FRAMES OF EXPONENTIALS AND SUB–MULTITILES IN LCA GROUPS.

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Abstract. In this note we investigate the existence of frames of exponentials for $L^2(\Omega)$ in the setting of LCA groups. Our main result shows that sub–multitiling properties of $\Omega \subset \widehat{G}$ with respect to a uniform lattice Γ of $\widehat{G}$ guarantee the existence of a frame of exponentials with frequencies in a finite number of translates of the annihilator of Γ. We also prove the converse of this result and provide conditions for the existence of these frames. These conditions extend recent results on Riesz bases of exponentials and multitilings to frames.

1. Introduction and main result

We begin by stating several known results.

- Let Ω be a measurable subset of $\mathbb{R}^d$ with positive, finite measure, let Λ be a complete lattice of $\mathbb{R}^d$ (i.e. $\Lambda = A\mathbb{Z}^d$ for some $d \times d$ invertible matrix A with real entries), and denote by Γ the annihilator of Λ. Recall that $\Gamma = \{\gamma \in \mathbb{R}^d : e^{2\pi i \langle \lambda, \gamma\rangle} = 1, \forall\ \lambda \in \Lambda\}$. In 1974, B. Fuglede ([5], Section 6) proved that $\{e^{2\pi i\langle \lambda, \cdot\rangle} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\Omega)$ if and only if (Ω, Γ) is a **tiling pair** for $\mathbb{R}^d$, that is $\displaystyle\sum_{\gamma\in\Gamma} \chi_\Omega(x+\gamma) = 1$, a. e. $x \in \mathbb{R}^d$.
- The result of B. Fuglede just stated also holds in the setting of locally compact abelian (LCA) groups. Let G be a second countable LCA group, and let Λ be a uniform lattice in G (i.e. Λ is a discrete and co-compact subgroup of G). Denote by $\widehat{G}$ the dual group of G. For a character $\omega \in \widehat{G}$ we use the notation $e_g(\omega) = \omega(g)$, for $g \in G$. Let Γ be the annihilator of Λ. (i.e. $\Gamma = \{\gamma \in \widehat{G} : e_\lambda(\gamma) = 1 \text{ for all } \lambda \in \Lambda\}$). The dual group $\widehat{G}$ of G is also a second countable LCA group, and Γ is also a uniform lattice. Let Ω be a measurable subset of $\widehat{G}$ with positive and finite measure. In 1987, S. Pedersen ([10], Theorem 3.6) proved that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\Omega)$ if and only if (Ω, Γ) is a tiling pair for $\widehat{G}$, that is $\sum_{\gamma\in\Gamma} \chi_\Omega(\omega+\gamma) = 1$, a. e. $\omega \in \widehat{G}$.
- Recent results in this area focused on *multitiling pairs*. Let Ω be a bounded, measurable subset of $\mathbb{R}^d$, and let Γ be a lattice of $\mathbb{R}^d$. If there exists a positive

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integer ℓ such that

$$\sum_{\gamma\in\Gamma}\chi_\Omega(x+\gamma)=\ell\,,\quad \text{a.e } x\in\mathbb{R}^d\,,$$

we will say that (Ω,Γ) is a **multitiling pair**, or an **ℓ-tiling pair** for $\mathbb{R}^d$. For a lattice $\Lambda\subset\mathbb{R}^d$ and $a_1,\dots,a_\ell\in\mathbb{R}^d$, let

$$E_\Lambda(a_1,\dots,a_\ell):=\{e^{2\pi i\langle a_j+\lambda,\cdot\rangle}: j=1,\dots,\ell;\lambda\in\Lambda\}\,.$$

S. Gresptad and N. Lev ([6], Theorem 1) proved in 2014 that if Γ is the annihilator of Λ, Ω is a bounded, measurable subset of $\mathbb{R}^d$ whose boundary has measure zero, and (Ω,Γ) is an ℓ-tiling pair for $\mathbb{R}^d$, there exist $a_1,\dots,a_\ell\in\mathbb{R}^d$ such that $E_\Lambda(a_1,\dots,a_\ell)$ is a *Riesz basis* for $L^2(\Omega)$. The proof of this result in [6] uses Meyer's quasicrystals. In 2015 M. Kolountzakis ([9], Theorem 1) found a simpler and shorter proof without the assumption on the boundary of Ω.

For the reader's convenience we recall that a countable collection of elements $\Phi=\{\phi_j: j\in J\}$ of a Hilbert space $\mathbb{H}$ is a **Riesz basis** for $\mathbb{H}$ if it is the image of an orthonormal basis of $\mathbb{H}$ under a bounded, invertible operator $T\in\mathcal{L}(\mathbb{H})$. Riesz bases provide stable representations of elements of $\mathbb{H}$.

- This result has been extended to second countable LCA groups by E. Agora, J. Antezana, and C. Cabrelli ([1], Theorem 4.1). Moreover, they prove the converse ([1], Theorem 4.4): with the same notation as in the second item of this section, given a relatively compact subset Ω of $\widehat{G}$, if $L^2(\Omega)$ admits a Riesz basis of the form

$$E_\Lambda(a_1,\dots,a_\ell):=\{e_{a_j+\lambda}: j=1,2,\dots,\ell;\lambda\in\Lambda\}$$

for some $a_1,\dots,a_\ell\in G$, then (Ω,Γ) is an ℓ-tiling pair for $\widehat{G}$.

The purpose of this note is to investigate the situation when (Ω,Γ) is a *sub-multitiling pair* for $\widehat{G}$. Let Ω be a measurable set in $\widehat{G}$ with positive and finite Haar measure. For Γ a lattice in $\widehat{G}$ and $\omega\in\widehat{G}$ define

$$F_{\Omega,\Gamma}(\omega):=\sum_{\gamma\in\Gamma}\chi_\Omega(\omega+\gamma)\,.$$

If there exists a positive integer ℓ such that

$$\operatorname{ess\,sup}_{\omega\in\widehat{G}}F_{\Omega,\Gamma}(\omega)=\ell\,, \tag{1.1}$$

we will say that (Ω,Γ) is a **sub-multitiling pair** or an **ℓ-subtiling pair**.

Denote by Q_Γ a fundamental domain of the lattice Γ in $\widehat{G}$, i.e. it is a Borel measurable section of the quotient group $\widehat{G}/\Gamma$. (Its existence is guaranteed by Theorem 1 in [4]). Since $F_{\Omega,\Gamma}(\omega)$ is a Γ-periodic function, it is enough to compute the ess sup in (1.1) over a fundamental domain Q_Γ. Observe that (Ω,Γ) is an ℓ-tiling pair for $\widehat{G}$ if $F_{\Omega,\Gamma}(\omega)=\ell$ for a. e. $\omega\in Q_\Gamma$.

Another structure that allows for stable representations, besides orthonormal and Riesz bases, is that of a *frame*. A collection of elements $\Phi=\{\phi_j: j\in J\}$ of a Hilbert space $\mathbb{H}$ is a **frame** for $\mathbb{H}$ if it is the image of an orthonormal basis of $\mathbb{H}$ under a bounded, surjective operator $T \in \mathcal{L}(\mathbb{H})$ or, equivalently, if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, \phi_j\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathbb{H}.$$

(See [11], Chapter 4, Section 7.) The numbers A and B are called **frame bounds** of Φ.

In this note we prove the following relationship between frames of exponentials in LCA groups and ℓ-subtiling pairs.

Theorem 1.1. *Let G be a second countable LCA group and let Λ be a uniform lattice of G. Let $\widehat{G}$ be the dual group of G, and let Γ be the annihilator of Λ. Let $\Omega \subset \widehat{G}$ be a measurable set of positive, finite measure, and let ℓ be a positive integer.*

1. *If for some $a_1, \dots, a_\ell \in G$ the collection $E_\Lambda(a_1, \dots, a_\ell)$ is a frame of $L^2(\Omega)$, then (Ω, Γ) must be an m-subtiling pair of $\widehat{G}$ for some positive integer $m \leq \ell$.*
2. *If $\Omega \subseteq \widehat{G}$ is a measurable,* **bounded** *set and (Ω, Γ) is an ℓ-subtiling pair of $\widehat{G}$, then there exist $a_1, \dots, a_\ell \in G$ such that $E_\Lambda(a_1, \dots, a_\ell)$ is a frame of $L^2(\Omega)$.*

Remark 1.2. *Recall that any locally compact and second countable group is metrizable, and its metric can be chosen to be invariant under the group action (see [8], Theorem 8.3). Thus, it makes sense to talk about bounded sets in the group $\widehat{G}$.*

The proof of Theorem 1.1 will be given in Section 2. In Section 3 we give other conditions for a set of exponentials of the form $E_\Lambda(a_1, \dots, a_\ell)$ to be a frame of $L^2(\Omega)$ and provide expressions to compute the frame bounds.

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2. Proof of Theorem 1.1

We start with a result that will be used in the proof of part (2) of Theorem 1.1

Proposition 2.1. *If Ω is a measurable,* **bounded** *set in $\widehat{G}$ and Γ is a uniform lattice in $\widehat{G}$ such that (Ω, Γ) is an ℓ-subtiling pair for $\widehat{G}$, there exists a* **bounded** *measurable set $\Delta \subset \widehat{G}$ such that $\Omega \subset \Delta$ and (Δ, Γ) is an ℓ-tiling pair for $\widehat{G}$.*

Proof. Let Q_Γ be a fundamental domain of Γ in $\widehat{G}$. Modifying Ω in a set of measure zero, we can assume that $\sup_{\omega\in Q_\Gamma} F_{\Omega,\Gamma}(\omega) = \ell$. Define $\widetilde{\Gamma} = \{\gamma \in \Gamma : \omega + \gamma \in \Omega \text{ for some } \omega \in Q_\Gamma\}$. Since Ω is bounded, the set $\widetilde{\Gamma}$ is finite and, by the definition of ℓ-subtiling pair, has at least ℓ different elements.

Set $Q_k = \{\omega \in Q_\Gamma : F_{\Omega,\Gamma}(\omega) = k\}$ for $k = 0, 1, ..., \ell$. Clearly

$$Q_\Gamma = \bigcup_{k=0}^{\ell} Q_k \,,$$

and the union is disjoint.

Now, for $k = 1, \dots, \ell$, let $\mathcal{B}_k = \{B \subset \widetilde{\Gamma} : \#B = k\}$. For $B \in \mathcal{B}_k$ set

$$Q_k(B) = \{\omega \in Q_k : \omega + \gamma \in \Omega, \quad \text{for all } \gamma \in B\}.$$

Since Ω is measurable, Q_k is measurable and since $Q_k(B) = \bigcap_{\gamma \in B}((\Omega - \gamma) \cap Q_k)$, then $Q_k(B)$ is also measurable. Observe that the collection $\mathcal{B}_k$ is finite since $\widetilde{\Gamma}$ is finite. Also if B and B' are different sets in $\mathcal{B}_k$ then $Q_k(B) \cap Q_k(B') = \emptyset$. Indeed, if $\omega \in Q_k(B) \cap Q_k(B')$, $\omega + \gamma \in \Omega$ for all $\gamma \in B$ and $\omega + \gamma' \in \Omega$ for all $\gamma' \in B'$. Since $B \neq B'$, there exists $\gamma_1 \in B' \setminus B$. Then, since $\omega \in Q_k$,

$$k = \sum_{\gamma \in \Gamma} \chi_\Omega(\omega + \gamma) \geq \sum_{\gamma \in B} \chi_\Omega(\omega + \gamma) + \chi_\Omega(\omega + \gamma_1) = k + 1\,,$$

which is a contradiction. Observe that $Q_k = \bigcup_{B \in \mathcal{B}_k} Q_k(B)$, $k = 1, \dots, \ell$, and the union is disjoint. Therefore,

$$\Omega = \bigcup_{k=1}^{\ell} \bigcup_{B \in \mathcal{B}_k} \bigcup_{\gamma \in B} Q_k(B) + \gamma\,, \tag{2.1}$$

and the union is disjoint.

For $k = 1, \dots, \ell$ and $B \in \mathcal{B}_k$, we extend $B \subseteq \widetilde{\Gamma}$ to $\widetilde{B}$ by inserting $\ell - k$ distinct elements from $\widetilde{\Gamma} \setminus B$ into B. Let $\widetilde{B}_0$ be a set of ℓ different elements from $\widetilde{\Gamma}$. We recall here that $\#\widetilde{\Gamma} \geq \ell$ since $\sup F_{\Omega,\Gamma} = \ell$.

Finally we define:

$$\Delta = \Big(\bigcup_{\gamma \in \widetilde{B}_0} Q_0 + \gamma\Big) \cup \Big(\bigcup_{k=1}^{\ell} \bigcup_{B \in \mathcal{B}_k} \bigcup_{\gamma \in \widetilde{B}} Q_k(B) + \gamma\Big).$$

The set Δ is measurable since it is a finite union of measurable sets. From (2.1) it is clear that $\Omega \subset \Delta$. Moreover, if $\omega \in Q_k(B)$, for some $B \in \mathcal{B}_k$, $\omega + \gamma \in \Omega$ only when $\gamma \in B$. Hence, if $\omega \in Q_k(B)$, $\omega + \widetilde{\gamma} \in \Delta$ only when $\widetilde{\gamma} \in \widetilde{B}$. Since $\widetilde{B}$ has precisely ℓ elements, if $\omega \in Q_k(B)$,

$$\sum_{\gamma \in \Gamma} \chi_\Delta(\omega + \gamma) = \sum_{\widetilde{\gamma} \in \widetilde{B}} \chi_\Delta(\omega + \widetilde{\gamma}) = \ell\,.$$

Also, if $\omega \in Q_0$

$$\sum_{\gamma \in \Gamma} \chi_\Delta(\omega + \gamma) = \sum_{\widetilde{\gamma} \in \widetilde{B}_0} \chi_\Delta(\omega + \widetilde{\gamma}) = \ell\,.$$

Taking into account that $Q_\Gamma = \bigcup_{k=0}^{\ell} Q_k = Q_0 \cup \Big(\bigcup_{k=1}^{\ell} \bigcup_{B \in \mathcal{B}_k} Q_k(B)\Big)$ is a disjoint union, we conclude that for $\omega \in Q_\Gamma$, $\sum_{\gamma \in \Gamma} \chi_\Delta(\omega + \gamma) = \ell$, proving that (Δ, Γ) is an ℓ-tiling pair for $\widehat{G}$. □

Remark 2.2. *The ℓ-tile found in Proposition 2.1 is not necessarily unique. It depends on the choice of the sets $\widetilde{B}$ and $\widetilde{B}_0$.*

For the proof of part (2) of Theorem 1.1 we will use the fiberization mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(Q_\Gamma, \ell^2(\Gamma))$ given by

$$\mathcal{T}f(\omega) = \{\widehat{f}(\omega + \gamma)\}_{\gamma \in \Gamma} \in \ell^2(\Gamma)\,, \quad \omega \in Q_\Gamma. \tag{2.2}$$

The mapping $\mathcal{T}$ is an isometry and satisfies

$$\mathcal{T}(t_\lambda f)(\omega) = e_{-\lambda}(\omega)\mathcal{T}f(\omega)\,, \quad \lambda \in \Lambda\,, f \in L^2(G)\,, \tag{2.3}$$

(see Proposition 3.3 and Remark 3.12 in [3]), where t_λ denotes the translation by λ that is $t_\lambda f(g) = f(g-\lambda)$.

The next result is Theorem 4.1 of [3] adapted to our situation. For $\varphi_1, \dots, \varphi_\ell \in L^2(G)$ denote by

$$S_\Lambda(\varphi_1, \dots, \varphi_\ell) := \overline{\operatorname{span}}\{t_\lambda \varphi_j : \lambda \in \Lambda, j = 1, \dots, \ell\}$$

the Λ-invariant space generated by $\{\varphi_1, \dots, \varphi_\ell\}$. The measurable range function associated to $S_\Lambda(\varphi_1, \dots, \varphi_\ell)$ is

$$J(\omega) = \overline{\operatorname{span}}\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_\ell(\omega)\} \subset \ell^2(\Gamma)\,, \quad \omega \in Q_\Gamma\,. \tag{2.4}$$

Proposition 2.3. *Let $\varphi_1, \dots, \varphi_\ell \in L^2(G)$ and let $J(\omega)$ be the measurable range function associated to $S_\Lambda(\varphi_1, \dots, \varphi_\ell)$ as in* (2.4)*. Let $0 < A \le B < \infty$. The following statements are equivalent:*

(i) The set $\{t_\lambda \varphi_j : \lambda \in \Lambda, j = 1, \dots, \ell\}$ is a frame for $S_\Lambda(\varphi_1, \dots, \varphi_\ell)$ with frame bounds A and B.

(ii) For almost every $\omega \in Q_\Gamma$ the set $\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_\ell(\omega)\} \subset \ell^2(\Gamma)$ is a frame for $J(\omega)$ with frame bounds $A|Q_\Gamma|^{-1}$ and $B|Q_\Gamma|^{-1}$.

Proof. Let $f \in S_\Lambda(\varphi_1, \dots, \varphi_\ell)$. Use that the fiberization mapping given in (2.2) is an isometry satisfying (2.3) to write

$$\begin{aligned}
\sum_{\lambda\in\Lambda}\sum_{j=1}^{\ell} |\langle t_\lambda \varphi_j, f\rangle_{L^2(G)}|^2 &= \sum_{\lambda\in\Lambda}\sum_{j=1}^{\ell} |\langle \mathcal{T}(t_\lambda \varphi_j), \mathcal{T}f\rangle_{L^2(Q_\Gamma, \ell^2(\Gamma))}|^2 \\
&= \sum_{j=1}^{\ell}\sum_{\lambda\in\Lambda} \Big| \int_{Q_\Gamma} e_{-\lambda}(\omega) \langle \mathcal{T}(\varphi_j)(\omega), \mathcal{T}f(\omega)\rangle_{\ell^2(\Gamma)}\, d\omega \Big|^2 .
\end{aligned}$$

Since $\{\frac{1}{\sqrt{|Q_\Gamma|}} e_\lambda(\omega) : \lambda \in \Lambda\}$ is an orthonormal basis of $L^2(Q_\Gamma)$ it follows that

$$\sum_{\lambda\in\Lambda}\sum_{j=1}^{\ell} |\langle t_\lambda \varphi_j, f\rangle_{L^2(G)}|^2 = |Q_\Gamma| \sum_{j=1}^{\ell} \int_{Q_\Gamma} |\langle \mathcal{T}\varphi_j(\omega), \mathcal{T}f(\omega)\rangle_{\ell^2(\Gamma)}|^2\, d\omega\,.$$

From here, the proof continues as in the proof of Theorem 4.1 in [3]. Details are left to the reader. □

Remark 2.4. *Notice that the factor $|Q_\Gamma|^{-1}$ that appears in (ii) of Proposition 2.3 does not appear in Theorem 4.1 of* [3]*. This is due to the fact that in* [3] *the measure of Q_Γ is normalized (see the beginning of Section 3 in* [3]*). Although this fact is not important to prove (2) of Theorem 1.1, it will be crucial in Section 3 to obtain optimal frame bounds of sets of exponentials.*

Proof of Theorem 1.1

(1) Assume that $E_\Lambda(a_1, \dots, a_\ell)$ is a frame for $L^2(\Omega)$. We define $\varphi \in L^2(G)$ by

$$\widehat{\varphi} := \chi_\Omega, \quad \text{and} \quad \varphi_j := t_{-a_j}\varphi, \quad j = 1, \dots, \ell,$$

where t_{a_j} denotes the translation by a_j, that is $t_{a_j}\varphi(g) = \varphi(g - a_j)$.

Since $E_\Lambda(a_1,\ldots,a_\ell)$ is a frame of $L^2(\Omega)$, we have that $\{t_\lambda\varphi_j\colon \lambda\in\Lambda, j=1,\ldots,\ell\}$ is a frame of the Paley-Wiener space $PW_\Omega := \{f\in L^2(G)\colon \widehat{f}\in L^2(\Omega)\} = \{f\in L^2(G)\colon \widehat{f}(\omega)=0, a.e.\ w\in\widehat{G}\backslash\Omega\}$. This follows from the definition of frame and the fact that for $f\in PW_\Omega$ one has $\|f\|_{L^2(G)} = \|\widehat{f}\|_{L^2(\Omega)}$ and $\langle f, t_\lambda\varphi_j\rangle_{L^2(G)} = \langle \widehat{f}, e_{-\lambda+a_j}\rangle_{L^2(\Omega)}$.

In particular,

$$PW_\Omega = S_\Lambda(\varphi_1,\ldots,\varphi_\ell) := \overline{\mathrm{span}}\{t_\lambda\varphi_j\colon \lambda\in\Lambda, j=1,\ldots,\ell\}.$$

That is, $V := PW_\Omega$ is a finitely generated Λ-invariant space. Denote by J_V the measurable range function of V as given in (2.4) (see also [3], Section 3, for details). We now use the fiberization mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(Q_\Gamma, \ell^2(\Gamma))$ defined in (2.2).

By Proposition 2.3, for a.e. $\omega\in Q_\Gamma$ the sequences $\{\mathcal{T}\varphi_1(\omega),\ldots,\mathcal{T}\varphi_\ell(\omega)\}$ form a frame of $J_V(\omega)\subseteq \ell^2(\Gamma)$. Therefore, $\dim(J_V(\omega))\le \ell$, for a.e. $\omega\in Q_\Gamma$.

In our particular situation there is another description of the range function $J_V(\omega)$ associated to V. For each $\omega\in Q_\Gamma$, define

$$\theta_\omega := \{\gamma\in\Gamma\colon \chi_\Omega(\omega+\gamma)\neq 0\}, \text{ and } \ell_\omega := \#\theta_\omega.$$

Write $\ell_\omega = 0$ if $\theta_\omega=\emptyset$. Then, there exist $\gamma_1(\omega),\ldots,\gamma_{\ell_\omega}(\omega)\in\Gamma$ such that $w+\gamma_j(\omega)\in\Omega$, for all $j=1,\ldots,\ell_\omega$, which implies that $J_V(\omega)\subseteq \ell^2(\{\delta_{\gamma_1(\omega)},\ldots,\delta_{\gamma_{\ell_\omega}(\omega)}\})$, for a.e. $\omega\in Q_\Gamma$. Moreover, as in Corollary 2.8. of [1], $J_V(\omega) = \ell^2(\{\delta_{\gamma_1(\omega)},\ldots,\delta_{\gamma_{\ell_\omega}(\omega)}\})$, for a.e. $\omega\in Q_\Gamma$. Thus, $\dim(J_V(\omega)) = \ell_\omega$, which implies that $\ell_\omega\le\ell$, for a.e. $\omega\in Q_\Gamma$, and therefore we obtain that

$$F_{\Omega,\Gamma}(\omega) = \sum_{\gamma\in\Gamma}\chi_\Omega(\omega+\gamma)\le\ell, \quad \text{for a.e. } \omega\in Q_\Gamma\,.$$

This shows that (Ω,Γ) is an m-subtiling pair for $\widehat{G}$ with $m\le\ell$.

(2) Since Ω is bounded, by Proposition 2.1 there exists a bounded set Δ containing Ω which is an ℓ-tile of $\widehat{G}$ by Γ. Now using Theorem 4.1 of [1], there exist $a_1,\ldots,a_\ell\in G$ such that $E_\Lambda(a_1,\ldots,a_\ell)$ is a Riesz basis of $L^2(\Delta)$. As a consequence, $E_\Lambda(a_1,\ldots,a_\ell)$ is a frame of $L^2(\Omega)$.

□

Remark 2.5. *Note that Ω does not need to be bounded: for example, $E_{\mathbb{Z}}(0) = \{e^{2\pi ikx} : k\in\mathbb{Z}\}$ is an orthonormal basis for $L^2(\Omega)$ for $\Omega = \bigcup_{n=0}^\infty n + (\frac{1}{2^{n+1}}, \frac{1}{2^n}]\subset\mathbb{R}$ and Ω is not bounded. However, for the proof of part (2) of Theorem 1.1 we need Ω to be bounded since the proof uses Proposition 2.1.*

Remark 2.6. *Theorem 1.1 for the case $\ell=1$ can be found in* [2]. *In this case, the proof does not require making use of either the Paley-Wiener space of Ω or the range function associated to it as in the proof given above.*

Remark 2.7. *In Part (1) of Theorem 1.1 the inequality $m\le\ell$ can be strict as the following example shows: choose $\Omega\subset\mathbb{R}^d$ such that $(\Omega,\mathbb{Z}^d)$ is an ℓ-tiling pair for $\mathbb{R}^d$ and pick $a_1,\ldots,a_\ell$ such that $E_{\mathbb{Z}^d}(a_1,\ldots,a_\ell)$ is a Riesz basis of $L^2(\Omega)$. Let $\Omega_0\subset\Omega$ be any subset of Ω such that $(\Omega_0,\mathbb{Z}^d)$ is an $(\ell$-1$)$-tiling pair of $\mathbb{R}^d$ (for example, remove from Ω a fundamental domain of $\mathbb{Z}^d$ in $\mathbb{R}^d$). Then $E_{\mathbb{Z}^d}(a_1,\ldots,a_\ell)$ is a frame for $L^2(\Omega_0)$, and $(\Omega_0,\mathbb{Z}^d)$ is not an ℓ-subtiling pair for $\mathbb{R}^d$.*

3. Optimal frame bounds for sets of exponentials.

The purpose of this section is to develop another condition guaranteeing when a set of exponentials of the form

$$E_\Lambda(a_1, \dots, a_m) := \{e_{a_j+\lambda} : j = 1, 2, \dots, m, \lambda \in \Lambda\}$$

forms a frame for $L^2(\Omega)$, where (Ω, Γ) is an ℓ-subtiling pair for $\widehat{G}$, as well as to find optimal frame bounds for this frame.

For the ℓ-subtiling pair (Ω, Γ) of $\widehat{G}$, let E be the set of measure zero in Q_Γ such that $F_{\Omega,\Gamma} > \ell$, and let $Q_0 := \{\omega \in Q_\Gamma : F_{\Omega,\Gamma}(\omega) = 0\}$. Let

$$\widetilde{Q_\Gamma} := Q_\Gamma \setminus (Q_0 \cup E)\,.$$

For each $\omega \in \widetilde{Q_\Gamma}$ there exist $\ell_\omega \leq \ell$ and $\gamma_1(\omega), \dots, \gamma_{\ell_\omega}(\omega) \in \Gamma$ such that $\omega + \gamma_j(\omega) \in \Omega$ for all $j = 1, \dots, \ell_\omega$ (see the proof of Theorem 1.1). Recall that

$$\ell_\omega := \#\{\gamma \in \Gamma \colon \chi_\Omega(\omega + \gamma) \neq 0\}\,. \tag{3.1}$$

Given $\varphi_1, \dots, \varphi_m \in PW_\Omega = \{f \in L^2(G) : \widehat{f} \in L^2(\Omega)\}$, and $\omega \in \widetilde{Q}_\Gamma$, consider the matrix

$$T_\omega = \begin{pmatrix} \widehat{\varphi}_1(\omega + \gamma_1(\omega)) & \dots & \widehat{\varphi}_m(\omega + \gamma_1(\omega)) \\ \vdots & & \vdots \\ \widehat{\varphi}_1(\omega + \gamma_{\ell_\omega}(\omega)) & \dots & \widehat{\varphi}_m(\omega + \gamma_{\ell_\omega}(\omega)) \end{pmatrix} \tag{3.2}$$

of size $\ell_\omega \times m$. Assume that

$$\Phi_\Lambda := \{t_\lambda \varphi_j : \lambda \in \Lambda, j = 1, \dots, m\}$$

is a frame for $S_\Lambda(\varphi_1, \cdots, \varphi_m)$. By Proposition 2.3, this is equivalent to having that for a.e. $\omega \in Q_\Gamma$ the set

$$\Phi_\omega := \{\mathcal{T}\varphi_j(\omega) : j = 1, \dots, m\} \subset \ell^2(\Gamma)$$

is a frame for $J(\omega) = \overline{\operatorname{span}}\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_m(\omega)\} \subset \ell^2(\Gamma)$. Moreover, as in the proof of Theorem 1.1, for a. e. $\omega \in Q_\Gamma$, $J(\omega) = \ell^2(\{\delta_{\gamma_1(\omega)}, \dots, \delta_{\gamma_{\ell_\omega}(\omega)}\})$ is a subspace of $\ell^2(\Gamma)$ of dimension ℓ_ω. (Notice that this implies $m \geq \ell$.)

It is well known (see, for example, Proposition 3.18 in [7]) that a frame in a finite dimensional Hilbert space is nothing but a generating set. Since the non-zero elements of $\mathcal{T}\varphi_j(\omega)$ are precisely the j-th column of T_ω, $j = 1, \dots, m$, it follows that Φ_Λ is a frame for $S_\Lambda(\varphi_1, \cdots, \varphi_m)$ if and only if $\operatorname{rank}(T_\omega) = \ell_\omega$ for a.e. $\omega \in \widetilde{Q}_\Gamma$.

For $\omega \in \widetilde{Q_\Gamma}$, let $\lambda_{min}(T_\omega T_\omega^*)$ and $\lambda_{max}(T_\omega T_\omega^*)$ respectively the minimal and maximal eigenvalues of $T_\omega T_\omega^*$. It is well known (see Proposition 3.27 in [7]) that the optimal lower and upper frame bounds of Φ_ω are precisely $\lambda_{min}(T_\omega T_\omega^*)$ and $\lambda_{max}(T_\omega T_\omega^*)$ respectively. By Proposition 2.3 the optimal frame bounds for Φ_Λ are

$$A = |Q_\Gamma| \operatorname{ess\,inf}_{\omega \in \widetilde{Q}_\Gamma} \lambda_{min}(T_\omega T_\omega^*) \qquad \text{and} \qquad B = |Q_\Gamma| \operatorname{ess\,sup}_{\omega \in \widetilde{Q}_\Gamma} \lambda_{max}(T_\omega T_\omega^*)\,. \tag{3.3}$$

We have proved the following result:

Proposition 3.1. *With the notation and definitions as above, the following are equivalent:*

(i) The set $\Phi_\Lambda := \{t_\lambda \varphi_j : \lambda \in \Lambda, j = 1, \dots, m\}$ *is a frame for* $S_\Lambda(\varphi_1, \dots, \varphi_m)$.

(ii) The matrix T_ω given in (3.2) *has rank ℓ_ω (see* (3.1)*) for a.e. $\omega \in \widetilde{Q}_\Gamma$.*
Moreover, in this situation, the optimal frame bounds A and B of Φ_Λ are given by (3.3).

Consider now the set of exponentials

$$E_\Lambda(a_1, \dots, a_m) := \{e_{\lambda + a_j} : \lambda \in \Lambda, j = 1, \dots, m\}$$

with $a_1, \dots, a_m \in G$. Let $\varphi \in L^2(G)$ given by $\widehat{\varphi} = \chi_\Omega$. Consider

$$\varphi_j := t_{-a_j}\varphi\,, \quad j = 1, \dots, m.$$

As in the proof of Theorem 1.1, $E_\Lambda(a_1, \dots, a_m)$ is a frame for $L^2(\Omega)$ with frame bounds A and B if and only if the set

$$\Phi_\Lambda := \{t_\lambda \varphi_j : \lambda \in \Lambda, j = 1, \dots, m\}$$

is a frame for $PW_\Omega = S_\Lambda(\varphi_1, \cdots, \varphi_m)$ with the same frame bounds.

For our particular situation, if $\omega \in \widetilde{Q_\Gamma}$,

$$T_\omega = \begin{pmatrix} e_{a_1}(\omega + \gamma_1(\omega)) & \dots & e_{a_m}(\omega + \gamma_1(\omega)) \\ \vdots & & \vdots \\ e_{a_1}(\omega + \gamma_{\ell_\omega}(\omega)) & \dots & e_{a_m}(\omega + \gamma_{\ell_\omega}(\omega)) \end{pmatrix}. \tag{3.4}$$

As in Theorem 2.9 of [1] the matrix T_ω, for $\omega \in \widetilde{Q}_\Gamma$, can be factored as

$$T_\omega = E_\omega U_\omega := \begin{pmatrix} e_{a_1}(\gamma_1(\omega)) & \dots & e_{a_m}(\gamma_1(\omega)) \\ \vdots & & \vdots \\ e_{a_1}(\gamma_{\ell_\omega}(\omega)) & \dots & e_{a_m}(\gamma_{\ell_\omega}(\omega)) \end{pmatrix} \begin{pmatrix} e_{a_1}(\omega) & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & e_{a_m}(w) \end{pmatrix}. \tag{3.5}$$

Since U_ω is unitary and $T_\omega T_\omega^* = E_\omega E_\omega^*$, we have proved the following result:

Proposition 3.2. *With the notation and definitions as above, the following are equivalent:*
(i) The set $E_\Lambda(a_1, \dots, a_m)$ is a frame for $L^2(\Omega)$.
(ii) The matrix E_ω given in (3.5) *has rank ℓ_ω (see* (3.1)*) for a. e. $\omega \in \widetilde{Q_\Gamma}$.*
Moreover, in this situation, the optimal frame bounds A and B of $E_\Lambda(a_1, \dots, a_m)$ are given by

$$A = |Q_\Gamma|\, ess\, inf_{\omega \in \widetilde{Q}_\Gamma} \lambda_{min}(E_\omega E_\omega^*) \qquad and \qquad B = |Q_\Gamma|\, ess\, sup_{\omega \in \widetilde{Q}_\Gamma} \lambda_{max}(E_\omega E_\omega^*)\,.$$

Remark 3.3. *Proposition 3.2 can be found in* [1] *when Ω is an ℓ-tile and "frame" is replaced by "Riesz basis".*

Example 3.4. *In this example we work with the additive group $G = \mathbb{R}^d$ and the lattice $\Lambda = \mathbb{Z}^d$. Recall that $\widehat{G} = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$. Let $\Omega_0 \subset \Omega_1 \subset [0,1)^d$ be two measurable sets in $\mathbb{R}^d$ and let $\gamma_0 \in \mathbb{Z}^d$ ($\gamma_0 \neq 0$). Take*

$$\Omega = \Omega_1 \cup (\gamma_0 + \Omega_0)\,,$$

so that $(\Omega, \mathbb{Z}^d)$ is a 2-subtiling pair of $\mathbb{R}^d$.

For $a_1, a_2, \dots, a_m \in \mathbb{R}^d$ consider the set of exponentials

$$E_{\mathbb{Z}^d}(a_1, \dots, a_m) = \{e^{2\pi i\langle k+a_j, \bullet\rangle} : k \in \mathbb{Z}^d, j = 1, \dots, m\}.$$

By factoring out $e^{2\pi i\langle a_1, x\rangle}$ we can assume $a_1 = 0$.

According to Proposition 3.2, to determine the values of $a_1 = 0, a_2, \dots, a_m$ for which the set $E_{\mathbb{Z}^d}(0, a_2, \dots, a_m)$ is a frame for $L^2(\Omega)$, we need to compute the ranks of the matrices E_ω given in (3.5).

For $\omega \in \Omega_1 \setminus \Omega_0$, $\ell_\omega = 1$, $E_w = (1, 1, \dots, 1)$, and rank$(E_\omega) = 1 = \ell_\omega$. For $\omega \in \Omega_0$, $\ell_\omega = 2$, and

$$E_\omega = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{2\pi i\langle a_2, \gamma_0\rangle} & \dots & e^{2\pi i\langle a_m, \gamma_0\rangle} \end{pmatrix}. \tag{3.6}$$

Let $H := \bigcup_{k\in\mathbb{Z}}\{x \in \mathbb{R}^d : \langle x, \gamma_0\rangle = k\}$, that is a countable union of hyperplanes in $\mathbb{R}^d$ perpendicular to the vector γ_0. The rank of the matrix given in (3.6) *is 2 when at least one of the a_j does not belong to H. In this case, $E_{\mathbb{Z}^d}(0, a_2, \dots, a_m)$ is a frame for $L^2(\Omega)$ as an application of Proposition 3.2.*

We now compute the optimal frame bounds. For $\omega \in \Omega_1 \setminus \Omega_0$, $E_\omega E_\omega^* = (m)$, so that $\lambda_{min}(E_\omega E_\omega^*) = \lambda_{max}(E_\omega E_\omega^*) = m$. For $\omega \in \Omega_0$,

$$E_\omega E_\omega^* = \begin{pmatrix} m & 1 + \sum_{j=2}^m e^{-2\pi i\langle a_j, \gamma_0\rangle} \\ 1 + \sum_{j=2}^m e^{2\pi i\langle a_j, \gamma_0\rangle} & m \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda = m \pm \left|1 + \sum_{j=2}^m e^{2\pi i\langle a_j, \gamma_0\rangle}\right|.$$

Therefore, the optimal lower and upper frame bounds of $E_{\mathbb{Z}^d}(0, a_2, \dots, a_m)$ in $L^2(\Omega)$ are

$$A = m - \left|1 + \sum_{j=2}^m e^{2\pi i\langle a_j, \gamma_0\rangle}\right| \quad \text{and} \quad B = m + \left|1 + \sum_{j=2}^m e^{2\pi i\langle a_j, \gamma_0\rangle}\right|$$

when $a_j \notin H$ for some $j \in \{2, \dots, m\}$. Observe that the frame $E_{\mathbb{Z}^d}(0, a_2, \dots, a_m)$ in $L^2(\Omega)$ is tight (with tight frame bound m) if and only if $1 + \sum_{j=2}^m e^{2\pi i\langle a_j, \gamma_0\rangle} = 0$. This occurs, for example, if the complex numbers $\{1, e^{2\pi i\langle a_2, \gamma_0\rangle}, \dots, e^{2\pi i\langle a_m, \gamma_0\rangle}\}$ are the vertices of a regular m-gon inscribed in the unit circle.

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