A Unified Characterization of Reproducing Systems Generated by a Finite Family, II

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Abstract

By a "reproducing" method for $\mathcal{H} = L^2(\mathbb{R}^n)$ we mean the use of two countable families $\{e_\alpha : \alpha \in \mathcal{A}\}$, $\{f_\alpha : \alpha \in \mathcal{A}\}$, in $\mathcal{H}$, so that the first "analyzes" a function $h \in \mathcal{H}$ by forming the inner products $\{< h, e_\alpha >: \alpha \in \mathcal{A}\}$, and the second "reconstructs" h from this information: $h = \sum_{\alpha\in\mathcal{A}} < h, e_\alpha > f_\alpha$.

A variety of such systems have been used successfully in both pure and applied mathematics. They have the following feature in common: they are generated by a single or a finite collection of functions by applying to the generators two countable families of operators that consist of two of the following three actions: dilations, modulations, and translations. The **Gabor systems**, for example, involve a countable collection of modulations and translations; the **affine systems** (that produce a variety of wavelets) involve translations and dilations.

Considerable amount of research has been conducted in order to characterize those generators of such systems. In this paper we establish a result that "unifies" all of these characterizations by means of a relatively simple system of equalities. Such unification has been presented in a work by one of us. One of the novelties here is the use of a different approach that provides us with a considerably more general class of such reproducing systems; for example, in the affine case, we need not to restrict the dilation matrices to ones that preserve the integer lattice and are expanding on $\mathbb{R}^n$. Another novelty is a detailed analysis, in the case of affine and quasi-affine systems, of the characterizing equations for different kinds of dilation matrices.

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1 Introduction

The terms **reproducing systems** or **reproducing formulae** are applied to any of several methods that "analyze" a vector v (or function) and, then "reconstructs" v in terms of this analysis. In order to fix our ideas, let us consider a specific way in which this procedure is carried out that will help us explain the principal features of this paper.

A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **frame** if there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|v\|^2 \leq \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \leq B\|v\|^2$$

for all $v \in \mathcal{H}$. If only the right hand side inequality holds, we say that $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a **Bessel system** with constant B. A frame is a **tight frame** if A and B can be chosen so that $A = B$, and is a **normalized tight frame** if $A = B = 1$. Thus, if $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a normalized tight frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \tag{1.1}$$

for each $v \in \mathcal{H}$. This is equivalent to

$$v = \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle \, e_\alpha \tag{1.2}$$

for all $v \in \mathcal{H}$, where the series in (1.2) converges in the norm of $\mathcal{H}$ (we refer the reader to [18], Chapters 7 and 8, for the basic properties of frames that we shall use). We shall also consider **dual systems** $\{e_\alpha : \alpha \in \mathcal{A}\}$, $\{f_\alpha : \alpha \in \mathcal{A}\}$, where the first system is used for analyzing v and the second for reconstructing v. In this case the reproducing formula has the form

$$v = \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle \, f_\alpha \,, \tag{1.3}$$

which is clearly more general than (1.2).

For the moment, in order to explain the scope of this paper, let us restrict ourselves to the case of normalized tight frames. Examples of systems that we intend to examine are the **Gabor systems**, which have the form

$$\mathcal{G}_{B,C}(g) = \{e^{2\pi i Bm\cdot x} g(x - Ck) : m, k \in \mathbb{Z}^n\} \,, \tag{1.4}$$

where $g \in L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Another class of examples is given by the **affine systems**

$$\mathcal{F}_A(\psi) = \{\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \tag{1.5}$$

where $\psi \in L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$. There are relatively simple characterizations of those functions g and ψ for which these systems (for appropriate A, B and C) are normalized tight frames for $L^2(\mathbb{R}^n)$. It is fair to say that, though the adjective "simple" is appropriate for describing the characterizations, it is not at all appropriate for a description of the proofs found in the literature (see [18, 15, 17, 29, 6, 2, 3, 8, 7]). The characterizations of those g that generate a Gabor system that is a normalized tight frame can be given by a system of equalities, and the same is true for those ψ generating affine systems that are normalized tight frames. Though these equalities are different, there are certain similarities that makes it plausible to ask if there exists a general result that contains these two characterizations as special cases. This is one of the novelties of this paper: we formulate and prove such a result (Theorem 2.1, below). Another new feature is the method of proof. It relies on an idea that appears in [19] and [23] that converts the expression on the right of equality (1.1) into a function of $x \in \mathbb{R}^n$ (here $\mathcal{H} = L^2(\mathbb{R}^n)$) by applying to v translations that depend on x; this function can then be written as an (almost periodic) Fourier series. Finally, we obtain the characterization result as a consequence of the uniqueness property for this (almost periodic) Fourier series. By these means, we obtain results that are more general than those that appear in the literature.

Perhaps, as an illustration of the type of characterization equations we are considering, it is useful to consider the affine systems (1.5) generated by a function $\psi \in L^2(\mathbb{R}^n)$. If they are a normalized tight frame, then ψ is called a normalized **tight frame wavelet** (TFW); if, in addition, $\|\psi\|_2 = 1$, the system is an orthonormal basis for $L^2(\mathbb{R}^n)$ and ψ is called an **orthonormal wavelet** or, simply, a **wavelet**. The first characterization results for such systems were obtained independently by G. Gripenberg ([16]) and X. Wang ([35]) in one dimension, and the dilation A was, simply, multiplication by 2:

Theorem 1.1. (G. Gripenberg ([16]), X. Wang ([35])) *A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if $\|\psi\|_2 = 1$,*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \qquad \text{for a. e. } \xi \in \mathbb{R}, \tag{1.6}$$

and

$$t_q(\xi) = \sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+q))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.7}$$

whenever q is an odd integer.

Remarks.

1. In this paper, the form of the Fourier transform we use is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\xi\cdot x}\, dx\,.$$

2. Without the condition $\|\psi\|_2 = 1$, the two equalities (1.6) and (1.7) characterize the normalized tight frame wavelets (as explained in [18, Chapter 7]).

Many extensions of this result were obtained in higher dimensions: for $A = 2I$ this was done in [15] and more general dilation matrices A were introduced in the references we presented after equality (1.5). Many of the proofs involve the theory of shift invariant spaces and, as a consequence, this limits the dilations A to be matrices that preserve the integer lattice $\mathbb{Z}^n$. Another assumption about A that is made in these articles is that A is **expanding** (i.e. each proper value λ satisfies $|\lambda| > 1$). As we shall see later on, we will only need a somewhat more general hypothesis for A and do not assume that the lattice $\mathbb{Z}^n$ is preserved by A. We thus obtain a result that is more general than the characterization in [7], in which A did not have to preserve the integer lattice, but had to be expanding. In addition, we present an analysis of how the characterizing equations depend on the dilation matrix A.

The second author of this article wrote a paper ([21]) that focuses on the "unified approach" we have just described. The methods of proof in his article were based on the ideas from shift invariant spaces we mentioned above; consequently, the results obtained are less general because of the more restrictive assumptions we described in the last paragraph. The new approach also presents a good perspective of the history of the subject. For these reasons we chose the same title for this paper as the one used in [21] and added "II "at the end.

We end this introduction by indicating that the general result, Theorem 2.1, includes and leads to several applications that are more general than the ones we described above. For example, the **Gabor** and **affine** systems can be generated by finite families $\{g^1, \dots, g^L\}$ and $\{\psi^1, \dots, \psi^L\}$ of functions in $L^2(\mathbb{R}^n)$. Moreover, special cases involve yet other systems generated by the translation, modulations and dilations. These features are best described when we present the various applications of Theorem 2.1.

2 The main result

Let $\mathcal{P}$ be a countable collection of indices, $\{g_p : p \in \mathcal{P}\}$ be a family of functions in $L^2(\mathbb{R}^n)$ and $\{C_p : p \in \mathcal{P}\}$ be a corresponding collection of matrices in $GL_n(\mathbb{R})$. For $y \in \mathbb{R}^n$, let T_y be the translation (by y) operator defined by $T_y f = f(\cdot - y)$. The main result of this paper presents a characterization of all those families of the form

$$\big\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\big\}, \tag{2.1}$$

that are normalized tight frames for $L^2(\mathbb{R}^n)$. We introduce the following notation:

$$\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I(\mathbb{Z}^n), \tag{2.2}$$

where $C_p^I = (C_p^t)^{-1}$ (= the inverse of the transpose of C_p), and for $\alpha \in \Lambda$,

$$\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^t \alpha \in \mathbb{Z}^n\}. \tag{2.3}$$

If $\alpha = 0 \in \Lambda$, then $\mathcal{P}_0 = \mathcal{P}$ (since $C_p^t 0 = 0$ for all $p \in \mathcal{P}$); otherwise the best we can say is that $\mathcal{P}_\alpha \subset \mathcal{P}$.

Let N be defined on $L^2(\mathbb{R}^n)$ by letting

$$N^2(f) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 \tag{2.4}$$

for $f \in L^2(\mathbb{R}^n)$. By (1.1), the system (2.1) is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if N is the $L^2(\mathbb{R}^n)$-norm of f:

$$N^2(f) = \|f\|_2^2 \tag{2.5}$$

for all $f \in L^2(\mathbb{R}^n)$. Our main result, therefore, involves conditions on the system (2.1) that are equivalent to equality (2.5).

Since equalities (1.2) and (1.1) are valid for all $v \in \mathcal{H}$ if and only if they hold for a dense subspace of $\mathcal{H}$ (see [18, Chapter 7]), we will find it useful to introduce the set

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact} \right\},$$

which is dense in $L^2(\mathbb{R}^n)$.

Here is the statement of our main result:

Theorem 2.1. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that*

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + C_p^I m)|^2 \, \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \, d\xi < \infty \tag{2.6}$$

for all $f \in \mathcal{D}$, where $C_p^I = (C_p^t)^{-1}$. Then the system (2.1) is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \, \overline{\hat{g}_p(\xi)} \, \hat{g}_p(\xi + \alpha) = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.7}$$

for each $\alpha \in \Lambda$, where δ is the Kronecker delta for $\mathbb{R}^n$.

The proof of this result will be derived from some lemmas that will be established in this section. In the course of doing so we shall also indicate why the hypothesis (2.6) is plausible and discuss the convergence of some of the series we shall encounter. As a first observation along these lines, note that if equality (2.7) is valid for $\alpha = 0$, so that

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.8}$$

then if follows from Schwarz's inequality that the other series in (2.7) are a.e. absolutely convergent (recall that $\mathcal{P}_\alpha \subset \mathcal{P}$).

Let C be an $n \times n$ real matrix and $f, g \in L^2(\mathbb{R}^n)$. The **C-bracket product** of f and g is defined as

$$[f,g](x;C) \;=\; \sum_{k\in\mathbb{Z}^n} f(x - Ck)\, \overline{g(x - Ck)}. \tag{2.9}$$

This is an extension of the notion and notation introduced in [11] when $C = I$. It is clear that $[f,g]$ is CZ^n- periodic; that is, $[f,g](x + Cm; C) = [f,g](x;C)$ for each $m \in \mathbb{Z}^n$.

Lemma 2.2. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. If $f \in \mathcal{D}$ and $g \in L^2(\mathbb{R}^n)$, then*

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{Ck}\, g\rangle|^2 = \frac{1}{|\det C|} \int_{C^I\mathbb{T}^n} |[\hat{f},\hat{g}](\xi;C^I)|^2\, d\,\xi, \tag{2.10}$$

where $\mathbb{T}^n = [0,1)^n$.

Proof. Since $(T_{Ck}\, g)^\wedge(\xi) = e^{-2\pi i Ck\cdot\xi}\, \hat{g}(\xi)$, it follows from the Plancherel theorem that the left side of (2.10) equals

$$\sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, e^{2\pi i Ck\cdot\xi}\, d\xi \Big|^2 . \tag{2.11}$$

Since $\mathbb{R}^n = \bigcup_{l\in\mathbb{Z}^n}\{C^I(\mathbb{T}^n - l)\}$ is a disjoint union, the integral in (2.11) can be written in the form

$$\sum_{l\in\mathbb{Z}^n} \int_{C^I(\mathbb{T}^n)} \hat{f}(\xi - C^I l)\, \overline{\hat{g}(\xi - C^I l)}\, e^{2\pi i Ck\cdot\xi}\, d\xi = \int_{C^I(\mathbb{T}^n)} [\hat{f},\hat{g}](\xi;C^I)\, e^{2\pi i Ck\cdot\xi}\, d\xi\, .$$

But $[\hat{f},\hat{g}](\xi;C^I)$ is a $C^I\mathbb{Z}^n$-periodic function belonging to $L^2(C^I\mathbb{T}^n)$ (since $f \in \mathcal{D}$). Thus, the expression (2.11) is, up to a constant, the square of the l^2-norm of the Fourier coefficients of this $C^I\mathbb{Z}^n$-periodic function with respect to the orthonormal basis

$$\{\sqrt{|\det C|}\, e^{2\pi i Ck\cdot\xi} : k \in \mathbb{Z}^n\}$$

of $L^2(C^I\mathbb{T}^n)$. Equality (2.10) now follows immediately from this observation. $\square$

Lemma 2.3. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. For each $f \in \mathcal{D}$ and $g \in L^2(\mathbb{R}^n)$, the function*

$$H(x) \;=\; \sum_{k\in\mathbb{Z}^n} |\langle T_x\, f, T_{Ck}\, g\rangle|^2 \tag{2.12}$$

is the trigonometric polynomial

$$H(x) = \sum_{m\in\mathbb{Z}^n} \hat{H}(m)\, e^{2\pi i (C^I m)\cdot x},$$

where

$$\hat{H}(m) \;=\; \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I m)}\, \overline{\hat{g}(\xi)}\, \hat{g}(\xi + C^I m)\, d\xi, \tag{2.13}$$

and only a finite number of these expressions is non-zero.

Proof. If we do establish (2.13), the fact that $\hat{H}(m) = 0$ for all but finitely many m is an immediate consequence of the fact that $\hat{f}(\xi)$ and $\hat{f}(\xi + C^I m)$ must have disjoint support if $|m|$ is sufficiently large.

By Lemma 2.2,

$$\begin{aligned}
|\det C|\, H(x) &= \int_{C^I\mathbb{T}^n} |[(T_x\, f)^\wedge, \hat{g}](\xi; C^I)|^2\, d\,\xi \\
&= \int_{C^I\mathbb{T}^n} \Big| e^{-2\pi i \xi\cdot x} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i C^I m\cdot x} \hat{f}(\xi + C^I m)\, \overline{\hat{g}(\xi + C^I m)} \Big|^2\, d\xi \\
&= \int_{C^I\mathbb{T}^n} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i C^I m\cdot x} \hat{f}(\xi + C^I m)\, \overline{\hat{g}(\xi + C^I m)} \sum_{l\in\mathbb{Z}^n} e^{2\pi i C^I l\cdot x}\, \overline{\hat{f}(\xi + C^I l)}\, \hat{g}(\xi + C^I l)\, d\xi\,.
\end{aligned}$$

Let $k = l - m$ and express the above integrand function as a sum over k and m. We obtain the expression

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} \int_{C^I\mathbb{T}^n} \hat{f}(\xi + C^I m)\, \overline{\hat{g}(\xi + C^I m)} \sum_{k\in\mathbb{Z}^n} e^{2\pi i C^I k\cdot x}\, \overline{\hat{f}(\xi + C^I m + C^I k)}\, \hat{g}(\xi + C^I m + C^I k)\; d\xi \\
&= \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}(\xi)} \sum_{k\in\mathbb{Z}^n} e^{2\pi i C^I k\cdot x}\, \overline{\hat{f}(\xi + C^I k)}\, \hat{g}(\xi + C^I k)\; d\xi \\
&= \sum_{k\in\mathbb{Z}^n} \Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I k)}\, \overline{\hat{g}(\xi)}\, \hat{g}(\xi + C^I k)\, d\xi \Big)\, e^{2\pi i C^I k\cdot x}.
\end{aligned}$$

The various exchanges of summations and integration are justified by the fact that $f \in \mathcal{D}$. Equality (2.13) is obtained by dividing by $|\det C|$. □

We are now ready to state and prove the principal result that we shall use to establish Theorem 2.1:

Proposition 2.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$, $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$, and let $C_p^I = (C_p^t)^{-1}$. Assume that, for $f \in \mathcal{D}$, (2.6) is valid. Then, the function*

$$w(x) = N^2(T_x\, f) \;=\; \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_p k}\, g_p\rangle|^2$$

is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$\sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x}\ ,$$

where

$$\hat{w}(\alpha) \;=\; \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha)\, d\xi, \tag{2.14}$$

and the integral in (2.14) converges absolutely.

Remark. The function $w(x)$ given in the above proposition is an almost periodic function since these are characterized as uniform limits of generalized trigonometric polynomials (see [1]).

Proof. Observe that

$$w(x) = N^2(T_x\, f) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p(k - C_p^{-1}x)}\, g_p\rangle|^2.$$

For a fixed $p \in \mathcal{P}$, let $w_p(x)$ denote the above sum over $k \in \mathbb{Z}^n$. By Lemma 2.3, $w_p(x)$ is the $C_p\mathbb{Z}^n$-periodic trigonometric polynomial

$$w_p(x) = \sum_{m\in\mathbb{Z}^n} \hat{w}_p(m)\, e^{2\pi i C_p^I m\cdot x},$$

where

$$\hat{w}_p(m) = \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C_p^I m)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi. \tag{2.15}$$

We claim that $\{\hat{w}_p(m) : p \in \mathcal{P}, m \in \mathbb{Z}^n\}$ belongs to $\ell^1(\mathcal{P}\times\mathbb{Z}^n)$. To see this, let $K = \operatorname{supp}\hat{f}$ (recall that $f \in \mathcal{D}$ and, thus, K is compact) and $K(m) = K - C_p^I\, m$, so that $\hat{f}(\xi)\overline{\hat{f}(\xi + C_p^I\, m)} \neq 0$ only if $\xi \in K \cap K(m)$. Thus, the integral over $\mathbb{R}^n$ in (2.15) is really over this intersection. An application of Schwarz's inequality then gives us the fact that this integral does not exceed

$$\left(\int_{K(m)} |\hat{f}(\xi)\, \hat{g}_p(\xi + C_p^I m)|^2\, d\xi\right)^{1/2} \left(\int_K |\hat{f}(\xi + C_p^I m)\, \hat{g}_p(\xi)|^2\, d\xi\right)^{1/2}$$

and the change of variables $\xi = \eta - C_p^I m$ in the first integral makes this expression equal to

$$\left(\int_K |\hat{f}(\eta - C_p^I m)\, \hat{g}_p(\eta)|^2\, d\eta\right)^{1/2} \left(\int_K |\hat{f}(\xi + C_p^I m)\, \hat{g}_p(\xi)|^2\, d\xi\right)^{1/2}.$$

Then the inequality $2|cd| \le |c|^2 + |d|^2$ together with condition (2.6) proves

$$\sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} |\hat{w}_p(m)| < \infty,$$

which is our claim. It follows that

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \hat{w}_p(m)\, e^{2\pi i C_p^I m\cdot x}$$

where the convergence is absolute and uniform. In terms of the notation introduced in (2.2) and (2.3), we can write this last equality in the form

$$\begin{aligned} w(x) &= \sum_{\alpha\in\Lambda} \Big\{ \sum_{p\in\mathcal{P}_\alpha} \int_{\mathbb{R}^n} \frac{1}{|\det C_p|}\, \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha)\, d\xi \Big\}\, e^{2\pi i\alpha\cdot x} \\ &= \sum_{\alpha\in\Lambda} \Big\{ \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha)\, d\xi \Big\}\, e^{2\pi i\alpha\cdot x} \\ &= \sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x}, \end{aligned} \tag{2.16}$$

where $\hat{w}(\alpha)$ is the sum of some of the coefficients $\hat{w}_p(m)$, as indicated within the curly bracket. Since, as we have shown, $\{\hat{w}_p(m) : p \in \mathcal{P}, m \in \mathbb{Z}^n\}$ belongs to $\ell^1(\mathcal{P}\times\mathbb{Z}^n)$, it follows that $\{\hat{w}(\alpha) : \alpha \in \Lambda\}$ belongs to $\ell^1(\Lambda)$. Then it immediately follows that the last series in (2.16) is absolutely convergent. This finishes the proof of the proposition. □

Remark. Notice that condition (2.6) has been used to prove that $\{\hat{w}(\alpha) : \alpha \in \Lambda\}$ belongs to $\ell^1(\Lambda)$. As we shall see, in most cases, when we apply Theorem 2.1 we do not need to assume condition (2.6); for example, it will be shown that the Gabor systems, the affine systems and some related systems do satisfy this property.

The following lemma, that will be needed in the proof of Theorem 2.1, is a simple fact about uniqueness of the coefficients of an almost periodic Fourier series, as the one in (2.16).

Lemma 2.5. *Suppose* $\{c_\alpha : \alpha \in \Lambda\} \in \ell^1(\Lambda)$ *where* $\Lambda \subset \mathbb{R}^n$ *is countable. Then,* $v(x) = \sum_{\alpha\in\Lambda} c_\alpha\, e^{2\pi i\alpha\cdot x} = 0$ *for all* $x \in \mathbb{R}^n$ *if and only if* $c_\alpha = 0$ *for all* $\alpha \in \Lambda$.

Proof. It is clear that if $c_\alpha = 0$ for all $\alpha \in \Lambda$, then $v(x) \equiv 0$. Suppose $v(x) = 0$ for all $x \in \mathbb{R}^n$. Fix $\beta \in \Lambda$ and let $Q(R) = [-R, R]^n$, $R > 0$. Then

$$0 = \lim_{R\to\infty} \frac{1}{(2R)^n} \int_{Q(R)} v(x)\, e^{-2\pi i\beta\cdot x}\, dx = \lim_{R\to\infty} \sum_{\alpha\in\Lambda} c_\alpha \frac{1}{(2R)^n} \int_{Q(R)} e^{2\pi i\alpha\cdot x}\, e^{-2\pi i\beta\cdot x}\, dx\,.$$

Let us examine each of the above integral means. If $\alpha = \beta$, then the mean is 1. If $\alpha \neq \beta$, then

$$\frac{1}{(2R)^n} \int_{Q(R)} e^{2\pi i(\alpha-\beta)\cdot x}\, dx = \prod_{j=1}^{n} \Big\{ \frac{1}{2R} \int_{-R}^{R} e^{-2\pi i(\alpha j-\beta_j)x}\, dx \Big\}\,.$$

For at least one j, $\alpha_j - \beta_j \neq 0$. Thus, this factor is equal to

$$\frac{1}{2R} \frac{2\sin(2\pi(\alpha_j - \beta_j)R)}{2\pi(\alpha_j - \beta_j)}\,,$$

which tends to zero as $R \to \infty$. $\square$

Proof of Theorem 2.1. As observed before the statement of Theorem 2.1, it suffices to prove the result for a dense subset of $L^2(\mathbb{R}^n)$. Let us assume that condition (2.6) holds for all $f \in \mathcal{D}$ and that (2.7) is true. By Proposition 2.4,

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} |\langle T_x\, f, T_{C_p m}\, g_p\rangle|^2 = \sum_{\alpha\in\Lambda} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x},$$

where the last series converges absolutely (thus, $w(x)$ is continuous) and, by (2.7) and (2.14),

$$\hat{w}(\alpha) \;=\; \Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)}\, d\xi \Big)\, \delta_{\alpha,0}$$

for each $f \in \mathcal{D}$. The desired tight frame property (2.5) follows by letting $x = 0$.

Now let us assume that we have the tight frame property $N^2(f) = \|f\|^2$ for all $f \in L^2(\mathbb{R}^n)$. By Proposition 2.4, if $f \in \mathcal{D}$, then the function $z(x) = w(x) - \|f\|^2$ is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\hat{z}(0) = \hat{w}(0) - \|f\|^2, \qquad \text{and} \qquad \hat{z}(\alpha) = \hat{w}(\alpha),\ \alpha \neq 0.$$

Since $z(x) = 0$, it follows from Lemma 2.5 that all coefficients $\hat{z}(\alpha)$ must be 0. Thus, for $\alpha \in \Lambda$ and $f \in \mathcal{D}$, we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+\alpha)} \Big(\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi+\alpha) \Big)\, d\xi = \delta_{\alpha,0}\, \|f\|^2. \tag{2.17}$$

Consider the case $\alpha = 0$ and let

$$h_0(\xi) = \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\,.$$

By (2.6), h_0 is locally integrable; choose ξ_0 to be a point of differentiability of the integral of this function. Letting $B(\epsilon)$ denote the ball of radius $\epsilon > 0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}}\, \chi_{B(\epsilon)}(\xi - \xi_0)\,.$$

Then $\|f_\epsilon\|_2 = 1$ and $f_\epsilon \in \mathcal{D}$. By (2.17) with $f = f_\epsilon$, we have

$$1 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|}\, h_0(\xi)\, d\xi = h_0(\xi_0).$$

This shows that $h_0(\xi) = 1$, a.e. $\xi \in \mathbb{R}^n$, and (2.7) is satisfied for $\alpha = 0$. When $\alpha \neq 0$, let

$$h_\alpha(\xi) = \sum_{p \in \mathcal{P}(\alpha)} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + \alpha).$$

By polarization of (2.17) we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{\phi}(\xi + \alpha)}\, h_\alpha(\xi)\, d\xi = 0 \tag{2.18}$$

for all $f, \phi \in \mathcal{D}$. By Schwarz's inequality and (2.6), h_α is locally integrable. We can choose, again, a point of differentiability ξ_0 of the integral of h_α, and choose f_ϵ and ϕ_ϵ such that

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}}\, \chi_{B(\epsilon)}(\xi - \xi_0), \qquad \hat{\phi}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}}\, \chi_{B(\epsilon)}(\xi - \xi_0 - \alpha).$$

Hence $\|f_\epsilon\|_2 = \|\phi_\epsilon\|_2 = 1$, $f_\epsilon\ , \phi_\epsilon \in D$ and, by (2.18),

$$0 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|}\, h_\alpha(\xi)\, d\xi = h_\alpha(\xi_0).$$

Hence $h_\alpha(\xi) = 0$, a.e. $\xi \in \mathbb{R}^n$, and (2.7) is satisfied for $\alpha \neq 0$. □

Remark. In some applications, namely in the case of affine systems, it will be useful to replace the dense set $\mathcal{D}$ that appears in the statement of Theorem 2.1 by smaller dense sets of the form

$$\mathcal{D}_E = \{f \in \mathcal{D} : (\operatorname{supp} \hat{f}) \cap E = \emptyset\},$$

for any linear subspace E of $\mathbb{R}^n$ of dimension smaller than n. The result of Theorem 2.1 still holds true if the set $\mathcal{D}$ is replaced by any of these smaller dense sets.

3 The Gabor systems

Given a function $g \in L^2(\mathbb{R})$ and $b, c \in \mathbb{R} \setminus \{0\}$, then the classical **Gabor system** on $\mathbb{R}$ generated by g with parameters b and c is the collection

$$\mathcal{G}_{b,c}(g) = \{e^{2\pi i b m x} g(x - ck) : m, k \in \mathbb{Z}\}\,. \tag{3.1}$$

Many results are known that determine conditions on g and relations between the parameters for such systems to be a frame (see, for example, [18], where the Balian-Low theorem is presented, the density theorem of Rieffel ([27, 22, 32]) and the duality condition ([19, 12, 30])). We begin by showing that Theorem 2.1 can be applied directly for obtaining a characterization of those n-dimensional extensions of the system (3.1) that are normalized tight frames. The results we obtain include characterizations obtained by different authors ([30, 10, 21]). In order to describe these systems we will use the **translation operators** (as defined in Section 2) and the **modulation operators** M_z, $z \in \mathbb{R}^n$ defined by

$$(M_z f)(x) = e^{2\pi i z \cdot x} f(x),$$

for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The Gabor systems will be generated by a finite family $G = \{g^1, g^2, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ and a pair of matrices $B, C \in GL_n(\mathbb{R})$ so that they have the form

$$\mathcal{G} = \mathcal{G}_{B,C}(G) = \{M_{Bm} T_{Ck}\, g^\ell : m, k \in \mathbb{Z}^n,\ \ell = 1, 2, \cdots, L\}. \qquad (3.2)$$

If we change the order in which the translation and modulation operators are applied we also have the system

$$\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{B,C}(G) = \{T_{Ck} M_{Bm}\, g^\ell : m, k \in \mathbb{Z}^n,\ \ell = 1, 2, \cdots, L\}. \qquad (3.3)$$

A simple calculation shows that

$$T_{Ck} M_{Bm}\, g^\ell = e^{-2\pi i Bm \cdot Ck} M_{Bm} T_{Ck}\, g^\ell \qquad (3.4)$$

$m, k \in \mathbb{Z}^n$, and it follows immediately that

Lemma 3.1. *(**a**) $\mathcal{G}$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\widetilde{\mathcal{G}}$ is a frame for $L^2(\mathbb{R}^n)$; furthermore, the frame constants A and B can be taken to be the same in the two cases.*

*(**b**) $\mathcal{G}$ is an orthonormal system if and only if $\widetilde{\mathcal{G}}$ is an orthonormal system.*

We begin by observing that our main result, Theorem 2.1, easily implies the following characterization theorem:

Theorem 3.2. *The system $\mathcal{G} = \mathcal{G}_{B,C}(G)$ (or $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{B,C}(G)$) is a normalized tight frame if and only if*

$$\sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^n} \frac{1}{|\det C|}\, \hat{g}^\ell(\xi - Bk)\, \overline{\hat{g}^\ell(\xi - Bk + C^I m)} = \delta_{m,0} \qquad (3.5)$$

for a.e. $\xi \in \mathbb{R}^n$, all $m \in \mathbb{Z}^n$, where $C^I = (C^t)^{-1}$.

Proof. It will be clear from our proof that we can reduce the argument by assuming $L=1$; in any case, we shall address this issue after we show how to apply Theorem 2.1. By Lemma 3.1, it suffices to consider the system $\widetilde{\mathcal{G}}$. When we do this, we can write it in the form (2.1) by letting $g_p = M_{Bp}\, g$ for $p \in \mathcal{P} = \mathbb{Z}^n$ and $C_p = C$. Condition (2.6) follows: for $f \in \mathcal{D}$, only a finite number of terms of the form $\hat{f}(\xi + C^I\, m)$ can be non-zero if ξ is restricted to $K = \operatorname{supp} \hat{f}$ (recall that C and, therefore, C^I, are invertible and that K is bounded). Hence the integrability over K of

$$\sum_{p\in\mathbb{Z}^n} \frac{1}{|\det(C)|} \sum_{m\in\mathbb{Z}^n} |\hat{f}(\xi + C^I\, m)\, \hat{g}(\xi - B\, p)|^2$$

follows from the integrability over K of

$$\sum_{p\in\mathbb{Z}^n} |\hat{f}(\xi + C^I\, m)\, \hat{g}(\xi - B\, p)|^2$$

for each $m \in \mathbb{Z}^n$ (since all but a finite number of these expressions is non-zero; also recall that $\mathcal{P} = \mathbb{Z}^n$ in our present case). Furthermore, the fact that $\|\hat{f}\|_\infty < \infty$ reduces our task to showing that

$$\int_K \sum_{p\in\mathbb{Z}^n} |\hat{g}(\xi - Bp)|^2 < \infty\,. \tag{3.6}$$

For each $j \in \mathbb{Z}^n$, the collection $\{B(\mathbb{T}^n + j - p) : p \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$. Thus,

$$\|g\|_2^2 = \int_{\bigcup_{p\in\mathbb{Z}^n} B(\mathbb{T}^n+j-p)} |\hat{g}(\eta)|^2\, d\eta = \int_{B(\mathbb{T}^n+j)} \sum_{p\in\mathbb{Z}^n} |\hat{g}(\xi - B\, p)|^2\, d\xi$$

which shows the integrability of the integrand in (3.6) over the set $B(\mathbb{T}^n + j)$ for each $j \in \mathbb{Z}^n$. Since any bounded subset of $\mathbb{R}^n$ is contained in a finite number of such sets, we have the desired integrability. Incidentally, if we had $L > 1$, this proves the local integrability of L sums of the form (3.6). Theorem 3.2 now follows from Theorem 2.1 using $(M_{Bp}\, g)^\wedge(\xi) = \hat{g}(\xi - Bp)$. $\square$

4 The Calderón condition and reproducing systems

As mentioned after the statement of Theorem 2.1, the case $\alpha = 0$ of (2.7) is

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.1}$$

This formula is valid when the systems described in (2.1) are normalized tight frames and satisfy condition (2.6). When applying this result to the affine system (1.5), a simple calculation (see also Section 6) shows that (4.1) becomes

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.2}$$

for $\psi \in L^2(\mathbb{R}^n)$ and $B = A^t \in GL_n(\mathbb{R})$. Versions of the "resolution of the identity"(4.2) have appeared in works of A. P. Calderón and has become known as the Calderón condition in the area of orthonormal wavelets. For this reason we shall say that (4.1) is a **Calderón condition**.

Under the assumption (2.6), Theorem 2.1 shows that the Calderón condition (4.1) is necessary for the system $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$, given by (2.1), to be a normalized tight frame. Together with the cases $\alpha \neq 0$ of (2.7) we obtain a necessary and sufficient condition. We will show in this section that other type of conditions can replace the cases $\alpha \neq 0$ of (2.7).

If we remove condition (2.6) we can still prove a weaker version of (4.1) where the equality is replaced by an inequality. This result, which will play a major role in Section 5, is a consequence of Lemma 2.3 and it is given below. The result is stated and proved for Bessel systems as defined in Section 1.

Proposition 4.1. *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$, and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. If the system $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is Bessel with constant B, then*

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 \leq B \quad \textit{for a.e. } \xi \in \mathbb{R}^n. \tag{4.3}$$

Proof. In most applications of this Proposition, $\mathcal{P}$ will be a subset of $\mathbb{Z}^r$ for some $r \in \mathbb{N}$. For simplicity we assume this to be the case here. However, the reader can easily check that this is not a loss of generality.

Assume that $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a Bessel sequence with constant B. Then, for every $M \in \mathbb{N}$

$$\sum_{p\in\mathcal{P},|p|\leq M} \; \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_pk}\, g_p\rangle|^2 \leq B\|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$. Applying Lemmas 2.2 and 2.3 to each $p \in \mathcal{P}$ (letting $x = 0$), we can write

$$\sum_{p\in\mathcal{P},|p|\leq M} \; \sum_{k\in\mathbb{Z}^n} \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C_p^I\, k)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I\, k)\, d\xi \leq B\|f\|_2^2 \tag{4.4}$$

for all $f \in \mathcal{D}$, $M \in \mathbb{N}$ (also recall that $C^I = (C^t)^{-1}$). For each $M \in \mathbb{N}$ let

$$h_{0,M} = \sum_{p\in\mathcal{P},|p|\le M} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 .$$

Since each $g_p \in L^2(\mathbb{R}^n)$ and there is only a finite number of elements of $\mathcal{P}$ in the above sum, $h_{0,M} \in L^1(\mathbb{R}^n)$. Let L_M be the set of differentiability points of the integral of $h_{0,M}$ and take $\xi_0 \in L_M$. Letting $B(\epsilon)$ denote the ball of radius $\epsilon > 0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}}\, \chi_{B(\epsilon)}(\xi - \xi_0) .$$

Then $\|f_\epsilon\|_2 = 1$ and $f_\epsilon \in \mathcal{D}$. For $M \in \mathbb{N}$, let

$$\Lambda_{0,M} = \bigcup_{p\in\mathcal{P},|p|\le M} C_p^I(\mathbb{Z}^n \setminus \{0\}) \qquad \text{and} \qquad \delta_M = \inf\{|C_p^I\, k| : C_p^I\, k \in \Lambda_{0,M}\}.$$

Observe that $\delta_M > 0$ since each C_p^I is invertible, $k \neq 0$, and there is only a finite number of elements of $\mathcal{P}$ in the set $\Lambda_{0,M}$. For $\epsilon < \delta_M/2$, $|\xi - \xi_0| < \epsilon$, and $C_p^I \in \Lambda_{0,M}$ we have

$$|\xi + C_p^I\, k - \xi_0| \ge |C_p^I\, k| - |\xi - \xi_0| \ge \delta_M - \epsilon > \delta_M - \frac{\delta_M}{2} = \frac{\delta_M}{2} > \epsilon ,$$

so that $\xi + C_p^I\, k - \xi_0$ does not belong to $B(\epsilon)$. This means that $\hat{f}_\epsilon(\xi + C_p^I\, k) = 0$ for all $k \neq 0$, $\epsilon < \delta_M/2$, and $|p| < M$, and, thus, all the terms in (4.4) equal 0 except the one corresponding to $k = 0$. Thus,

$$\lim_{\epsilon\to 0} \int_{|\xi-\xi_0|\le\epsilon} \frac{1}{|B(\epsilon)|}\, h_{0,M}(\xi)\, d\xi = \lim_{\epsilon\to 0} \int_{|\xi-\xi_0|\le\epsilon} \frac{1}{|B(\epsilon)|} \sum_{p\in\mathcal{P},|p|\le M} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi \le B.$$

Since the left hand side of this formula coincides with $h_{0,M}(\xi_0)$, we deduce that $h_{0,M}(\xi_0) \le B$ for all $\xi_0 \in L_M$.
Since

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2 \; = \lim_{M\to\infty} h_{0,M}(\xi),$$

we obtain the desired result for all ξ in the intersection of all L_M, which is a dense set in $\mathbb{R}^n$. □

We now present the main result of this section, which follows from the arguments presented in Section 2.

Theorem 4.2. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that (2.6) holds for all $f \in \mathcal{D}$. Then the system $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (4.1) holds.*

Proof. Under condition (2.6), if $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame, then it is clearly Bessel with constant 1, and, by Theorem 2.1, the Calderón condition (4.1) holds (take $\alpha = 0$ in (2.7)).

For the converse we need to recall the following fact about almost periodic functions which can be found in [1, Satz XXXVI] (see also [36, page 111]):

Lemma 4.3. *Suppose that h is a non-negative almost periodic function defined in $\mathbb{R}^n$, and let*

$$M(h) \equiv \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(x)\, dx$$

be the mean of h, where $Q(R) = [-R, R]^d$. Then, $M(h) = 0$ if and only if $h \equiv 0$.

Let

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_pk}\, g_p\rangle|^2$$

as in Proposition 2.4. Since $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is Bessel with constant 1 we have $w(x) \le \|T_x f\|_2^2 = \|f\|_2^2$ for all $f \in L^2(\mathbb{R}^n)$. Thus, for any $f \in \mathcal{D}$, the function $h(x) = \|f\|_2^2 - w(x)$ is non-negative, and, by Proposition 2.4 and the remark that follows its proof, is continuous and almost periodic. Taking the mean value of $h(x)$ and using, again, Proposition 2.4 to write $w(x)$ as an absolutely convergent (generalized) Fourier series with coefficients $\hat{w}(\alpha)$, given by (2.14), we obtain

$$M(h) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(x)\, dx = \|f\|_2^2 - \sum_{\alpha\in\Lambda} \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} \hat{w}(\alpha)\, e^{2\pi i\alpha\cdot x}\, dx\, .$$

As in the proof of Lemma 2.5 all the above integrals are zero except the one corresponding to $\alpha = 0$ that becomes $\hat{w}(0)$. Thus, $M(h) = \|f\|_2^2 - \hat{w}(0)$ for all $f \in \mathcal{D}$. By the Calderón condition (4.1), we have

$$\hat{w}(0) = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2\, d\xi = \|f\|_2^2\, .$$

Hence, $M(h) = 0$ for all $f \in \mathcal{D}$. By Lemma 4.3, $h(x) = 0$ for all $x \in \mathbb{R}^n$ and all $f \in \mathcal{D}$. Taking $x = 0$ (recall that h is continuous) we deduce that $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame, as desired, since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$. □

Remarks. (**1**) That Theorem 4.2 follows from our main work in Section 2 was pointed out to us by S. Xiao. The method that we use follows the line of argument presented in [24]. In the case of wavelet systems, like the types described by (1.5), Theorem 4.2 has been proved by M. Bownik [3] for expanding dilation matrices with integer entries, and by R. Laugesen [23, 24] for expanding dilation matrices with real entries.

(**2**) As in the remark given at the end of Section 2, the set $\mathcal{D}$ that appears in Theorem 4.2 can be replaced by the smaller dense subsets $\mathcal{D}_E$ and Theorem 4.2 still holds.

For orthonormal systems, we have the following simple corollary of Theorem 4.2:

Corollary 4.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that (2.6) holds for all $f \in \mathcal{D}$ and that the system $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is an orthonormal system in $L^2(\mathbb{R}^n)$. Then $\{T_{C_pk}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is complete in $L^2(\mathbb{R}^n)$ if and only if the Calderón condition (4.1) holds.*

When applied to wavelet systems, like the types described in (1.5), Corollary 4.4 shows that an orthonormal wavelet system is complete if and only if the Calderón condition for wavelets (4.2) holds. This has been proved in [3, 33] for expanding dilation matrices with integer entries and in [24] and for expanding dilation matrices with real entries.

In the next section, we shall explain how Theorem 4.2 and Corollary 4.4 can be applied to the affine systems. For the moment, we restrict our attention to the Gabor systems and establish other consequences of Theorem 4.2 and Corollary 4.4.

Consider the Gabor systems $\mathcal{G}_{B,C}(G)$, given by (3.2), and $\widetilde{\mathcal{G}}_{B,C}(G)$, given by (3.3). Since $(M_{B\,p}\, g)^\wedge(\xi) = \hat{g}(\xi - Bp)$, the Calderón condition (4.1) for the system $\widetilde{\mathcal{G}}_{B,C}(G)$ becomes

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} |\hat{g^\ell}(\xi - B\,k)|^2 = |\det C| \qquad \text{for a. e. } \xi \in \mathbb{R}^n. \tag{4.5}$$

From Theorem (4.2) and Corollary (4.4) we obtain:

Corollary 4.5. *Let $G = \{g^1, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Then the Gabor system $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is a normalized tight frame if and only if $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is a Bessel system with constant 1 and (4.5) holds.*

Corollary 4.6. *Let $G = \{g^1, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ and $B, C \in GL_n(\mathbb{R})$. Suppose that the Gabor system $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is an orthonormal system in $L^2(\mathbb{R}^n)$. Then $\mathcal{G}$ (or $\widetilde{\mathcal{G}}$) is complete if and only if (4.5) holds.*

The results obtained in the above corollaries are contained in [30, 20, 10]. Thus, neither of these two results are new, but the point is that each follows easily from our general framework.

Using Proposition 4.1, we obtain the following special case of Theorem 2.1, where we assume $C_p = C$ for every $p \in \mathcal{P}$. This result can also be found in [28, 21].

Theorem 4.7. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ be a collection of functions in $L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. Then, the system $\big\{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\big\}$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{p\in\mathcal{P}} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C^I m) = |\det C|\, \delta_{m,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.6}$$

for every $m \in \mathbb{Z}^n$, where δ is the Kronecker delta in $\mathbb{R}^n$.

Proof. Since (4.6) follows immediately from (2.7) when $C_p = C$ for all $p \in \mathcal{P}$, then we only need to show that condition (2.6) is always satisfied in Theorem 2.1 under these conditions. Indeed, since $C_p = C$ for every $p \in \mathcal{P}$, then the sum with respect to m in (2.6) is finite (since $f \in \mathcal{D}$). If the system $\big\{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\big\}$ is a normalized tight frame, then, by Proposition 4.1,

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C|}\, |\hat{g}_p(\xi)|^2 \leq 1. \tag{4.7}$$

Together with the fact that the sum with respect to m is finite, this implies (2.6). Similarly, if (4.6) holds, then we have inequality (4.7). Together with the fact that the sum with respect to m is finite, this implies (2.6), as in the previous case. □

5 Affine systems and wavelets

The classical **affine system** on $\mathbb{R}$ generated by $\psi \in L^2(\mathbb{R})$ is the collection

$$\mathcal{F}_2(\psi) = \{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}. \tag{5.1}$$

This is the system (1.5) when the dimension is 1 and $A = 2$. As mentioned in Section 1, the characterization of those functions ψ for which $\mathcal{F}_2(\psi)$ is a normalized tight frame in $L^2(\mathbb{R})$ was accomplished by G. Gripenberg ([16]) and X. Wang ([35]), and this result has been extended to general dilations $a \in \mathbb{R}$, $a > 1$, (cf. [8, Th. 1]), and to $\mathbb{R}^n$ where dilations are performed by real expanding matrices (cf. [7, Cor. 2.4] and [24, Th. 5.1]).

To define these more general systems, we use the **translation operators** (as defined in Section 2) and the **dilation operators** D_A, $A \in GL_n(\mathbb{R})$, defined by

$$(D_A\, f)(x) = |\det A|^{1/2}\psi(Ax),$$

for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Our affine systems will be generated by applying these operators to a finite family $\Psi = \{\psi^1, \ldots \psi^L\} \subset L^2(\mathbb{R}^n)$, and they have the form

$$\mathcal{F}_A(\Psi) = \{D_A^j\, T_k\, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \ldots, L\}. \tag{5.2}$$

A simple calculation shows that

$$D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell\,,$$

so that, in order to apply Theorem 2.1, we are led to consider

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1, 2, \ldots, L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell, \qquad \text{and} \qquad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1, \ldots, L.$$

There are good reasons for the fact that, in the literature, the characterizations of the systems $\mathcal{F}_A(\Psi)$, given by (5.2), that are normalized tight frames assume that the dilation matrices are **expanding**. In a private communication, D. Speegle has presented us with examples of dilation matrices which are not expanding for which there cannot exist any tight frame wavelets.

By definition, a matrix $M \in GL_n(R)$ is **expanding** on $\mathbb{R}^n$ if and only if all the eigenvalues of M have modulus greater than 1. There is an equivalent definition of expanding matrices (which we present in Lemma 5.2), that will be most useful for our purposes. To show this equivalence we need the following result:

Lemma 5.1. *Suppose $M \in GL_n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < |\lambda| < \beta < \infty$ for all eigenvalues λ of M. There exists $C = C(M, \alpha, \beta) \geq 1$ such that*

$$\frac{1}{C}\alpha^j |x| \leq |M^j\, x| \leq C\beta^j |x|, \tag{5.3}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$.

Remark. By applying (5.3) to $x = M^{-j}y$, we obtain

$$\frac{1}{C}\beta^{-j}|y| \leq |M^{-j}\, y| \leq C\alpha^{-j}|y|, \tag{5.4}$$

when $y \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$.

Proof. We make use of the following fact involving the spectral radius, $\rho(M) = \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$:

$$\rho(M) = \lim_{n\to\infty} \|M^n\|^{1/n}$$

(see [31, p. 235]). Since $\rho(M) < \beta$, there exists $J_0 \in \mathbb{N}$ such that

$$|M^j\, x| \leq \|M^j\||x| \leq \beta^j |x|$$

for $j \geq J_0$ and $x \in \mathbb{R}^n$. For $0 \leq j < J_0$ we have

$$|M^j\, x| \leq \|M^j\|\, |x| = \frac{\|M^j\|}{\beta^j}\beta^j\, |x| \leq \Big(\max_{0\leq j<J_0} \big\{\frac{\|M^j\|}{\beta^j}\big\}\Big)\beta^j\, |x|.$$

Hence, letting $C = \max_{0\leq j<J_0}\big\{1, \frac{\|M^j\|}{\beta^j}\big\}$ we have $|M^j\, x| \leq C\beta^j\, |x|$, for all $j \in \mathbb{Z}$, $j \geq 0$ and $x \in \mathbb{R}^n$. This gives us the right hand side inequality in (5.3). For the left hand side inequality of this formula, apply the result just proved to $N = M^{-1}$; since $\rho(N) < 1/\alpha$, we deduce $|N^j\, y| \leq C(1/\alpha)^j\, |y|$, for all $y \in \mathbb{R}^n$, $j > 0$, $j \in \mathbb{Z}$. The result follows by writing $y = M^j x$, for $x \in \mathbb{R}^n$. □

Remark. Lemma (5.1) appears without proof in a paper by P.G. Lemarié-Rieusset ([26]). We thank G. Garrigós for pointing this reference to us.

Lemma 5.2. *A matrix* $M \in GL_n(\mathbb{R})$ *is expanding if and only if there exist* $0 < k \leq 1 < \gamma < \infty$ *such that*

$$|M^j\, x| \geq k\gamma^j |x| \tag{5.5}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$. *Moreover, if* $M \in GL_n(\mathbb{R})$ *is expanding, then we also have*

$$|M^{-j}\, x| \leq \frac{1}{k}\gamma^{-j}|x| \tag{5.6}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$.

Proof. If M is an expanding matrix, then we can choose $\alpha > 1$ in Lemma 5.1. Thus, (5.5) follows immediately from the left hand side inequality of (5.3). The inequality (5.6) follows by applying (5.5) to $y = M^{-j}x$.

Assume now that (5.5) holds and suppose that λ is an eigenvalue of M. If $\lambda \in \mathbb{R}$, let $x \in \mathbb{R}^n$ be an eigenvector corresponding to λ. By (5.5) we have

$$|\lambda|^j |x| = |\lambda^j x| = |M^j x| \geq k\gamma^j |x|$$

for all $j \in \mathbb{Z}$, $j \geq 0$. It follows that $|\lambda| \geq k^{1/j}\gamma$ for all $j \geq 0$. Hence, $|\lambda| \geq \gamma > 1$.

If $\lambda = \alpha + i\beta \in \mathbb{C}^n$, choose a corresponding eigenvector $u = x + iy \in \mathbb{C}^n$. Since M is expanding and $x \in \mathbb{R}^n$,

$$k\gamma^j\, |x| \leq |M^j\, x| \leq |M^j\, x + i\, M^j\, y| = |M^j(x+iy)| = |\lambda^j u| = |\lambda|^j |u|.$$

Without loss of generality we can assume $|y| \leq |x|$, so that $|u| \leq \sqrt{2}\, |x|$. It follows that $k\, \gamma^j\, |x| \leq |\lambda|^j \sqrt{2}\, |x|$. Since $x \neq 0$ (otherwise $u = 0$), we have $k^{1/j}\, \gamma \leq |\lambda|\, 2^{1/j}$ for all $j \geq 0$. Hence, $1 < \gamma \leq |\lambda|$. □

The dilation matrices we are going to use are more general than the expanding ones: they could have some, but not all, of its eigenvalues with modulus 1, while the rest have modulus strictly larger than 1. Here we must notice that, if $|\det A| = 1$, then there is no $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ such that the Calderón condition:

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\widehat{\psi^\ell}(B^{-j}\xi)|^2 = 1\,, \quad \text{for a. e. } \xi \in \mathbb{R}^n, \tag{5.7}$$

where $B = A^t$, holds. This result follows by an argument similar to one presented in [25], where continuous wavelets are studied. It is also known that, in some cases, when some of the eigenvalues of A have modulus greater than 1 and others have modulus smaller than 1, there is no $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ such that (5.7) hold, even if $|\det A| \neq 1$. As we pointed out before Lemma 5.1, this non-existence has been shown to us by D. Speegle, in a personal communication, for the case of diagonal dilation matrices. At the moment we are not aware of this (negative) result for general matrices A with these properties.

The dilation matrices we are going to use must have the properties described below:

Definition 5.1. Given $M \in GL_n(\mathbb{R})$ and a non-zero linear subspace F of $\mathbb{R}^n$, we say that M **is expanding on** F if there exists a complementary (not necessarily orthogonal) linear subspace E of $\mathbb{R}^n$ with the following properties:

(i) $\mathbb{R}^n = F + E$ and $F \cap E = \{0\}$;

(ii) $M(F) = F$ and $M(E) = E$, that is, F and E are invariant under M;

(iii) condition (5.5) (and therefore (5.6)) holds for all $x \in F$;

(iv) given $r \in \mathbb{N}$, there exists $C = C(M, r)$ such that, for all $j \in \mathbb{Z}$, the set

$$\mathcal{Z}_r^j(E) = \{m \in E \cap \mathbb{Z}^n : |M^j\, m| < r\}$$

has less than C elements.

Example 1. When M is an expanding matrix, Definition 5.1 is satisfied with $F = \mathbb{R}^n$ and $E = \{0\}$.

Example 2. For $a \in \mathbb{R}$, $|a| > 1$, the matrix

$$M = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues a and 1. Letting F be the eigenspace corresponding to the eigenvalue a, and E the eigenspace corresponding to the eigenvalue 1, it is clear that M is expanding on F, in the sense of Definition 5.1. It is easy to obtain analogous, higher dimensional, diagonal matrices, even allowing some of the elements of the diagonal to be -1, that satisfy "expanding on F".

Example 3. More generally, given $a \in \mathbb{R}$, $|a| > 1$ and two independent vectors $u, v \in \mathbb{R}^2$, let M be a matrix for which u is an eigenvector corresponding to the eigenvalue a and v is an eigenvector corresponding to the eigenvalue 1. By taking $F = \{tu : t \in \mathbb{R}\}$ and $E = \{tv : t \in \mathbb{R}\}$ is easy to see that the conditions of Definition 5.1 are satisfied.

Example 4. For $a \in \mathbb{R}$, $|a| > 1$, and $\theta \in \mathbb{R}$, consider the matrix

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

which corresponds to a dilation on the X–axis and a rotation around the origin in the YZ–plane. The matrix M is expanding on $F = \mathbb{R} \times \{0\} \times \{0\}$, with $E = \{0\} \times \mathbb{R} \times \mathbb{R}$.

Example 5. For $a, b \in \mathbb{R}$, $|a| > 1$, consider

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

With $F = \mathbb{R} \times \{0\} \times \{0\}$, and $E = \{0\} \times \mathbb{R} \times \mathbb{R}$, properties (i), (ii), and (iii) of Definition 5.1 are obvious. A little bit of work is required to prove property (iv); however it is straightforward. Write $m \in E \cap \mathbb{Z}^3$ as $m = (0, m_2, m_3)$ with $m_2, m_3 \in \mathbb{Z}$. Since

$$M^j = \begin{pmatrix} a^j & 0 & 0 \\ 0 & 1 & jb \\ 0 & 0 & 1 \end{pmatrix}, \qquad j \in \mathbb{Z},$$

$|M^j\, m| < r$ implies $|m_2 + jbm_3|^2 + |m_3|^2 < r^2$. Hence $|m_3| < r$ and $|m_2 + jbm_3| < r$. For each $m_3 \in \mathbb{Z}$ fixed, there are at most $2r$ elements $m_2 \in \mathbb{Z}$ such that $|m_2 + jbm_3| < r$. Since there are at most $2r$ elements $m_3 \in \mathbb{Z}$ such that $|m_3| < r$, it follows that the number of elements in $\mathcal{Z}_r^j(E)$ does not exceed $4r^2$ for all $j \in \mathbb{Z}$.

The main result of this section is the following characterization of the affine systems $\mathcal{F}_A(\Psi)$, which will be obtained as a consequence of Theorem 2.1. As we mentioned in Section 1, this result is related and extends several other results that are in the literature. We will later show (Theorem 5.7) that there is an equivalent formulation of the following theorem, where (5.8) is replaced by a simpler expression.

Theorem 5.3. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then, the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{5.8}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} :\, B^{-j}\alpha \in \mathbb{Z}^n\}$.

Proof. Recall that $D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell$. Apply Theorem 2.1 with

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1,2,\ldots,L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell, \qquad \text{and} \qquad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1,\ldots,L.$$

Since

$$\hat{g}_p(\xi) = (D_A^j \psi^\ell)^\wedge(\xi) = D_B^{-j} \hat{\psi}^\ell(\xi) = |\det B|^{-j/2} \hat{\psi}^\ell(B^{-j}\,\xi),$$

(5.8) follows from (2.7) in Theorem 2.1, provided the hypothesis (2.6) in this Theorem is satisfied. Therefore, all that it is left to prove is that the hypothesis (2.6) is satisfied in this particular case. Thus, we need to show that $L(f) < \infty$ for f in an appropriate dense set of $L^2(\mathbb{R}^n)$, where

$$\begin{aligned} L(f) =& \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\det A^j|\, |(D_A^j \psi^\ell)^\wedge(\xi)|^2\, d\xi \\ =& \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi\,. \end{aligned} \tag{5.9}$$

The dense set we choose is the following: since $B = A^t$ is expanding on F, we can then take E a complementary subspace to F as in Definition 5.1, and consider

$$\mathcal{D}_E = \{f \in \mathcal{D} : (\operatorname{supp}\hat{f}) \cap E = \emptyset\} \tag{5.10}$$

where $\mathcal{D} = \{f \in L^2(\mathbb{R}^n) :\, \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp}\hat{f} \text{ is compact}\}$. This set $\mathcal{D}_E$ is dense in $L^2(\mathbb{R}^n)$, since E has measures zero.

The proof that $L(f) < \infty$ if $f \in \mathcal{D}_E$ is given in Proposition 5.6 below. To prove this delicate result we need some preparation and two lemmas.

Since B is expanding on F, by property (i) of Definition 5.1, given $x \in \mathbb{R}^n$, there exist unique $x_F \in F$ and $x_E \in E$ such that $x = x_F + x_E$. For $r, s \in \mathbb{R}$, define

$$Q(r,s) = \Big\{x = x_F + x_E : x_F \in F,\ x_E \in E,\ \frac{1}{r} < |x_F| < r,\ |x_E| < s\Big\}, \tag{5.11}$$

and write $Q(r) = Q(r,r)$ (see Figure 5). It is clear that given any $f \in \mathcal{D}_E$ there exists $r \in \mathbb{N}$ such that $\operatorname{supp}\hat{f} \subset Q(r)$.

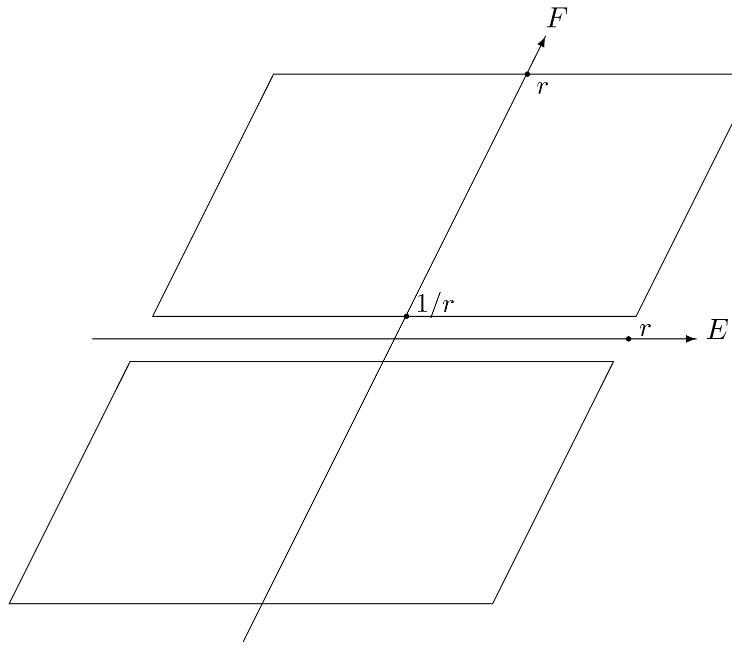

Figure 1: The set $Q(r)$ $(n = 2)$.

Lemma 5.4. *Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, and $r \in \mathbb{R}$. There exists $N = N(M, r) \in \mathbb{N}$ such that*

$$\#\{j \in \mathbb{Z} : M^j \eta \in Q(r)\} \leq N$$

for all $\eta \in \mathbb{R}^n$.

Proof. Choose E to be a complementary subspace for F as in Definition 5.1. If $\eta \in E$, we can choose $N = 1$. For $\eta \notin E$, write $\eta = \eta_F + \eta_E$ with $\eta_F \in F$, $\eta_E \in E$ and $\eta_F \neq 0$. Then, by (ii) of Definition 5.1, for any $j \in \mathbb{Z}$, we have $M^j \eta = M^j \eta_F + M^j \eta_E$ with $M^j \eta_F \in F$ and $M^j \eta_E \in E$.
Choose $j_0 = j_0(\eta)$ to be the smallest integer such that $|M^{j_0} \eta_F| > 1/r$. This is possible since, by property (iii) of the matrix M, there exist $0 < k \leq 1 < \gamma < \infty$ such that

$$|M^j \eta_F| \geq k\gamma^j |\eta_F| \quad \text{if} \quad j \in \mathbb{Z},\, j \geq 0\,,$$

and

$$|M^{-j} \eta_F| \leq \frac{1}{k} \gamma^{-j} |\eta_F| \quad \text{if} \quad j \in \mathbb{Z},\, j \geq 0\,.$$

Thus, if $j < j_0$, then $|M^j \eta_F| \leq 1/r$, which implies that $M^j \eta \notin Q(r)$ by the definition of $Q(r)$.

Choose $N_0 = 1 + [\log_\gamma(r^2/k)]$ (observe that $k/r \le 1/r < r$ implies $r^2/k > 1$, so that $[\log_\gamma(r^2/k)] \ge 0$). Since M is expanding on F, if $j \ge N_0 \ge 1$, we have

$$|M^{j+j_0}\eta_F| = |M^j M^{j_0}\eta_F| \ge k\gamma^j |M^{j_0}\eta_F| > k\gamma^j \frac{1}{r}\,,$$

by the choice of j_0. Thus,

$$|M^{j+j_0}\eta_F| > k\gamma^j \frac{1}{r} \ge k\gamma^N \frac{1}{r} \ge k\frac{r^2}{k}\frac{1}{r} = r\,.$$

This shows that if $j \ge N_0$, *then*$M^{j+j_0}\eta_F \notin Q(r)$. Hence,

$$\{j \in \mathbb{Z} : M^j\eta \in Q(r)\} \subset \{j_0, j_0+1, \cdots, j_0+N_0-1\}.$$

By taking $N = N_0$ the proof is finished. □

Remark. Lemma 5.4 is adapted from [2, Lemma 2.3], where the result is proved only for expanding matrices on $\mathbb{R}^n$.

For $r, s \in \mathbb{R}$, define

$$\widetilde{Q}(r,s) = \{x = x_F + x_E : x_F \in F,\, x_E \in E,\, |x_F| < r,\, |x_E| < s\}\,,$$

and write $\widetilde{Q}(r) = \widetilde{Q}(r,r)$. These sets will be used in the statement and the proof of the next lemma.

Lemma 5.5. *Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, $r \in \mathbb{R}$, and E be a complementary subspace of F as in Definition 5.1. There exists $\widetilde{C} = \widetilde{C}(M,r) \in \mathbb{R}$ such that*

$$\#\{m \in \mathbb{Z}^n \setminus E : M^j m \in \widetilde{Q}(r)\} \le \widetilde{C}|\det M|^{-j}$$

for all $j \in \mathbb{Z}$.

Proof. For $m \in \mathbb{Z}^n \setminus E$, write $m = m_F + m_E$ with $m_F \in F$, $m_E \in E$ and $m_F \ne 0$. Let

$$T_r = \inf\{|m_F| : m \in (\mathbb{Z}^n \setminus E) \cap \widetilde{Q}(r)\} > 0.$$

Take j_1 to be the smallest positive integer greater than $\log_\gamma(r/(kT_r))$, where k and γ are as in Lemma 5.2 (adapted to Definition 5.1). If $j \ge j_1$ and $m \in \mathbb{Z}^n \setminus E$, then, by (iii) of Definition 5.1, we have $|M^j m_F| \ge k\gamma^j |m_F| \ge k\frac{r}{kT_r}T_r = r$. Hence, for $j \ge j_1$,

$$\#\{m \in \mathbb{Z}^n \setminus E : M^j m \in \widetilde{Q}(r)\} = 0. \tag{5.12}$$

Thus, we only need to consider $j < j_1$. Choose $m \in \mathbb{Z}^n \setminus E$ with $M^j m \in \widetilde{Q}(r)$, $\xi \in [0,1)^n$ and $j < j_1$. Write $\xi = \xi_F + \xi_E$ with $\xi_F \in F$ and $\xi_E \in E$. Since M is expanding on F,

$$\begin{aligned}|M^{-j_1+j}(m_F + \xi_F)| \leq & |M^{-j_1+j}(m_F)| + |M^{-j_1+j}\xi_F)| \\ \leq & \frac{1}{k}\gamma^{-j_1}|M^j(m_F)| + \frac{1}{k}\gamma^{-j_1+j}|\xi_F| \\ < & \frac{1}{k}\gamma^{-j_1}r + \frac{1}{k}|\xi_F| \leq \frac{1}{k}\gamma^{-j_1}r + \frac{1}{k}S_1 \equiv R_1\end{aligned}$$

where $S_1 = \sup\{|\xi_F| : \xi \in [0,1)^n\}$. Also, since $\|M\| \geq \rho(M) \geq 1$, we have

$$|M^{-j_1+j}(m_E + \xi_E)| \leq \|M\|^{-j_1}r + \|M\|^{-j_1+j}|\xi_E| \leq \|M\|^{-j_1}r + S_2 \equiv R_2,$$

where $S_2 = \sup\{|\xi_E| : \xi \in [0,1)^n\}$. We have just shown that

$$\{m \in \mathbb{Z}^n \setminus E : M^j\eta \in \widetilde{Q}(r)\} \subset \{m \in Z^n : m + [0,1)^n \subset M^{j_1-j}(\widetilde{Q}(R_1,R_2))\} \equiv \mathcal{M}^j_{R_1,R_2}\,.$$

Since the sets $m + [0,1)^n$, $m \in Z^n$, are disjoint,

$$\begin{aligned}\#\{m \in \mathbb{Z}^n \setminus E : M^j\eta \in Q(r)\} \leq \#\mathcal{M}^j_{R_1,R_2} = & \Big|\bigcup_{m\in\mathcal{M}^j_{R_1,R_2}} (m + [0,1)^n)\Big| \\ \leq & \left|M^{j_1-j}(\widetilde{Q}(R_1,R_2))\right| = |\widetilde{Q}(R_1,R_2)|\,|\det M|^{j_1}\,|\det M|^{-j} \end{aligned} \tag{5.13}$$

The Lemma then follows from (5.12) and (5.13) by taking $\widetilde{C} = |\widetilde{Q}(R_1,R_2)|\,|\det M|^{j_1}$. □

We can now go back to our task of showing that $L(f) < \infty$. The situation here is different from the case we encountered in Section 3, where we showed that the integrability condition (2.6) follows from the fact that $g \in L^2(\mathbb{R}^n)$ (recall that, in the case of Gabor systems, the matrices C_p are independent of p). We will show in the following Proposition that if

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^\ell(B^{-j}\xi)|^2 \leq 1 \qquad \text{for a. e.} \qquad \xi \in \mathbb{R}^n, \tag{5.14}$$

then the integrability condition (2.6) is satisfied for the affine system $\mathcal{F}_A$ (where the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$). Observe that if $\mathcal{F}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then, by Proposition 4.1 applied to the affine system (5.2), we deduce inequality (5.14). This inequality also holds if we assume (5.8) (take $\alpha = 0$). Therefore, the following Proposition implies that if either $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame or if the case $\alpha = 0$ of (5.8) holds, then $L(f) < \infty$, where $L(f)$ is given by (5.9). This is all we need to finish the proof of Theorem 5.3.

Proposition 5.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. If (5.14) holds, then $L(f) < \infty$, where $L(f)$ is given by (5.9).*

Proof. Let $f \in \mathcal{D}_E$ and choose $r \in \mathbb{N}$ such that $\operatorname{supp} \hat{f} \subset Q(r)$. Then

$$\begin{aligned} L(f) &\leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{Q(r)} |\hat{f}(\xi + B^j m)|^2 \, |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi \\ &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{B^j\eta\in Q(r)} |\hat{f}(B^j(\eta + m)|^2 \, |\hat{\psi}^\ell(\eta)|^2 \, |\det B|^j \, d\eta \, . \end{aligned} \tag{5.15}$$

We write $L_0(f)$ for the sum of the terms in (5.15) for which $m \in E \cap Z^n$, and $L_1(f)$ for the sum of the terms in the same expression for which $m \in Z^n \setminus E$. Then, $L(f) = L_0(f) + L_1(f)$.

We first estimate $L_0(f)$. For $m \in E \cap Z^n$, if $\xi \in Q(r)$ and $\xi + B^j m \in Q(r)$, then, for $j \in \mathbb{Z}$, we have

$$|B^j m| \leq |\xi_E + B^j m| + |\xi_E| < r + r = 2r \, ,$$

where $\xi = \xi_F + \xi_E$, with $\xi_F \in F$ and $\xi_E \in E$. Thus, using the notation introduced in property (iv) of Definition 5.1, we have:

$$\{m \in E \cap Z^n : \xi \in Q(r) \text{ and } \xi + B^j m \in Q(r)\} \subset \mathcal{Z}^j_{2r}(E),$$

for every $j \in \mathbb{Z}$. By property (iv) of Definition 5.1, the number of elements in $\mathcal{Z}^j_{2r}(E)$ is less than $C = C(B, 2r)$ for all $j \in \mathbb{Z}$. Thus

$$L_0(f) \leq C(B, 2r) \|\hat{f}\|^2_\infty \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{Q(r)} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi.$$

Using (5.14), it follows that

$$L_0(f) \leq C(B, 2r) \|\hat{f}\|^2_\infty |Q(r)| < \infty. \tag{5.16}$$

We now estimate $L_1(f)$. For $m \in Z^n \setminus E$, if $B^j\eta \in Q(r)$ and $B^j(\eta + m) \in Q(r)$, then, for $j \in \mathbb{Z}$, we have that

$$|B^j m_F| \leq |B^j(\eta_F + m_F)| + |B^j \eta_F| < r + r = 2r,$$

and

$$|B^j m_E| \leq |B^j(\eta_E + m_E)| + |B^j \eta_E| < r + r = 2r,$$

where we decomposed m and η as a unique sum of elements in F and E. Thus, with the notation introduced before Lemma 5.5,

$$\{m \in Z^n \setminus E : B^j\eta \in Q(r) \text{ and } B^j(\eta + m) \in Q(r)\} \subset \{m \in Z^n \setminus E : B^j m \in \widetilde{Q}(2r)\},$$

for every $j \in \mathbb{Z}$. By Lemma 5.5, the number of elements in the last set does not exceed $\widetilde{C}(B, 2r)|\det B|^{-j}$, for all $j \in \mathbb{Z}$. Thus,

$$L_1(f) \leq \widetilde{C}(B, 2r)\|\hat{f}\|_\infty^2 \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{B^j\eta \in Q(r)} |\hat{\psi}^\ell(\eta)|^2 \, d\eta.$$

By Lemma 5.4, the number of $j \in \mathbb{Z}$ such that $B^j\eta \in Q(r)$ does not exceed a fixed number, $N(B, r)$, independently of $\eta \in \mathbb{R}^n$. Hence,

$$L_1(f) \leq \widetilde{C}(B, 2r)\|\hat{f}\|_\infty^2 \, N(B, r) \sum_{\ell=1}^{L} \|\hat{\psi}^\ell\|_2^2 < \infty. \tag{5.17}$$

From (5.15), (5.16), and (5.17) we deduce that, if $f \in \mathcal{D}_E$, then $L(f) < \infty$. □

Equality (5.8) in Theorem 5.3 can be written in a simpler form involving the lattice points $m \in \mathbb{Z}^n$ instead of the elements $\alpha \in \Lambda$. This shows that there is a redundancy in the original condition (5.8). We have the following:

Theorem 5.7. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{5.18}$$

and all $m \in \mathbb{Z}^n$, where $\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^n\}$.

Proof. It is enough to show that (5.8) is valid for each $\alpha \in \Lambda$ if and only if (5.18) is valid for each $m \in \mathbb{Z}^n$. Each lattice point $m \in \mathbb{Z}^n$ belongs to Λ since $\mathbb{Z}^n = B^0(\mathbb{Z}^n) \subset \Lambda = \cup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, and, therefore, (5.8) implies (5.18). Now, suppose that (5.18) is true for all $m \in \mathbb{Z}^n \setminus \{0\}$ (the case $m = 0$ in (5.18) is equal to the case $\alpha = 0$ in (5.8), and so we only have to consider the case $m \neq 0$). For any $\alpha \in \Lambda \setminus \{0\}$, we have $\alpha = B^{j_0}m_0$ for some $j_0 \in \mathbb{Z}$ and some $m_0 \in \mathbb{Z}^n \setminus \{0\}$. By making the change of variables $\xi = B^{j_0}\eta$ in the left hand side of (5.8), we obtain

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + \alpha))} = \sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_{B^{j_0}m_0}} \hat{\psi}^\ell(B^{-j+j_0}\eta)\, \overline{\hat{\psi}^\ell(B^{-j+j_0}(\eta + m_0))} \tag{5.19}$$

Let $k = j - j_0$ and observe that $\mathcal{P}_\alpha = \mathcal{P}_{B^{j_0}m_0} = \{j \in \mathbb{Z} : B^{-j}(B^{j_0}m) \in \mathbb{Z}^n\}$. Since $B^{-(k+j_0)}(B^{j_0}m_0) = B^{-k}m_0$, it follows that $j = k + j_0 \in \mathcal{P}_{B^{j_0}m_0}$ if and only if $k \in \mathcal{P}_k$.

Replacing $-j + j_0$ by $-k$ in the second sum of (5.19), we obtain

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = \sum_{\ell=1}^{L} \sum_{k \in \mathcal{P}_{m_0}} \hat{\psi}^\ell(B^{-k}\eta) \, \overline{\hat{\psi}^\ell(B^{-k}(\eta+m_0))}$$

and the last expression is zero for a.e. $\eta \in \mathbb{R}^n$ by (5.18) (recall that $m_0 \neq 0$). So the left hand side is also zero for a.e. $\xi \in \mathbb{R}^n$ when $\alpha \in \Lambda \setminus \{0\}$. □

Examples of **orthonormal A-wavelets** (that is, systems $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, such that $\mathcal{F}_A(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^n)$) for expanding matrices on $\mathbb{R}^n$ can be found in [13, 34, 14]. Here we show how Theorem 5.3 can be applied to obtain examples of **orthonormal A-wavelets** for some dilation matrices A for which $B = A^t$ satisfy Definition 5.1, but is not necessarily expanding on $\mathbb{R}^n$.

In order to construct these examples, observe that if $j \in \mathcal{P}_k$ (see Theorem 5.7 for the definition of the set $\mathcal{P}_k$ we use here), then $B^{-j}k = m \in \mathbb{Z}^n$, so that if $\psi^\ell \in L^2(\mathbb{R}^n)$ and $(\text{supp}\, \hat{\psi}^\ell) \cap (\text{supp}\, \hat{\psi}^\ell(\cdot - m)) = \emptyset$ (a.e) for all $m \in \mathbb{Z}^n \setminus \{0\}$, $\ell = 1, \cdots, L$, then all the equations in (5.18) with $k \neq 0$ are trivially true. Since $\mathcal{P}_0 = \mathbb{Z}$, we have the following:

Corollary 5.8. *Assume the same set up as in Theorem 5.7, and suppose that* $(\text{supp}\, \hat{\psi}^\ell) \cap (\text{supp}\, \hat{\psi}^\ell(\cdot - m)) = \emptyset$ *(a.e) for all* $m \in \mathbb{Z}^n \setminus \{0\}$, $\ell = 1, \cdots, L$. *If*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e} \quad \xi \in \mathbb{R}^n, \tag{5.20}$$

then the system $\mathcal{F}_A(\Psi)$ *is a normalized tight frame for* $L^2(\mathbb{R}^n)$. *If, in addition,* $\|\psi^\ell\|_2 = 1$ *for all* $\ell = 1, \cdots, L$, *then* $\Psi = \{\psi^1, \cdots, \psi^L\}$ *is an orthonormal A-wavelet for* $L^2(\mathbb{R}^n)$.

Example 6. For $a \in \mathbb{R}, a > 1$, let

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

as in Example 2. We construct a single function $\psi \in L^2(\mathbb{R}^2)$, with $\|\psi\|_2 = 1$, such that ψ is an orthonormal A-wavelet. The vertical strips

$$V = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2a} \leq |x| < \frac{1}{2}\}$$

are tiled by the sets

$$E_n = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2a} \leq |x| < \frac{1}{2}, \, \frac{n}{2} \leq |y| < \frac{n+1}{2}\} \quad n = 0, 1, 2, \ldots$$

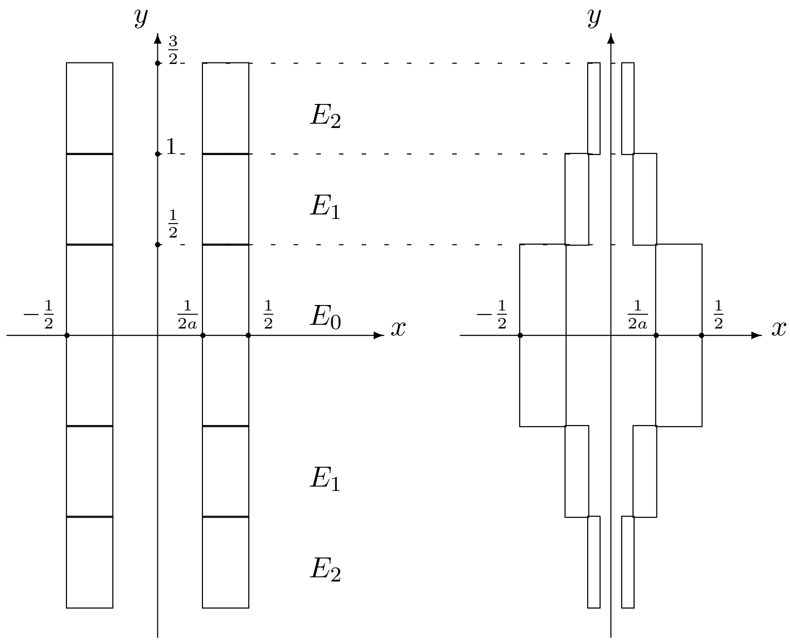

Figure 2: Example 6.

(see Figure 2). Define

$$S_n = A^{-n} E_n, \quad n = 0, 1, 2, \ldots \qquad \text{and} \qquad W = \bigcup_{n=0}^{\infty} S_n.$$

Observe that W is a disjoint union of the sets S_n . Thus, we have

$$|W| = \sum_{n=0}^{\infty} 4\frac{1}{2}(\frac{1}{2a^n} - \frac{1}{2a^{n+1}}) = (1 - \frac{1}{a}) \sum_{n=0}^{\infty} \frac{1}{a^n} = 1\,.$$

Define $\psi \in L^2(\mathbb{R}^2)$ by $\hat{\psi} = \chi_W$. The above computation shows that $\|\psi\|_2 = 1$. Since $\bigcup_{n=0}^{\infty} A^n S_n = \bigcup_{n=0}^{\infty} E_n = V$, and $\{A^j V : j \in \mathbb{Z}\}$ is a tiling of $\mathbb{R}^2$ by the vertical strips $A^j V$, (5.20) follows. Finally, observe that horizontal and vertical translations of W by non zero integers do not overlap. Hence, ψ is an orthonormal A-wavelet. (An example similar to this one has been exhibited in [5] for the case $a = 2$.)

Remarks

(**1**) Applying Theorem 4.2 to the affine system $\mathcal{F}_A(\Psi)$, it follows that $\mathcal{F}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (5.20) holds. In particular, if $\mathcal{F}_A(\Psi)$ is an orthonormal system, then $\mathcal{F}_A(\Psi)$ is complete if and only if (5.20) holds. See Remark (1) following Theorem 4.2 for appropriate references to this result.

(**2**) The ideas presented in this section apply to more general affine systems than (5.2). For $\{\psi^1, \cdots, \psi^L\} \in L^2(\mathbb{R}^n)$, $A_1, \ldots, A_L \in GL_n(\mathbb{R})$ and $N_1, \ldots N_L \in GL_n(\mathbb{R})$, consider the affine systems

$$\{D^j_{A_\ell}\, T_{N_\ell k}\, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \ldots, L\}\,. \tag{5.21}$$

Since $D^j_{A_\ell}\, T_{N_\ell k}\, \psi^\ell(x) = T_{A_\ell^{-j} N_\ell k}\, D^j_{A_\ell}\, \psi^\ell(x)$, the system (5.21) can be described as a collection of the form (2.1) for appropriate choices of $\mathcal{P}$, g_p and C_p. Then Theorems 2.1 and 4.2 can be applied to characterize normalized tight frames for the affine systems given by (5.21). Since the study of this case is very similar to Theorem 5.3, the details will be omitted. The results one obtains in the case of expansive dilation matrices $A_1, \ldots, A_L \in GL_n(\mathbb{R})$ can be found in [24].

6 Affine systems and wavelets: special dilation matrices

In this section, we are going to analyze the forms that the characterization equations (5.18) assume for different values of $m \in \mathbb{Z}^n$, depending on the dilation matrices A: for example, a corollary of our work in this section is that, for affine systems in one dimension with $A = 2$, the equations (5.18) in Theorem 5.7 are the equations (1.6) and (1.7) in the classical Theorem 1.1.

Observe that the major difference between these two equations is that, in the first, we encounter the sum over all $j \in \mathbb{Z}$, while, in the second, the sum is over all $j \geq 0$. In terms of the notation used for the general case in (5.18), (1.6) and (1.7) represent the two types of equations obtained when $m = 0$ (the Calderón condition we already discussed) and the case when $m \neq 0$.

We will present different classes of dilation matrices where there are, in fact, three or more types of equalities. We always have the case $m = 0$, which, in terms of the notation in Theorem 5.7), gives $\mathcal{P}_0 = \mathbb{Z}$ and represents the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n\,. \tag{6.1}$$

As we shall see, the case $m \neq 0$ can assume several different forms. In the simple example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, there are two different types of equalities besides the case corresponding to $m = 0$. If $m = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$, $m_2 \neq 0$, we have $\mathcal{P}_m = \mathbb{Z}$ since $A^{-j}k = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$ for all $j \in \mathbb{Z}$. On the other hand, if m is not of the above form, then one obtains an equation similar to equation (1.7) (see Example 8 for details).

To better understand how these cases arise, let us first consider the intersection $I(B) = \bigcap_{i\in\mathbb{Z}} B^i(\mathbb{Z}^n)$, where $B = A^t \in GL_n(\mathbb{R})$, and B is expanding on a subspace F of $\mathbb{R}^n$. If B is expanding on $\mathbb{R}^n$, then $I(B) = \{0\}$. In general, $I(B) \subset \mathbb{Z}^n$. When $I(B)$ is not empty and $m \in I(B)$, $m \neq 0$, then we have $\mathcal{P}_m = \mathbb{Z}$, and (5.18) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = 0 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n\,. \tag{6.2}$$

Observe that (6.2) is void for dilation matrices expanding on $\mathbb{R}^n$. For the matrices of Example 2, we have $I(B) = \{0\}\times\mathbb{Z}$. For the matrices of Example 3, we have $I(B) = E\cap\mathbb{Z}^2$. For the matrices of Example 4, the set $I(B)$ depends on the angle of rotation θ. For the matrices of Example 5, we have $I(B) = \{0\} \times \mathbb{Z} \times \mathbb{Z}$ when b is an integer.

We describe further how equation(5.18) assumes different forms by selecting three types of the dilation matrix $B = A^t \in GL_n(\mathbb{R})$.

6.1 Matrices of Type-I

Definition 6.1. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-I** if

$$M^j(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(M) = \bigcap_{i\in\mathbb{Z}} M^i(\mathbb{Z}^n) \tag{6.3}$$

for all $j \in \mathbb{Z} \setminus \{0\}$.

Examples of matrices of Type-I are the matrices $M = aI_n$ with $a \in \mathbb{R}$ such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$; in this case, $I(M) = \{0\}$. More generally, any diagonal matrix whose diagonal entries a_{ii} are such that $a_{ii}^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z} \setminus \{0\}$ is a matrix of Type-I. The matrices of Example 2 are also of this type when $a \in \mathbb{R}$ is such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z}\setminus\{0\}$; in this case $I(M) = \{0\} \times \mathbb{Z}$.

We now apply Theorem 5.7 to characterize the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), where $B = A^t$ is a matrix of Type-I that is expanding on a subspace of $\mathbb{R}^n$.

Proposition 6.1. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-I which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2) and*

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\,\overline{\hat{\psi}^\ell(\xi+m)} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{6.4}$$

and all $m \in \mathbb{Z}^n \setminus I(B)$.

Proof. We have already observed that (6.1) and (6.2) correspond to the cases $m = 0$ and $m \in I(B) \setminus \{0\}$ of (5.18). Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I(B)$. It is clear that the set $\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^n\}$ contains the element $j = 0$. We now show that it does not contain any other element. If there exist $j \in \mathbb{Z}$, with $j \neq 0$, such that $j \in \mathcal{P}_m$, then we must have $B^{-j}m \in \mathbb{Z}^n$. Since $-j \neq 0$ we deduce from (6.3) that $m \in I(B)$, contrary to the properties of m. Hence, $\mathcal{P}_m = \{0\}$ and (5.18) gives

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\overline{\hat{\psi}^\ell(\xi + m)} = 0 \qquad \text{for a. e.} \qquad \xi \in \mathbb{R}^n,$$

which is what we wanted to prove. $\square$

Example 7. For the matrix $A = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.1 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j\in\mathbb{Z}} \mid \hat{\psi}(\pi^j\xi_1,\ \xi_2) \mid^2 = 1 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R},$$
$$\sum_{j\in\mathbb{Z}} \hat{\psi}(\pi^j\xi_1,\ \xi_2)\,\overline{\hat{\psi}(\pi^j\xi_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_2 \in \mathbb{Z} \setminus \{0\},$$
$$\hat{\psi}(\xi_1,\ \xi_2)\,\overline{\hat{\psi}(\xi_1 + m_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_1 \in \mathbb{Z} \setminus \{0\}, m_2 \in \mathbb{Z}.$$

6.2 Matrices of Type-II

Definition 6.2. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-II** if $M(\mathbb{Z}^n) \subset \mathbb{Z}^n$ (equivalently, all the entries of M are integers).

Before we describe the equations that characterize affine systems for dilation matrices of Type-II that are expanding on a subspace of $\mathbb{R}^n$, we make the following observation.

Lemma 6.2. *Let $M \in GL_n(\mathbb{Z})$. If $m \in \mathbb{Z}^n \setminus I(M)$, there exist unique $d \in \mathbb{Z}^+ \cup \{0\}$ and $r \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$ such that $m = M^d r$.*

Proof. If $m \notin M(\mathbb{Z}^n)$, then write $m = M^0 m$ and the result follows by taking $d = 0$, $r = m$. If $m \in M(\mathbb{Z}^n)$, write $m = Mm_1$ with $m_1 \in \mathbb{Z}^n$; while, if $m_1 \notin M(\mathbb{Z}^n)$, the result follows by taking $d = 1$, $r = m_1$. If $m_1 \in M(\mathbb{Z}^n)$, write $m_1 = Mm_2$ with $m_2 \in \mathbb{Z}^n$. Thus, $m = M^2 m_2$. If $m_2 \notin M(\mathbb{Z}^n)$, the result follows by taking $d = 2$, $r = m_2$. Continue in this way. This process stops. Otherwise, $m = M^j m_j$ for all $j \in \mathbb{Z}^+$, with $m_j \in \mathbb{Z}^n$. Since,

$$\cdots \subset M^2(\mathbb{Z}^n) \subset M(\mathbb{Z}^n) \subset \mathbb{Z}^n \subset M^{-1}(\mathbb{Z}^n) \subset M^{-2}(\mathbb{Z}^n) \subset \cdots, \qquad (6.5)$$

we deduce $m \in I(M)$, contrary to our assumption.
To show uniqueness, suppose that $M^d r = m = M^{d_1} m_1$ with $d_1 \geq d$ and $r, r_1 \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$. Then, $r = M^{d_1 - d} m_1$. Since $r \notin M(\mathbb{Z}^n)$, we deduce from (6.5) that $r \notin M^{d_1-d}(\mathbb{Z}^n)$ if $d_1 > d$. Hence, $d_1 = d$ and $r = r_1$. □

Proposition 6.3. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and let $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-II which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2) and*

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell(B^j \xi)\, \overline{\hat{\psi}^\ell(B^j(\xi + r))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.6}$$

and all $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ (observe that $r \notin I(B)$).

Proof. We have already observed that (6.1) and (6.2) are the cases $m = 0$ and $m \in I(B) \setminus \{0\}$ of (5.18). Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I(B)$.

We want to examine $\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j} m \in \mathbb{Z}^n\}$. If $j \in \mathcal{P}_m$, we have $B^{-j} m = s \in \mathbb{Z}^n$. By Lemma 6.2, there exist unique $d \in \mathbb{Z}^+ \cup \{0\}$ and $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ such that $m = B^d r$. Hence, $s = B^{-j+d} r$. We must have $-j + d \geq 0$ (otherwise, with $-j + d = -\ell < 0$, we deduce $s = B^{-\ell} r$, and $r = B^\ell s \in B(\mathbb{Z}^n)$).

Thus, for $m = B^d\, r \in \mathbb{Z}^n \setminus I(B)$, (5.18) of Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j \leq d} \hat{\psi}^\ell(B^{-j} \xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d\, r))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{6.7}$$

with $r \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. Replacing ξ by $B^d\, \eta$ in the above expression and then changing the index of summation to $k = d - j$ we obtain (6.6). □

Example 8. For the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.3 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi_1,\ \xi_2)|^2 = 1 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R},$$

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(2^j \xi_1,\ \xi_2)\, \overline{\hat{\psi}(2^j \xi_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_2 \in \mathbb{Z} \setminus \{0\},$$

$$\sum_{j \geq 0} \hat{\psi}(2^j \xi_1,\ \xi_2)\, \overline{\hat{\psi}(2^j(\xi_1 + q_1),\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } q_1 \in \mathbb{Z} \setminus 2\mathbb{Z}, m_2 \in \mathbb{Z}.$$

6.3 Matrices of Type-III

Definition 6.3. A matrix $M \in GL_n(\mathbb{R})$ is of **Type-III** if there exists $\delta \in \mathbb{N}$, $\delta > 1$, such that:

$$\text{(i) } M^\delta(\mathbb{Z}^n) \subset \mathbb{Z}^n,$$

and

$$\text{(ii) } M^r(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(M) = \bigcap_{i \in \mathbb{Z}} B^i(M) \quad \text{for all} \quad 0 < r < \delta,\ r \in \mathbb{Z}.$$

The following are examples of matrices of Type-III:

$$M_1 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Observe that the lower-right 2×2 matrix of M_3 is a rotation by $\pi/4$ radians. Since M_1^2, M_2^2, and M_3^2 are matrices with integer entries, the matrices M_1, M_2 and M_3 satisfy (i) of Definition 6.3. To verify condition (ii), notice that

$$I(M_1) = \{0\}, \qquad I(M_2) = \{0\} \times \mathbb{Z} \times \mathbb{Z} \qquad \text{and} \qquad I(M_3) = \{0\}\,.$$

Obvious substitutions of $\sqrt{2}$ by other roots and of the rotation by $\pi/4$ by other rotations give many more examples of matrices of this type.

Now we want to write down the form that equation (5.18) assumes for the affine system $\mathcal{F}_A(\Psi)$ when the dilation matrix is of Type-III. Besides (6.1) and (6.2), which correspond to the cases $m = 0$ and $m \in I(B) \setminus \{0\}$, we are led to consider the case $m \in I(B^\delta) \setminus I(B)$ (it is easy to see that this set is non empty for the matrix M_3). The details can be seen in the proof of Proposition 6.5 below. Before we present this proposition, we state the following lemma, which shows that, for matrices of Type-III, condition (ii) is true for any integer that is not divisible by δ.

Lemma 6.4. *Let M be a matrix of Type-III. If $\mathbb{Z}^n \cap M^s(\mathbb{Z}^n) \supsetneq I(M)$, then δ divides s.*

Proof. If $s = 0$ the result is obviously true. If $s < 0$, from $\mathbb{Z}^n \cap M^s(\mathbb{Z}^n) \supsetneq I(M)$ we deduce $M^{-s}\mathbb{Z}^n \cap \mathbb{Z}^n \supsetneq M^{-s}(I(M)) = I(M)$. Hence, without loss of generality we can assume $s > 0$. Write $s = c\delta + r,\ 0 \le r < \delta,\ r \in \mathbb{Z}$, with c a non-negative integer. Choose $m \in \mathbb{Z}^n \cap M^s(\mathbb{Z}^n)$ and $m \notin I(M)$. Using (i) of Definition 6.3 we obtain

$$m \in M^s(\mathbb{Z}^n) = M^r M^{c\delta}(\mathbb{Z}^n) \subset M^r(\mathbb{Z}^n)\,.$$

Hence $m \in \mathbb{Z}^n \cap M^r(\mathbb{Z}^n)$. By (ii) of Definition 6.3, $m \in I(M)$ if $0 < r < \delta$, contrary to our assumption. We deduce that r must be zero, showing that s has to be a multiple of δ. □

Proposition 6.5. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-III which is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: (6.1), (6.2),*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}^\ell(B^{-j\delta}\xi)\,\overline{\hat{\psi}^\ell(B^{-j\delta}(\xi+m))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.8}$$

and all $m \in I(B^\delta) \setminus I(B)$, and

$$\sum_{\ell=1}^{L} \sum_{j\geq 0} \hat{\psi}^\ell(B^{j\delta}\xi)\,\overline{\hat{\psi}^\ell(B^{j\delta}(\xi+q))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{6.9}$$

and all $q \in \mathbb{Z}^n \setminus B^\delta(\mathbb{Z}^n)$ (observe that $q \notin I(B^\delta)$).

Proof. The set $\mathbb{Z}^n$ is the disjoint union of the sets

$$\{0\}, \quad I(B)\setminus\{0\}, \quad I(B^\delta)\setminus I(B), \quad \text{and} \quad \mathbb{Z}^n \setminus I(B^\delta).$$

For $m \in I(B)$, condition (5.18) is equivalent to (6.1) if $m = 0$, and to (6.2) if $m \neq 0$.

Consider now

$$m \in I(B^\delta) \setminus I(B). \tag{6.10}$$

We claim that, for m as above, we have:

$$\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^n\} = \{i\delta : i \in \mathbb{Z}\}. \tag{6.11}$$

Since, for all $i \in \mathbb{Z}, B^{-i\delta}m \in B^{-i\delta}(I(B^\delta)) = I(B^\delta)$, and $I(B^\delta) \subset \mathbb{Z}^n$, it is clear that the set in the right hand side of (6.11) is contained in $\mathcal{P}_\alpha$m Suppose now that $j \in \mathcal{P}_m$ and $j \neq i\delta$ for each $i \in \mathbb{Z}^n$. We can then write $j = c\delta - s$ with $0 < s < \delta$. Therefore,

$$B^{-j}m = B^s B^{-c\delta}m \in B^s(B^{-c\delta}(I(B^\delta)) = B^s(I(B^\delta)) \subset B^s(\mathbb{Z}^n).$$

Also, $B^{-j}m \in \mathbb{Z}^n$ since $j \in \mathcal{P}_m$. Thus, $B^{-j}m \in \mathbb{Z}^n \cap B^s(\mathbb{Z}^n)$. Since B is of Type-III, by (ii) of Definition 6.3, $B^{-j}m \in I(B)$, and, consequently, $m \in B^j(I(B)) = I(B)$, contradicting the choice of m. This establishes (6.11).

For m as in (6.10), the equality (6.11) shows that (5.18) in Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L} \sum_{i\in\mathbb{Z}} \hat{\psi}^\ell(B^{-i\delta}\xi)\,\overline{\hat{\psi}^\ell(B^{-i\delta}(\xi+m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

which is (6.8).

Choose, finally,

$$m \in \mathbb{Z}^n \setminus I(B^\delta). \tag{6.12}$$

By Lemma 6.2 applied to $M = B^\delta$, we deduce the existence of unique $d \in Z^+ \cup \{0\}$ and $q \in \mathbb{Z}^n \setminus B^\delta(\mathbb{Z}^n)$ such that $m = B^{d\delta}q$. We claim that, for m as in (6.12), we have

$$\mathcal{P}_m = \{j \in \mathbb{Z} : B^{-j}\alpha \in \mathbb{Z}^n\} = \{k\delta : k \in \mathbb{Z},\, k \le d\}. \tag{6.13}$$

If $j = k\delta$ with $k \in \mathbb{Z}$, $k \le d$, then we can use (i) of Definition 6.3 to obtain $B^{-j}m = B^{-k\delta}B^{d\delta}q = B^{(d-k)\delta}q \in \mathbb{Z}^n$, since $d - k \ge 0$ and $q \in \mathbb{Z}^n$. This shows that the set on the right side of (6.13) is contained in $\mathcal{P}_m$. Choose now $j \in \mathcal{P}_m$ so that $B^{-j}m = s \in Z^n$. Then, $s = B^{-j}m = B^{-j+d\delta}q$. Hence, $q = B^{j-d\delta}s \in \mathbb{Z}^n \cap B^{j-d\delta}(\mathbb{Z}^n)$. Also, $q \notin B^\delta(\mathbb{Z}^n)$, which implies $q \notin I(B)$. By Lemma 6.4 applied to $M = B$, we deduce $j = k\delta$ for some $k \in \mathbb{Z}$. Then, $q = B^{(k-d)\delta}s$ and $k - d \le 0$ (otherwise, if $k - d = t > 0$, then $q = B^{\delta t}s \in B^{\delta t}(\mathbb{Z}^n) \subset B^\delta(\mathbb{Z}^n)$ by (i) of Definition 6.3). This establishes (6.13).

For m as in (6.12), the equality (6.13) shows that (5.18) in Theorem 5.7 is equivalent to

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z},\, k\le d} \hat{\psi}^\ell(B^{-k\delta}\xi)\, \overline{\hat{\psi}^\ell(B^{-k\delta}(\xi + B^{d\delta}q))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{6.14}$$

The change of variables $\xi = B^{d\delta}\eta$ shows that (6.14) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z},\, k\le d} \hat{\psi}^\ell(B^{(d-k)\delta}\eta)\, \overline{\hat{\psi}^\ell(B^{(d-k)\delta}(\eta + q))} = 0 \quad \text{for a.e. } \eta \in \mathbb{R}^n. \tag{6.15}$$

Finally, the change of indices $j = d - k$ in the summation shows that (6.15) is equivalent to (5.18) in Theorem 5.7 for the values of m given by (6.12). This finishes the proof of the Proposition. □

Example 9. For the matrix $A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$ and a single $\psi \in L^2(\mathbb{R}^2)$, it follows from Proposition 6.5 that the affine system $\mathcal{F}_A(\psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^2)$ if and only if

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^{j/2}\xi_1,\ \xi_2)|^2 = 1 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R},$$

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^{j/2}\xi_1,\ \xi_2)\, \overline{\hat{\psi}(2^{j/2}\xi_1,\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } m_2 \in \mathbb{Z} \setminus \{0\},$$

$$\sum_{j\ge 0} \hat{\psi}(2^j\xi_1,\ \xi_2)\, \overline{\hat{\psi}(2^j(\xi_1 + q_1),\ \xi_2 + m_2)} = 0 \quad \text{for a.e. } \xi_1, \xi_2 \in \mathbb{R}, \text{ and all } q_1 \in \mathbb{Z} \setminus 2\mathbb{Z}, m_2 \in \mathbb{Z}.$$

6.4 Matrices expanding on $\mathbb{R}^n$

If the dilation matrix $A \in GL_n(\mathbb{R})$ is expanding (i.e., it is expanding on $F = \mathbb{R}^n$), then the form of the equalities (5.18) can be expressed in a way that is yet more similar to the "classical" equalities (1.6) and (1.7). That is, one equality, corresponding to the case $q = 0$ in (6.16), is the Calderón condition, while the others have a form that is a direct generalization of (1.7).

Theorem 6.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $B = A^t$ and $A \in GL_n(\mathbb{R})$ be expanding on $\mathbb{R}^n$. Then the system $\mathcal{F}_A(\Psi)$, given by (5.2), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_q} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + q))} = \delta_{q,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{6.16}$$

and all $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$, where $\mathcal{P}_q = \{j \in \mathbb{Z} :\, B^{-j}q \in \mathbb{Z}^n\}$.

Proof. We must show the equivalence of (5.18) and (6.16) when B is expanding. Let us first observe that, in this case,

$$I^+(B) \;=\; \bigcap_{j \geq 0} B^j(\mathbb{Z}^n) \;=\; \{0\}.$$

This is an immediate consequence of inequality (5.6) in Lemma 5.2. Indeed, if $x \in I^+(B)$, then, for each $j \geq 0$, there exists $m_j \in \mathbb{Z}^n$ such that $x = B^j\, m_j$. By (5.6), we then have $|m_j| = |B^{-j}\, x| \leq \frac{1}{k}\, \gamma^{-j}\, |x|$ and the last expression tends to zero as $j \to \infty$ since $\gamma > 1$. Hence, m_j must be zero since the last expression must be strictly smaller than 1, the minimal norm for a non-zero lattice point, for j large enough.

It is clear that (5.18) implies (6.16), and that the two expressions are the same when $q = m = 0$. Therefore, we only have to show that equality (5.18), for $m \in \mathbb{Z}^n \setminus \{0\}$, is equivalent to one of the equalities (6.16), for an appropriate $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. We first claim that any such $m \in \mathbb{Z}^n \setminus \{0\}$ can be written as $m = B^d\, q$ for some $d \geq 0$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$, provided B is expanding. To prove this claim proceed as follows. If $m \notin B(\mathbb{Z}^n)$, then set $d = 0$ and $q = m$; while, if $m \in B(\mathbb{Z}^n)$, then set $m = B\, m_1$ and we reason for m_1 as we just did for m: either $m_1 \notin B(\mathbb{Z}^n)$, and we are done with $q = m_1$, or $m_1 \in B(\mathbb{Z}^n)$ in which case $m = B\, m_1 = B^2\, m_2$ with $m_2 \in \mathbb{Z}^n$. This process must stop after a finite number of steps; otherwise, $m = B^j\, m_j$ for a $m_j \in \mathbb{Z}^n$ for all $j \geq 0$. This would imply that $m \in I^+(B) = \cap_{j \geq 0} B^j(\mathbb{Z}^n)$ and we reach a contradiction. This establishes the last claim.

Thus, if (5.18) with $m \neq 0$ is true, then we can write $m = B^d\, q$ for some $d \geq 0$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$. Now, using the change of variables $\xi = B^d\eta$ and the fact that $j \in \mathcal{P}_{B^d q}$ if and only if $k = j - d \in \mathcal{P}_q$, we obtain

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_{B^d q}} \hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+B^d\, q))}$$

$$= \sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_{B^d q}} \hat{\psi}^\ell(B^{-j+d}\eta)\,\overline{\hat{\psi}^\ell(B^{-j+d}(\eta+q))} = \sum_{\ell=1}^{L} \sum_{k\in\mathcal{P}_q} \hat{\psi}^\ell(B^{-k}\eta)\,\overline{\hat{\psi}^\ell(B^{-k}(\eta+q))}. \quad \square$$

Remark. The types of matrices we have considered in this section do not cover all the possible matrices that are expanding on subspaces of $\mathbb{R}^n$. For example, the matrix

$$M = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

does not belong to any of the above types. It is a significant problem to understand the form that (5.8) in Theorem 5.3 assumes for all the dilation matrices expanding on subspaces of $\mathbb{R}^n$, in order to obtain expressions that do not involve the sets Λ and $\mathcal{P}_\alpha$, in the same spirit as done in Propositions 6.1, 6.3, and 6.5. The problem has been completely solved in [8] for dimension 1 (the matrix is a real number a, with $a > 1$), where they have considered the sets

$$E_1 = \{a \in \mathbb{R} : a > 1, \text{ and } a^j \in \mathbb{Z} \text{ for some integer } j > 0\},$$
$$E_2 = \{a \in \mathbb{R} : a > 1, \text{ and } a^j \in \mathbb{Q} \setminus \mathbb{Z} \text{ for some integer } j > 0\},$$
$$E_3 = \{a \in \mathbb{R} : a > 1, \text{ and } a^j \notin \mathbb{Q} \text{ for all integer } j > 0\},$$

and they have given simpler expressions for the characterization equations in each one of these cases. The general problem in dimension $n > 1$ remains open.

7 Quasi-affine systems

Let $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and $A \in GL_n(\mathbb{R})$. The **quasi-affine system** generated by Ψ, denoted as $\widetilde{\mathcal{F}}_A(\Psi)$, is defined by

$$\widetilde{\mathcal{F}}_A(\Psi) = \big\{\widetilde{\psi}^\ell_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}, \tag{7.1}$$

where

$$\widetilde{\psi}^\ell_{j,k} = \begin{cases} |\det A|^{\frac{j}{2}}\, T_k\, D_{A^j}\, \psi^\ell, & j < 0 \\ D_{A^j}\, T_k\, \psi^\ell, & j \geq 0. \end{cases}$$

The notion of quasi-affine system was introduced by A. Ron and Z. Shen in [28] under the assumption that $A \in GL_n(\mathbb{Z})$. It is easy to verify that when the dilation matrix A preserves the integer lattice (i.e., $A\,\mathbb{Z}^n \subset \mathbb{Z}^n$), then the quasi-affine systems $\widetilde{\mathcal{F}}_A$, unlike the affine systems $\mathcal{F}_A$, are invariant under integer translations. Ron and Shen discovered that there is some sort of equivalence between the affine systems $\mathcal{F}_A(\Psi)$ and the corresponding quasi-affine systems $\widetilde{\mathcal{F}}_A(\Psi)$. In particular, they obtained the following result (discovered in [28] under a mild decay assumption on ψ, and proved in full generality in [9]):

Theorem 7.1 ([9]). *Let $A \in GL_n(\mathbb{Z})$ be expanding. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ is a normalized tight frame if and only the corresponding affine system $\mathcal{F}_A(\Psi)$ is a normalized tight frame.*

More general notions of equivalence are also proved in [9], such as the fact that affine and quasi-affine frames are equivalent. It follows from Theorem 7.1 that, once the quasi-affine systems $\widetilde{\mathcal{F}}_A(\Psi)$ have been studied using techniques from the theory of shift-invariant spaces, then the results can be transferred to the corresponding affine systems $\mathcal{F}_A(\Psi)$ (cf. [29, 3, 21] for an application of this approach to the characterization of affine tight and dual frames).

The main result of this section is yet another application of Theorem 2.1, which gives the following characterization of normalized quasi-affine tight frames.

Theorem 7.2. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and let $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}^- \cup \mathcal{Q}_m} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{7.2}$$

for all $m \in \mathbb{Z}^n$, and

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{Q}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{7.3}$$

for all $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, where $\Lambda^q = \bigcup_{j \in \mathbb{Z}^+ \cup \{0\}} B^j(\mathbb{Z}^n)$ and $\mathcal{Q}_x = \{j \in \mathbb{Z}^+ \cup \{0\} : B^{-j}x \in \mathbb{Z}^n\}$.

Remarks.

1. Our result is not restricted to integer-valued expanding matrices, as is the classical result of A. Ron and Z. Shen, but is valid for real matrices expanding on subspaces, as defined in Section 5. As a corollary to our result, we will show the equivalence of affine and quasi-affine systems may not hold if the matrix is not integer-valued (see Example 11).

2. The expressions (7.2) and (7.3) share some features with equation (5.7) in Theorem 5.4. Observe that, if $A \in GL_n(\mathbb{Z})$ (hence, $A\mathbb{Z}^n \subset \mathbb{Z}^n$), then $\Lambda^q = \mathbb{Z}^n$ and equation (7.3) is void. We will show in Proposition 8.2 that, in the case of integer-valued matrices, the expressions (7.2) and (5.7) are equivalent, and this gives a new proof of Theorem 7.1.

3. If $\alpha = 0$, then $\mathbb{Z}^- \cup \mathcal{Q}_0 = \mathbb{Z}$, and (7.2) becomes the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

exactly as in the case of affine systems.

Proof of Theorem 7.2. Apply Theorem 2.1 with

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1, 2, \dots, L\big\},$$

$$g_p \equiv g_{(j,l)} = \begin{cases} |\det A|^{\frac{j}{2}}\, D_{A^j}\,\psi^\ell, & j\, <\, 0 \\ D_{A^j}\,\psi^\ell, & j\, \ge\, 0, \end{cases} \qquad C_p \equiv C_{(j,\ell)} = \begin{cases} I, & j\, <\, 0,\, \ell = 1,\dots,L \\ A^{-j}, & j\, \ge\, 0,\, \ell = 1,\dots,L. \end{cases}$$

With this choice for $\mathcal{P}$, g_p and C_p, using the relation $T_{A^{-j}k}\, D_{A^j}\,\psi = D_{A^j}\, T_k\,\psi$, it follows that the system $\{T_{C_p\,k}\, g_p :\, k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$.
With the same choices, the set Λ, given by (2.2), is

$$\Lambda \,=\, \Lambda^q \,=\, (\bigcup_{j\in\mathbb{Z}^-} I^j\,\mathbb{Z}^n) \cup (\bigcup_{j\in\mathbb{Z}^+\cup\{0\}} B^j\,\mathbb{Z}^n) \,=\, \bigcup_{j\in\mathbb{Z}^+\cup\{0\}} B^j\,\mathbb{Z}^n,$$

and the set $\mathcal{P}_\alpha$, given by (2.3), is

$$\begin{aligned}\mathcal{P}_\alpha =& \{(j,\ell) : j \in \mathbb{Z}^-,\, \ell = 1,\dots,L :\, \alpha \in \mathbb{Z}^n\} \cup \\ & \cup \{(j,\ell) : j \in \mathbb{Z}^+ \cup \{0\},\, \ell = 1,\dots,L :\, B^{-j}\alpha \in \mathbb{Z}^n\}.\end{aligned}$$

Thus, if $\alpha = m \in \mathbb{Z}^n$, then $\mathcal{P}_\alpha = (\mathbb{Z}^- \cup \mathcal{Q}_m) \times \{1,\dots,L\}$, and equation (7.2) follows from (2.6) in Theorem 2.1. Similarly, if $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, then $\mathcal{P}_\alpha = \mathcal{Q}_\alpha \times \{1,\dots,L\}$, and equation (7.3) follows from (2.6) in Theorem 2.1.

Therefore, all that is left to prove is that the hypothesis (2.6) is satisfied in this particular case. Choose $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is a dense subspace of $L^2(\mathbb{R}^n)$ defined by (5.10), and E is a complementary subspace to F as in Definition 5.1. Thus, we need to show that $L^q(f) < \infty$ for $f \in \mathcal{D}_E$, where

$$L^q(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}^-} \sum_{m\in\mathbb{Z}^n} \int_{\text{supp}\,\hat{f}} |\hat{f}(\xi+m)|^2\, |\det A^j|\, |(D_A^j\psi^\ell)^\wedge(\xi)|^2\, d\xi +$$

$$+\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^+\cup\{0\}}\sum_{m\in\mathbb{Z}^n}\int_{\operatorname{supp}\hat f}|\hat f(\xi+B^j m)|^2\,|\det A^j|\,|(D_A^j\psi^\ell)^\wedge(\xi)|^2\,d\xi.$$

Since $(D_A^j\psi)^\wedge(\xi)=|\det A|^{-1/2}\,\hat\psi(B^{-j}\xi)$, then

$$L^q(f)=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^-}\sum_{m\in\mathbb{Z}^n}\int_{\operatorname{supp}\hat f}|\hat f(\xi+m)|^2\,|\hat\psi^\ell(B^{-j}\xi)|^2\,d\xi+$$
$$+\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^+\cup\{0\}}\sum_{m\in\mathbb{Z}^n}\int_{\operatorname{supp}\hat f}|\hat f(\xi+B^j m)|^2\,|\hat\psi^\ell(B^{-j}\xi)|^2\,d\xi.$$

Write $L^q(f)=L^q_-(f)+L^q_+(f)$, where $L^q_-(f)$ and $L^q_+(f)$ denote the sums corresponding to $j\in\mathbb{Z}^-$ and $j\in\mathbb{Z}^+\cup\{0\}$, respectively.

Consider first the expression for $L^q_-(f)$. Since $f\in\mathcal{D}_E$, there exists an $R>0$ such that $\operatorname{supp}\hat f\subset B(R)$. In order to have $L^q(f)\neq 0$, we must have $|\xi|\le R$ and $|\xi+m|\le R$. Therefore, $|m|\le 2\,R$, and the sum with respect to m in $L^q_-(f)$ is finite, where the number of $m\in\mathbb{Z}^n$ is at most $(2\,R)^n$. Furthermore, if the quasi-affine system $\widetilde{\mathcal{F}_A}(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then, by Proposition 4.1 applied to the quasi-affine system (7.1), we deduce that

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat\psi^\ell(B^{-j}\xi)|^2\le 1\qquad\text{for a.e. }\ \xi\in\mathbb{R}^n\,.$$

This inequality also holds if we assume (7.3) (take $\alpha=0$). Together with the bound for the sum with respect to m, the last inequality shows that:

$$L^q_-(f)\le(2\,R)^n\,|B(R)|\,\|\hat f\|^2_\infty. \tag{7.4}$$

Finally, consider the expression for $L^q_+(f)$. It is clear that

$$L^q_+(f)\le L(f)=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n}\int_{\operatorname{supp}\hat f}|\hat f(\xi+B^j m)|^2\,|\hat\psi^\ell(B^{-j}\xi)|^2\,d\xi,$$

and $L(f)<\infty$, by Proposition 5.6. □

A simple application of Theorem 4.2 to the quasi-affine systems $\widetilde{\mathcal{F}}_A$ yields another characterization of quasi-affine normalized tight frames.

Theorem 7.3. *Let $\Psi=\{\psi^1,\cdots\psi^L\}\subset L^2(\mathbb{R}^n)$ and $A\in GL_n(\mathbb{R})$ be such that the matrix $B=A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if it is a Bessel system with constant 1 and the Calderón condition (6.1) holds.*

Proof. Apply Theorem 4.2 with $\mathcal{P}$, g_p and C_p as in the proof of Theorem 7.2. The fact that condition (2.6) is satisfied for all $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is a dense subspace of $L^2(\mathbb{R}^n)$ defined by (5.10), and E is a complementary subspace to F as in Definition 5.1, follows from the same argument as in the proof of Theorem 7.2. $\square$

Using Theorem 7.3, we make the following observation.

Corollary 7.4. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. If the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ is a normalized tight frame for $L^2(\mathbb{R}^n)$, then the corresponding affine system $\mathcal{F}_A(\Psi)$ is also a normalized tight frame for $L^2(\mathbb{R}^n)$.*

In order to prove Corollary 7.4, we need the following Lemma, which is adapted from [9, Theorem 2].

Lemma 7.5. *Let $\Psi = \{\psi^1, \cdots \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$. If the system $\mathcal{F}_A^+(\Psi) = \{D_{A^j}\, T_k\, \psi^\ell : j \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ is a Bessel system with constant B, then the affine system $\mathcal{F}_A(\Psi)$, given by (5.1), has the same property.*

Proof. Since $\mathcal{F}_A^+(\Psi)$ is a Bessel system with constant B, then

$$\sum_{\ell=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^j\, T_k\, \psi^\ell\rangle|^2 \leq B\, \|f\|_2^2,$$

for all $f \in L^2(\mathbb{R}^n)$. Thus, given $N \in \mathbb{N}$ and any $g \in L^2(\mathbb{R}^n)$, from the last inequality with $f = D_A^N\, g$ we deduce that

$$\sum_{\ell=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^n} |\langle D_A^N\, g, D_A^j\, T_k\, \psi^\ell\rangle|^2 \leq B\, \|D_A^N\, g\|_2^2 = B\, \|g\|_2^2, \tag{7.5}$$

for all $g \in L^2(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Since $\langle D_A^N\, g, D_A^j\, T_k\, \psi^\ell\rangle = \langle g, D_A^{j-N}\, T_k\, \psi^\ell\rangle$, then from (7.5) we have that

$$\sum_{\ell=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^n} |\langle g, D_A^{j-N}\, T_k\, \psi^\ell\rangle|^2 \leq B\, \|g\|_2^2,$$

for all $g \in L^2(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Thus, applying the change of indices $i = j - N$ we obtain

$$\sum_{\ell=1}^{L} \sum_{i\geq -N} \sum_{k\in\mathbb{Z}^n} |\langle g, D_A^{i}\, T_k\, \psi^\ell\rangle|^2 \leq B\, \|g\|_2^2,$$

and the result then follows by taking the limit for N approaching infinity. $\square$

Proof of Corollary 7.4. If the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a Bessel system with constant 1, then so is the system $\mathcal{F}_A^+(\Psi) = \{D_{A^j}\, T_k\, \psi^\ell \,:\, j \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$, and so, by Lemma 7.5, is the corresponding affine system $\mathcal{F}_A(\Psi)$. By the item **3** of the Remarks after Theorem 7.2, the systems $\widetilde{\mathcal{F}}_A(\Psi)$ and $\mathcal{F}_A(\Psi)$ satisfy the same Calderón condition, and this completes the proof. □

8 Quasi-affine systems: special dilation matrices

In this section, we are going to analyze, in the same spirit as in Section 6, the forms that the characterization equations (7.2) and (7.3) assume corresponding to different values of $m \in \mathbb{Z}^n$ and $\alpha \in \Lambda^q$. These differences will depend on the dilation matrix A, expanding on subspaces of $\mathbb{R}^n$, similarly to the situation we encountered in Section 6.

As a consequence of the results we discuss in this Section, we have that for $A \in GL_n(\mathbb{R})$ of Type-II (i.e., A has integer entries), the affine system $\mathcal{F}_A(\Psi)$ is a normalized tight frame if and only if the corresponding quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ **has** the same property (see Theorem 7.1 and the references given before its statement for this equivalence in the case of expanding dilation matrices in $GL_n(\mathbb{R})$). On the other hand, we give examples of matrices A of Type-I for which $\mathcal{F}_A(\Psi)$ is a normalized tight frame, but the corresponding quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$ **does not have** the same property.

If we take $m = 0$ in (7.2), we have $\mathcal{Q}_m = \mathbb{Z}^+ \cup \{0\}$ (the set $\mathcal{Q}_m$ is defined in Theorem 7.2). Then (7.2) is the Calderón condition

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 = 1 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n . \tag{8.1}$$

If $m \neq 0$ and $m \in I^q(B) = \bigcap_{\{i \in \mathbb{Z},\ i \geq 0\}} B^i(\mathbb{Z}^n)$, we have $\mathcal{Q}_m = \mathbb{Z}^+ \cup \{0\}$, and (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = 0 \quad \text{for a. e.} \quad \xi \in \mathbb{R}^n . \tag{8.2}$$

The set $I^q(B)$ is contained in $\mathbb{Z}^n$. If B is an expanding matrix on $\mathbb{R}^n$ we have $I^q(B) = \{0\}$ and, consequently, condition (8.2) is not present. For the matrices of Example 2 in Section 5, we have $I^q(B) = \{0\} \times \mathbb{Z}$. For the matrices of Example 3 in Section 5, we have $I^q(B) = E \cap \mathbb{Z}^2$. For the matrices of Example 4 in Section 5, the set $I^q(B)$ depends on the angle of rotation θ. For the matrices of Example 5 in Section 5, we have $I^q(B) = \{0\} \times \mathbb{Z} \times \mathbb{Z}$ when b is an integer.

For other values of $\alpha \in \Lambda$, the form that (7.2) and (7.3) assume depends on the dilation matrix $B = A^t \in GL_n(R)$. As special applications of Theorem 7.2 we treat below matrices of Type-I and Type-II, as defined in Section 6. In order to avoid excessive technical discussions, we leave to the reader the exploration of how dilation matrices of Type-III are involved in the quasi-affine case.

8.1 Matrices of Type-I

Recall that $B \in GL_n(\mathbb{R})$ is a matrix of **Type-I** if $B^j(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(B) = \bigcap_{i\in\mathbb{Z}} B^i(\mathbb{Z}^n)$ for all $j \in \mathbb{Z} \setminus \{0\}$, according to Definition 6.1. Then, in this situation, $I^q(B) \subset B(\mathbb{Z}^n) \cap \mathbb{Z}^n = I(B) \subset I^q(B)$, so that we have

$$I^q(B) = I(B). \tag{8.3}$$

In view of this equality, condition (6.2) for affine systems and condition (8.2) for quasi-affine systems range over the same values of m.

We can now give the form that the equations that appear in Theorem 7.2 assume in this situation.

Proposition 8.1. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-I which is expanding on a subspace of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: the Calderón condition (8.1), (8.2),*

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(\xi)\, \overline{\hat{\psi}^\ell(\xi+m)} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{8.4}$$

and all $m \in \mathbb{Z}^n \setminus I^q(B)$, and

$$\sum_{\ell=1}^{L} \sum_{j\geq 1} \hat{\psi}^\ell(B^j\xi)\, \overline{\hat{\psi}^\ell(B^j(\xi+m))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{8.5}$$

and all $m \in \mathbb{Z}^n \setminus I^q(B)$.

Proof. We apply Theorem 7.2. We have already observed that (8.1) and (8.2) are the cases $m = 0$ and $m \in I^q(B) \setminus \{0\}$ of (7.2). We now need to consider the cases of $m \in \mathbb{Z}^n \setminus I^q(B)$ and $\alpha \in \Lambda^q \setminus I^q(B)$.
For $m \in \mathbb{Z}^n \setminus I^q(B)$, since B is of Type-I, we obtain $\mathcal{Q}_m = \{0\}$. In this case, (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j\leq 0} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{8.6}$$

For $\alpha \in \Lambda^q \setminus \mathbb{Z}^n$, write $\alpha = B^{j_0} m$ for some $j_0 \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^n \setminus I^q(B)$. Then, $j \in \mathcal{Q}_\alpha$ if and only if $j \in \mathbb{Z}^+ \cup \{0\}$ and $B^{-j}\alpha = B^{-j+j_0} m \in \mathbb{Z}^n$. Since B is of Type-I, we deduce that $j = j_0$, so that $\mathcal{Q}_\alpha = \{j_0\}$. In this case,

$$\sum_{\ell=1}^{L} \hat{\psi}^\ell(B^{-j_0}\xi)\,\overline{\hat{\psi}^\ell(B^{-j_0}(\xi + B^{j_0} m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Change $B^{-j_0}\xi$ to η to obtain (8.4). Finally, observe that the terms with $j = 0$ in (8.6) add up to zero by the just proved equation (8.4), so that (8.6) becomes (8.5) after changing j to $-j$, $\square$

Example 10. If $a \in \mathbb{R},\ a > 1$, with $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z}^+$, and a single $\psi \in L^2(\mathbb{R})$, it follows from Proposition 8.1 that the quasi-affine system $\widetilde{\mathcal{F}}_A(\psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R})$ if and only if

$$\sum_{j\in\mathbb{Z}} \mid \hat{\psi}(a^j\xi) \mid^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{8.7}$$

$$\hat{\psi}(\xi)\overline{\hat{\psi}(\xi + m)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}, \text{ and all } m \in \mathbb{Z} \setminus \{0\}, \tag{8.8}$$

and

$$\sum_{j\geq 1} \hat{\psi}(a^j\xi)\overline{\hat{\psi}(a^j(\xi + m))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}, \text{ and all } m \in \mathbb{Z} \setminus \{0\}. \tag{8.9}$$

Remark. Comparing Propositions 6.1 and 8.1, it is clear that, for matrices of Type-I expanding on subspaces of $\mathbb{R}^n$, if the quasi-affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$, then also the corresponding affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$. Of course, this is in agreement with Corollary 7.4. On the other hand, (8.5) does not appear in Proposition 6.1: this fact will allow us to exhibit affine normalized tight frames for $L^2(\mathbb{R}^n)$, for which the corresponding quasi-affine system is not a normalized tight frame for $L^2(\mathbb{R}^n)$. This is presented in the following example.

Example 11. We carry out the construction in dimension $n = 1$. Let $a \in \mathbb{R}$, $a > 1$, and $a \notin \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $r - 1 < a < r$. Assume $r - (1/2) \leq a < r$ (the case $r - 1 < a < r - (1/2)$ requires only minor modifications from the example we present below). Choose $\beta \in \mathbb{R}$ such that $r - 1 < a < \beta < r$, and let $\epsilon = r - \beta > 0$.

Write $J = (r - \frac{1}{2}, \beta)$, and $I = (\beta, a(r - \frac{1}{2}))$, so that $J \cup I = (r - \frac{1}{2}, a(r - \frac{1}{2}))$. Observe that $a(r - \frac{1}{2}) > (r - \frac{1}{2})^2 = r^2 - r + \frac{1}{4} > \beta$, since $r \geq 2$. Choose j_0 to be a positive integer large enough so that

$$\frac{a(r - \frac{1}{2})}{a^{j_0}} < \min\{\epsilon, \beta - a\}. \tag{8.10}$$

Let $K = \frac{1}{a^{j_0}} I$, and consider $W_a = (\pm K) \cup (\pm J)$. Define ψ_a by

$$\hat{\psi}_a = \chi_{W_a}.$$

Since $a^{j_0} K \cup J = (\pm I) \cup (\pm J) = \pm(r - \frac{1}{2}, a(r - \frac{1}{2}))$, (8.7) holds for ψ_a. By (8.10), $W_a \cap (W_a + m) = \emptyset$ for all $m \in \mathbb{Z} \setminus \{0\}$. Hence (8.8) is true.

If we choose $a \in \mathbb{R},\ a > 1$, such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{Z}^+$, from Example 10 we deduce that the affine system $\mathcal{F}_a(\psi_a)$ is a normalized tight frame $L^2(\mathbb{R}^n)$. Moreover, by (8.10), if $\xi \in a^{-1} K$,

$$\hat{\psi}_a(a\xi)\, \overline{\hat{\psi}_a(a\xi + a)} = 1, \qquad \text{(since } K + a \subset J\text{)},$$

and for $j \in \mathbb{Z}^+$, $\hat{\psi}_a(a^j\xi)\, \overline{\hat{\psi}_a(a^j\xi + a^j)} = 0$, since $\hat{\psi}_a(a^j\xi) = 0$. Thus, (8.9) does not hold for $m = 1$ and, consequently, the quasi-affine system $\widetilde{\mathcal{F}}_a(\psi_a)$ is **not** a normalized tight frame for $L^2(\mathbb{R})$.

8.2 Matrices of Type-II

Recall that $B \in GL_n(\mathbb{R})$ is a matrix of **Type-II** if $B(\mathbb{Z}^n) \subset \mathbb{Z}^n$, according to Definition 6.2. Then, in this situation,

$$\cdots \subset B^2(\mathbb{Z}^n) \subset B(\mathbb{Z}^n) \subset \mathbb{Z}^n \subset B^{-1}(\mathbb{Z}^n) \subset B^{-2}(\mathbb{Z}^n) \subset \cdots$$

and, consequently,

$$I^q(B) = \bigcap_{\{i \in \mathbb{Z},\, i \geq 0\}} B^i(\mathbb{Z}^n) = \bigcap_{i \in \mathbb{Z}} B^i(\mathbb{Z}^n) = I(B)\,. \tag{8.11}$$

In view of this equality, condition (6.2) for affine systems and condition (8.2) for quasi-affine systems range over the same values of m.

We can now give the form that the equations that appear in Theorem 7.2 assume in this situation.

Proposition 8.2. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that $B = A^t$ is a matrix of Type-II which is expanding on a subspace of $\mathbb{R}^n$. Then the quasi-affine system $\widetilde{\mathcal{F}}_A(\Psi)$, given by (7.1), is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold: the Calderón condition (8.1), (8.2), and*

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell(B^j\xi)\, \overline{\hat{\psi}^\ell(B^j(\xi + q))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{8.12}$$

and all $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$.

Proof. We apply Theorem 7.2. We have already observed that (8.1) and (8.2) are the cases $m = 0$ and $m \in I^q(B) \setminus \{0\}$ of (7.2).
For matrices of Type-II, $\Lambda^q = \bigcup_{j \in \mathbb{Z}^+ \cup \{0\}} B^j(\mathbb{Z}^n) = \mathbb{Z}^n$ by the inclusions that precede (8.11), and (7.3) is void. Thus, we only need to consider the case $m \in \mathbb{Z}^n \setminus I^q(B)$. Observe that, since $I^q(B) = I(B)$, by (8.11), we can apply Lemma 6.2 to deduce the existence of unique $d \in \mathbb{Z} + \cup \{0\}$ and $q \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$ such that $m = B^d q$.
We want to examine $\mathcal{Q}_m = \{j \in \mathbb{Z}^+ \cup \{0\} : B^{-j} m \in \mathbb{Z}^n\}$. If $j \in \mathcal{Q}_m$, then $B^{-j} B^d q = B^{-j} m \in \mathbb{Z}^n$. We must have $-j + d \geq 0$ (otherwise, with $-j + d = -\ell < 0$, we deduce $B^{-\ell} q \in \mathbb{Z}^n$, which implies $q \in B^\ell(\mathbb{Z}^n) \subset B(\mathbb{Z}^n)$, contrary to our choice of q).
Also, if $0 \leq j \leq d$, then $j \in \mathcal{Q}_m$; in fact, since $-j + d \geq 0$ and B is of Type-II, we obtain $B^{-j} m = B^{-j+d} q \in \mathbb{Z}^n$. Thus, $\mathcal{Q}_m = \{0, 1, 2, \cdots, d\}$, and (7.2) is equivalent to

$$\sum_{\ell=1}^{L} \sum_{j<0} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d q))} + \sum_{\ell=1}^{L} \sum_{j=0}^{d} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + B^d q))} = 0$$

for a.e. $\xi \in \mathbb{R}^n$. Collecting the two sums with j ranging from $-\infty$ to d, doing the change of variables $\xi = B^d \eta$, and changing the index of summation to $k = -j + d$, we obtain (8.12). This finishes the proof of the Proposition. $\square$

Remarks.

(1) Comparing Propositions 6.3 and 8.2, and taking into account the equality (8.11), it is clear that for matrices of Type-II (i.e. matrices with integer entries), expanding on subspaces of $\mathbb{R}^n$, the quasi-affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$ if and only if the affine frame is a normalized tight frame for $L^2(\mathbb{R}^n)$. This generalizes Theorem 7.1 to the case of matrices which are not just expanding, but expanding on subspaces of $\mathbb{R}^n$.

(2) We have proved in Example 7 that the equivalence stated in the above remark does not carry over to matrices of Type-I. On the other hand, recently M. Bownik [4] has modified the quasi-affine system (7.1) to obtain this equivalence for rational dilation matrices expanding on $\mathbb{R}^n$ (that is, matrices with rational entries).

9 Dual systems

In this section, we consider the case of systems satisfying a reproducing formula of the form

$$v = \sum_{\alpha \in \mathcal{A}} \langle v, e_\alpha \rangle \, \eta_\alpha, \qquad v \in \mathcal{H}$$

where the "analyzing" family $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ differs from the "synthetizing" family $\{e_\alpha\}_{\alpha \in \mathcal{A}}$. Since the results that we shall present in this section follow for the most part by simple adaptations of the arguments used in the tight frame case, in order to avoid repetitions, we will omit or simply sketch some of the proofs.

Let $e = \{e_\alpha\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\eta_\alpha\}_{\alpha\in\mathcal{A}}$ be Bessel systems for $\mathcal{H}$. Then $\{\eta_\alpha\}_{\alpha\in\mathcal{A}}$ is called a **dual system** to $\{e_\alpha\}_{\alpha\in\mathcal{A}}$, if

$$K_{e,\eta}(v,w) \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle\, \langle \eta_\alpha, w\rangle \;=\; \langle v, w\rangle, \qquad \text{for all } v,w\in\mathcal{H}. \tag{9.13}$$

If this is the case, then we have:

$$v \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, \eta_\alpha\rangle\, e_\alpha \;=\; \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle\, \eta_\alpha, \qquad \text{for all } v\in\mathcal{H},$$

with convergence in $\mathcal{H}$. Note that, by the polarization identity for sesquilinear forms, we have $K_{e,\eta}(v,w) = \frac{1}{4}\sum_{n=0}^{3} i^n\, K_{e,\eta}(v+i^n w, v+i^n w)$. Therefore, (9.13) holds if and only if it holds for all $v = w\in\mathcal{H}$. Also, it is enough to prove (9.13) for $v = w$ in a dense subspace of $\mathcal{H}$ (cf. [15]).

We have the following general result, which characterizes a class of dual systems for the collections of the form $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$.

Theorem 9.1. *Let* $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ *and* $\{T_{C_p k}\, \gamma_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ *be Bessel systems for* $L^2(\mathbb{R}^n)$*, where* $\mathcal{P}$ *is countable,* $\{g_p\}_{p\in\mathcal{P}}$*,* $\{\gamma_p\}_{p\in\mathcal{P}}$*, are collections of functions in* $L^2(\mathbb{R}^n)$ *and* $\{C_p\}_{p\in\mathcal{P}}\subset GL_n(\mathbb{R})$*. Suppose that*

$$\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\int_{supp\,\hat f} |\hat f(\xi + C_p^I m)|^2\, \frac{1}{|\det C_p|}\, |\hat g_p(\xi)|^2\, d\xi < \infty. \tag{9.14}$$

and

$$\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\int_{supp\,\hat f} |\hat f(\xi + C_p^I m)|^2\, \frac{1}{|\det C_p|}\, |\hat \gamma_p(\xi)|^2\, d\xi < \infty. \tag{9.15}$$

for all $f\in\mathcal{D}$*, where* $C_p^I = (C_p^t)^{-1}$*. Then* $\{T_{C_p k}\,\gamma_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ *is a dual frame to* $\{T_{C_p k}\, g_p : p\in\mathcal{P}, k\in\mathbb{Z}^n\}$ *in* $L^2(\mathbb{R}^n)$ *if and only if*

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat g_p(\xi)}\, \hat\gamma_p(\xi+\alpha) = \delta_{\alpha,0} \;\; \textit{for a.e. } \xi\in\mathbb{R}^n, \tag{9.16}$$

for each $\alpha\in\Lambda$*, where* δ *is the Kronecker delta for* $\mathbb{R}^n$*, and* Λ*,* $\mathcal{P}_\alpha$ *are defined by (2.2) and (2.3).*

In order to prove Theorem 9.1, we need the following Lemmas, whose proofs can be easily adapted from those of Lemmas 2.2 and 2.3 (see also [21, Sec.4]). Recall that the dense subspace $\mathcal{D}\subset L^2(\mathbb{R}^n)$ is defined in Section 2.

Lemma 9.2. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. If $f \in \mathcal{D}$ and $g, \gamma \in L^2(\mathbb{R}^n)$, then*

$$\sum_{k \in \mathbb{Z}^n} \langle f, T_{Ck}\, g \rangle \, \langle T_{Ck}\, \gamma, f \rangle = \frac{1}{|\det C|} \int_{C^I \mathbb{T}^n} [\hat{f}, \hat{g}](\xi; C^I)\, [\hat{\gamma}, \hat{f}](\xi; C^I)\, d\,\xi, \tag{9.17}$$

where $\mathbb{T}^n = [0,1)^n$.

Lemma 9.3. *Let $C \in GL_n(\mathbb{R})$ and $C^I = (C^t)^{-1}$. For each $f \in \mathcal{D}$ and $g, \gamma \in L^2(\mathbb{R}^n)$, the function*

$$K(x) \;=\; \sum_{k \in \mathbb{Z}^n} \langle T_x\, f, T_{Ck}\, g \rangle \, \langle T_{Ck}\, \gamma, T_x\, f \rangle \tag{9.18}$$

is the trigonometric polynomial

$$K(x) = \sum_{m \in \mathbb{Z}^n} \hat{K}(m)\, e^{2\pi i (C^I m) \cdot x},$$

where

$$\hat{K}(m) \;=\; \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I m)}\, \overline{\hat{g}(\xi)}\, \hat{\gamma}(\xi + C^I m)\, d\xi, \tag{9.19}$$

and only a finite number of these expressions is non-zero.

The following Proposition is the principal result that we shall use to establish Theorem 9.1. The proof is very similar to the proof of Proposition 2.4 and will be omitted. Observe that, unlike Proposition 2.4 where only condition (9.14) was needed, in this case we need both (9.14) and (9.15) in order to show that the generalized Fourier series (9.20) converges absolutely.

Proposition 9.4. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p \in \mathcal{P}}, \{\gamma_p\}_{p \in \mathcal{P}}$ be collections of functions in $L^2(\mathbb{R}^n)$, $\{C_p\}_{p \in \mathcal{P}} \subset GL_n(\mathbb{R})$, and let $C_p^I = (C_p^t)^{-1}$. Assume that, for $f \in \mathcal{D}$, the conditions (9.14) and (9.15) hold. Then, the function*

$$w(x) = \sum_{p \in \mathcal{P}} \sum_{k \in \mathbb{Z}^n} \langle T_x f, T_{C_p k}\, g_p \rangle \, \langle T_{C_p k}\, \gamma_p, T_x f \rangle$$

is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$\sum_{\alpha \in \Lambda} \hat{w}(\alpha)\, e^{2\pi i \alpha \cdot x} \;, \tag{9.20}$$

where

$$\hat{w}(\alpha) \;=\; \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + \alpha)} \sum_{p \in \mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + \alpha)\, d\xi, \tag{9.21}$$

and the integral in (9.21) converges absolutely.

Remark. As in Proposition 2.4, the series for $w(x)$ given in Proposition 9.4 is an almost periodic function since these are characterized as uniform limits of generalized trigonometric polynomials (see [1]).

We can now prove Theorem 9.1.

Proof of Theorem 9.1. By the observation at the beginning of this section, it suffices to prove that

$$\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{Z}^n}\langle f, T_{C_pk}\, g_p\rangle\,\langle T_{C_pk}\,\gamma_p, f\rangle \;=\; \|f\|^2, \tag{9.22}$$

for f in a dense subset of $L^2(\mathbb{R}^n)$. Let us assume that conditions (9.14) and (9.15) hold for all $f\in\mathcal{D}$, where $\mathcal{D}$ is given in Section 2, and that (9.16) is true. By Proposition 9.4,

$$w(x)=\sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}^n}\langle T_x\, f, T_{C_pm}\, g_p\rangle\,\langle T_{C_pm}\,\gamma_p, T_x\, f\rangle=\sum_{\alpha\in\Lambda}\hat{w}(\alpha)\,e^{2\pi i\alpha\cdot x},$$

where the last series converges absolutely (thus, $w(x)$ is continuous) and, by (9.16),

$$\hat{w}(\alpha)\;=\;\Big(\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+\alpha)}\,d\xi\Big)\,\delta_{\alpha,0}$$

for each $f\in\mathcal{D}$. Then equation (9.22) follows by letting $x=0$.

Now let us assume that equation (9.22) holds for all $f\in L^2(\mathbb{R}^n)$. By Proposition 9.4, if $f\in\mathcal{D}$, then the function $z(x)=w(x)-\|f\|^2$ is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\hat{z}(0)=\hat{w}(0)-\|f\|^2, \qquad \text{and} \qquad \hat{z}(\alpha)=\hat{w}(\alpha),\ \alpha\neq 0.$$

Since $z(x)=0$, it follows from Lemma 2.5 that all coefficients $\hat{z}(\alpha)$ must be 0. Thus for $\alpha\in\Lambda$ and $f\in\mathcal{D}$

$$\int_{\mathbb{R}^n}\hat{f}(\xi)\,\overline{\hat{f}(\xi+\alpha)}\Big(\sum_{p\in\mathcal{P}_\alpha}\frac{1}{|\det C_p|}\,\overline{\hat{g}_p(\xi)}\,\hat{\gamma}_p(\xi+\alpha)\Big)\,d\xi=\delta_{\alpha,0}\,\|f\|^2, \tag{9.23}$$

Consider the case $\alpha=0$ and let

$$s_0(\xi)=\sum_{p\in\mathcal{P}}\frac{1}{|\det C_p|}\,\overline{\hat{g}_p(\xi)}\,\hat{\gamma}_p(\xi).$$

By (9.14) and (9.15), s_0 is locally integrable. Choose ξ_0 to be a point of differentiability of the integral of this function. Letting $B(\epsilon)$ denote the ball of radius $\epsilon>0$ about the origin, define f_ϵ by

$$\hat{f}_\epsilon(\xi)=\frac{1}{\sqrt{|B(\epsilon)|}}\,\chi_{B(\epsilon)}(\xi-\xi_0)\,.$$

Then $\|f_\epsilon\|_2 = 1$ and $f_\epsilon \in \mathcal{D}$. By (9.23) with $f = f_\epsilon$ we have

$$1 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|} \, s_0(\xi)\, d\xi = s_0(\xi_0).$$

This shows that $s_0(\xi) = 1$, a.e. $\xi \in \mathbb{R}^n$, and (9.16) is satisfied for $\alpha = 0$.
When $\alpha \ne 0$, let

$$s_\alpha(\xi) = \sum_{p \in \mathcal{P}(\alpha)} \frac{1}{|\det C_p|} \, \overline{\hat{g}_p(\xi)} \, \hat{\gamma}_p(\xi + \alpha).$$

By the polarization of (9.23), we have

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{h}(\xi + \alpha)} \, s_\alpha(\xi)\, d\xi = 0 \tag{9.24}$$

for all $f, h \in \mathcal{D}$. By Schwarz's inequality and conditions (9.14), (9.15), we have that s_α is locally integrable. We can choose, again, a point of differentiability ξ_0 of the integral of s_α, and choose f_ϵ and h_ϵ such that

$$\hat{f}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \, \chi_{B(\epsilon)}(\xi - \xi_0), \qquad \hat{h}_\epsilon(\xi) = \frac{1}{\sqrt{|B(\epsilon)|}} \, \chi_{B(\epsilon)}(\xi - \xi_0 - \alpha).$$

Hence $\|f_\epsilon\|_2 = \|g_\epsilon\|_2 = 1$, f_ϵ , $g_\epsilon \in \mathcal{D}$ and by (9.24),

$$0 = \lim_{\epsilon \to 0} \int_{|\xi - \xi_0| \le \epsilon} \frac{1}{|B(\epsilon)|} \, s_\alpha(\xi)\, d\xi = s_\alpha(\xi_0).$$

Hence $s_\alpha(\xi) = 0$, a.e. $\xi \in \mathbb{R}^n$, and (9.16) is satisfied for $\alpha \ne 0$. □

The application of Theorem 9.1 to the Gabor systems $\mathcal{G}_{B,C}(G)$, defined by (3.2), yields the following characterization of Gabor dual frames, known as the Wexler-Raz theorem (cf. [20, 30, 21]). Our proof, which is adapted from [21], will only be sketched.

Theorem 9.5 (Wexler-Raz). *Let $G = \{g^1, \cdots, g^L\}$, $\Gamma = \{\gamma^1, \cdots, \gamma^L\} \subset L^2(\mathbb{R}^n)$, $B, C \in GL_n(\mathbb{R})$, and assume that $\mathcal{G}_{B,C}(G)$ and $\mathcal{G}_{B,C}(\Gamma)$ are Bessel systems for $L^2(\mathbb{R}^n)$. Then the system $\mathcal{G}_{B,C}(\Gamma)$ is a dual system to $\mathcal{G}_{B,C}(G)$ if and only if*

$$\sum_{\ell=1}^{L} \langle g^\ell, T_{B^I v}\, M_{C^I u}\, \gamma^\ell \rangle = |\det B|\, |\det C|\, \delta_{u,0}\, \delta_{v,0} \tag{9.25}$$

for each u, $v \in \mathbb{Z}^n$, where δ is the product Kronecker delta in $\mathbb{Z}^n$, $B^I = (B^t)^{-1}$ and $C^I = (C^t)^{-1}$.

Proof. We apply Theorem 9.1 with $g_p = M_{Bp}\, g$, $\gamma_p = M_{Bp}\, \gamma$, $p \in \mathcal{P} = \mathbb{Z}^n$ and $C_p = C$. An argument similar to the proof of Theorem 3.2 shows that, if $\mathcal{G}_{B,C}(G)$ and $\mathcal{G}_{B,C}(\Gamma)$ are Bessel systems for $L^2(\mathbb{R}^n)$, then

$$\sum_{\ell=1}^{L} \sum_{k,m \in \mathbb{Z}^n} \langle f, T_{Ck}\, M_{Bm}\, g^\ell \rangle \langle T_{Ck}\, M_{Bm}\, \gamma^\ell, h \rangle \;=\; \langle f, h \rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n)$$

if and only if

$$F(\xi) = \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \frac{1}{|\det C|}\, \hat{g}^\ell(\xi - Bk)\, \overline{\hat{\gamma}^\ell(\xi - Bk + C^I m)} = \delta_{m,0}$$

for a.e. $\xi \in \mathbb{R}^n$, all $m \in \mathbb{Z}^n$.

The proof then follows by expanding the $B\mathbb{Z}^n$- periodic function $F(\xi)$ into a Fourier series, as in the argument used in [21, theorem 6.1]. □.

The application of Theorem 9.1 to the affine systems $\mathcal{F}_A(\Psi)$, defined by (5.2), yields the following characterization of affine dual systems, whose proof is similar to the proof of Theorem 2.1. This theorem generalizes previous results about affine dual systems, such as those in [15, 2, 7].

Theorem 9.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\}, \Phi = \{\phi^1, \cdots, \phi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ such that the matrix $B = A^t$ is expanding for a subspace F of $\mathbb{R}^n$. Assume that the systems $\mathcal{F}_A(\Psi)$ and $\mathcal{F}_A(\Phi)$ are Bessel systems for $L^2(\mathbb{R}^n)$. Then the system $\mathcal{F}_A(\Phi)$ is a dual system to $\mathcal{F}_A(\Psi)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\phi}^\ell(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{9.26}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j(\mathbb{Z}^n)$, where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} :\ B^{-j}\alpha \in \mathbb{Z}^n\}$.

Proof. Recall that $D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell$. We are going to apply Theorem 9.1 with

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\ \ell = 1,2,\ldots,L\big\},$$

$$g_p \equiv g_{(j,l)} = D_A^j\, \psi^\ell,\ \gamma_p \equiv \gamma_{(j,l)} = D_A^j\, \phi^\ell, \quad \text{and} \quad C_p \equiv C_{(j,\ell)} = A^{-j} \text{ for all } \ell = 1,\ldots,L.$$

Since we have that $\hat{g}_p(\xi) = (D_A^j\psi^\ell)^\wedge(\xi) = |\det B|^{-j/2}\hat{\psi}^\ell(B^{-j}\,\xi)$, and $\hat{\gamma}_p(\xi) = (D_A^j\phi^\ell)^\wedge(\xi) = |\det B|^{-j/2}\hat{\phi}^\ell(B^{-j}\,\xi)$, then (9.26) follows from (9.16) in Theorem 9.1, provided the conditions (9.14) and (9.15) in this Theorem are satisfied. Therefore, all that it is left to prove is that (9.14) and (9.15) are satisfied in this particular case. Thus, we need to show that:

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty \tag{9.27}$$

and

$$J(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\phi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty, \tag{9.28}$$

for f in an appropriate dense set of $L^2(\mathbb{R}^n)$. Like in the proof of Theorem 2.1, the dense set we choose is

$$\mathcal{D}_E = \{f \in \mathcal{D} : (\operatorname{supp} \hat{f}) \cap E = \emptyset\}$$

where $\mathcal{D} = \{f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact}\}$, and E is a complementary subset to F as in Definition 5.1. The proof that $L(f) < \infty$ and $J(f) < \infty$ now follows from Proposition 5.6. $\square$

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