptions there, the discussion is very informative). We also refer to [6] for these results. The approach used here is new as far as we know. For the properties of bases and similar systems we recommend the books [19] and [20]. The original paper [9] is a good source for the ideas involved in the study of frames (see also chapter 7 and

8 in [11] for their use in the theory of wavelets). The papers [15] and [16] contain methods that are very similar to the ones used in this article. Theorems 2.7 and 3.3 can be found in [14] (with different proofs). The results presented here about Schauder bases can be found in [14] and stem from ideas presented in [10]. In this last article there is a discussion concerning the important role played by the work in [12]. In [17] there is material that is related to Fact 3.4 and Fact 3.5.

We recommend reading the discussion about redundancy in [8]. Some of the ideas involved in the description of the material that extends this work (presented at the very end of §3) can also be found in [10]. The detailed classification of various subclasses of Parseval frame wavelets based partially on the results described here is given in [18].

## References


[1] Benedetto, J. and Li, S. *"The Theory of Multi-Resolution Analysis for Frames and Applications to Filter Banks"*, Appl. Comp. Harmonic An. 5 (1998), 389–427.

[2] de Boor, C., De Vore, R. A. and Ron, A. *"The structure of Finitely Generated Shift-invariant Spaces in $L^2(\mathbb{R}^d)$"*, J. of Funct. Anal. 119 (1994), 37–78

[3] de Boor, C., De Vore, R. A. and Ron, A. *"Approximation From Shift-invariant Subspaces of $L^2(\mathbb{R}^d)$"*, Trans. Amer. Math. Soc. 341 (1994), 787–806.

[4] Bownik, M. *"The Structure of Shift Invariant Spaces of $L^2(\mathbb{R}^n)$"*, J. of Funct. Anal. 177 (2000), no.2, 282-309.

[5] Bownik, M. and Speegle, D. *"Linear Independence of Parseval Wavelets"*, in preparation (personal communication) (2007).

[6] Casazza, P.G., Christensen, O. and Kalton, N.J. *"Frames of Translates"*, Collect. Math. 52 (2001), 35–74.

[7] Casazza, P. G., Kutyniok, G., Speegle,D. and Tremain, J.C. *"A Decomposition Theorem for Frames and the Feichtinger Conjecture"*, reprint (2007).

[8] Daubechies, I. *"The Wavelet Transform, Time-Frequency Localization and Signal Analysis"*, IEEE Trans. Inform. Theory 36[5] (1990), 961–1005.

[9] Duffin, R. and Schaeffer, A. *"A Class of Nonharmonic Fourier Series"*, Trans. Amer. Math. Soc. 72 (1952), 341–366.

[10] Heil, C. and Powell, A.M. *"Gabor Schauder Bases and the Balian-Low Theorem"*, J.Math.Phys. 47 (2006), no.11, 113506, 21 pgs.

[11] Hernandez, E. and Weiss, G. *"A First Course in Wavelets"*, CRC Press, Boca Raton (1996).

[12] Hunt, R., Muckenhoupt, B. and Wheeden, R. *"Weighted Norm Inequalities for the Conjugate Function and Hilbert Transform"*, Trans. Math. Soc. 176 (1973), 227–251.

[13] Jia, Rong-Qing and Micchelli, Charles A. *"Using the Refinement equations for the Construction of Pre-Wavelets II: Powers of Two"*. Curves and Surfaces, P. J. Laurent, A. Le Méhauté, L.L. Schumaker Eds, Academic Press, (1991) pgs. 209-246.

[14] Nielsen, M. and Šikić, H. *"Schauder Bases of Integer Translates"*, Appl. Comp. Harm. An. 23 (2007), no.2, 259-262.

[15] Paluszyński, M., Šikić, H., Weiss, G. and Xiao, S. *"Generalized Low Pass Filters and MRA Frame Wavelets"*, J. Geom. Anal. 11 (2001), no.2, 311–342.

[16] Paluszyński, M., Šikić, H., Weiss, G. and Xiao, S. *"Tight Frame Wavelets, Their Dimension Functions, MRA Tight Frame Wavelets and Connectivity Properties"* Adv. Comp. Math 18 (2-4)(2003), 297–327.

[17] Šikić, H. and Speegle, D. *"Dyadic PFW's and $W_0$ Bases"*, Functional Analysis IX, Var.Pub.Ser. (Aarhus), 48 (2007), 85-90.

[18] Šikić, H., Speegle, D. and Weiss, G. *"Structure of dyadic PFW's"*, Contemporary Mathematics, Vol. 451 (2008), 263–291.

[19] Singer, I. *"Bases in Banach Spaces I"*, Springer-Verlag, Berlin (1970).

[20] Young, R. *"An Introduction to Nonharmonic Fourier Series"*, Academic Press, New York (1980).

[21] Weiss, G. and Wilson, E.N. *"The Mathematical Theory of Wavelets"*, Twentieth Century Harmonic Analysis — a Celebration, Il Ciocco. NATO Sci. Ser. II Math., Phys., Chem., 33, Kluwer Acad. Publ. Dordrecht (2001), 329–366.

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain. e-mail address: eugenio.hernandez@uam.es

Dept. of Mathematics, Univ. of Zagreb, Bijenička 30, HR-10 000, Zagreb, Croatia. e-mail address: hsikic@math.hr

Dept. of Mathematics, Washington Univ., Box 1146, St. Louis, MO 63130.
e-mail address: guido@math.wustl.edu

Dept. of Mathematics, Washington Univ., Box 1146, St. Louis, MO 63130.
e-mail address: enwilson@math.wustl.edu