CYCLIC SUBSPACES FOR UNITARY REPRESENTATIONS OF LCA GROUPS: GENERALIZED ZAK TRANSFORM

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1. Introduction

Cyclic subspaces generated by integer translations of a single function $\psi \in L^2(\mathbb{R})$ have attracted attention as building blocks for more general translation invariant subspaces of $L^2(\mathbb{R})$. Also, they play a key role in the theory of Multiresolution Analysis (MRA) where the scaling spaces V_0 and the zero resolution wavelet space W_0 are both principal invariant subspaces of $L^2(\mathbb{R})$.

For $\psi \in L^2(\mathbb{R})$, let $< \psi >$ be the closure in $L^2(\mathbb{R})$ of the finite linear combinations of the integer translations $T_k\psi(x) = \psi(x-k)$. Properties of the collection $\{T_k\psi : k \in \mathbb{Z}\}$ in $< \psi >$ can be studied using the "periodization" function

$$P_\psi(\xi) = \sum_{l\in\mathbb{Z}} |\widehat{\psi}(\xi + l)|^2.$$

For example, $\{T_k\psi : k \in \mathbb{Z}\}$ is an orthonormal basis for $< \psi >$ if and only if $P_\psi(\xi) = 1$ a.e. (see [9], Proposition 1.11, Chapter 1.) More properties of the collection $\{T_k\psi : k \in \mathbb{Z}\}$ and its relation to the function $P_\psi(\xi)$ is the subject of [10]. The present paper includes the results of [10] and many more.

Now consider integer translations $T_k\psi(x) = \psi(x-k)$, $x \in \mathbb{R}$ and integer modulations $M_l\psi(x) = e^{2\pi ikx}\psi(x)$ of a single function $\psi \in L^2(\mathbb{R})$. We obtain the collection $\{T_kM_l\psi : k, l \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$, known as Gabor system, that can be studied using the Zak transform

$$Z\psi(x,\xi) = \sum_{l\in\mathbb{Z}} \psi(x+l)e^{2\pi il\xi},$$

as the analog of P_ψ. For example, $\{T_kM_l\psi : k, l \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ if and only if $|Z\psi(x,\xi)| = 1$ a.e. (see [8], Theorem 4.3.3 and the references given there.)

We show that these results are particular cases of a more general theory involving representations of locally compact abelian groups. Suppose that G is a locally compact

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Date: September 2, 2008.

2000 *Mathematics Subject Classification.* 42C40, 43A65,43A70.

Key words and phrases. Cyclic subspaces, invariant subspaces, unitary representations, locally compact abelian groups, translations, dilations, Gabor systems, Zak transform.

The research of E. Hernández is supported by grants MTM2007-60952 of Spain and SIMUMAT S-0505/ESP-0158 of the Madrid Community Region. The research of H. Šikić, G. Weiss and E. Wilson is supported by the US-Croatian grant NSF-INT-0245238. The research of H. Šikić is also supported by the MZOS grant 037-0372790-2799 of the Republic of Croatia.

abelian (LCA) group and $g \longrightarrow T_g$ is a unitary representation of G on a Hilbert space H. This representation is said to be **dual integrable** if there exist a function

$$[\cdot,\cdot] : H \times H \longrightarrow L^1(\widehat{G}, d\alpha)$$

such that

$$< \varphi, T_g\psi > = \int_{\widehat{G}} [\varphi,\psi]\, \overline{\alpha(g)}\, d\alpha\,, \text{ for all } \varphi,\psi \in H\,.$$

We prove that properties of the collection $\{T_g : g \in G\}$ correspond to properties of the collection of "exponentials" $\{e_g(\alpha) = \alpha(g) : g \in G\}$ in the weighted space $L^2(\widehat{G}, [\psi,\psi](\alpha)\, d\alpha)$. For example, if G is discrete, $\{T_g : g \in G\}$ is an orthonormal basis for $< \psi >$ if and only if $[\psi,\psi](\alpha) = 1$ a.e. $\alpha \in \widehat{G}$. Precise conditions on $[\psi,\psi]$ are derived that characterize when the collection $\{T_g : g \in G\}$ is a Riesz basis, a frame, or has an associated biorthogonal system in $< \psi >$.

This results not only apply to translations and modulations in dimension 1, but works in any dimension, and for systems of general, not necessarily isotropic, dilations in $\mathbb{R}^n$.

Notation, definitions, simple examples, and elementary properties are given in section 2. In particular, the central notion of **dual integrable** unitary representation is given here. We show that this notion is equivalent to other regularity definitions concerning unitary representations appearing in the literature. Section 3 contains the main results of this paper showing the isometric isomorphism that allows us to study properties of $\{T_g : g \in G\}$ in $< \psi >$ by looking at the corresponding properties of $\{e_g(\alpha) = \alpha(g) : g \in G\}$ in the weighted space $L^2(\widehat{G}, [\psi,\psi](\alpha)\, d\alpha)$. The translation and the translation-dilation systems in $\mathbb{R}^n$ are carefully studied in section 4 as important special cases of our more general situation. Detailed results relating properties of the collection $\{T_g : g \in G\}$ in $< \psi >$ to the function $[\psi,\psi]$ in $\widehat{G}$ are described in sections 5 and 6. Finally, section 7 shows that our theory can be used to describe properties of cyclic subspaces generated by general dilations and shear matrices.

2. Notation, definitions, examples, and properties.

2.1. **Notation, definitions and examples.** Let G be a locally compact abelian (LCA) group. We shall use aditive notation for G. A **character** of G is a continuous map $\alpha : G \longrightarrow \mathbb{C}$ for which

$$|\alpha(g)| = 1 \qquad \text{for all } g \in G \tag{2.1}$$

and

$$\alpha(g_1 + g_2) = \alpha(g_1)\cdot\alpha(g_2) \qquad \text{for all } g_1, g_2 \in G. \tag{2.2}$$

The **character group** of G is the multiplicative group of all characters. For theoretical purpose, we take the **dual group** $\widehat{G}$ to be the character group; for applications, we take $\widehat{G}$ to be an abelian group parametrizing the character group. Thus, $\mathbb{Z}^n$ and $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ are dual to each other in the sense that for $e_\xi(k) = e_k(\xi) = e^{2\pi i\xi\cdot k}$, $\{e_\xi : \xi \in \mathbb{T}\}$ is the character group of $\mathbb{Z}^n$ and $\{e_k : k \in \mathbb{Z}^n\}$ is the character group of $\mathbb{T}^n$.

The basic facts about Fourier Analysis in LCA groups, including the definition of the Fourier Transform and its properties, can be found in [15] or [2].

A **representation** of a LCA group G on a Hilbert space H is a strongly continuous map $g \longrightarrow T_g$ from G into the group $\mathcal{L}(H, H)$ of bounded linear operators on H with bounded inverses, such that $T_g \circ T_h = T_{g+h}$ for all $g, h \in G$. We say that a representation T is **unitary** if all the operators T_g are unitary, that is $< T_g\varphi, T_g\psi >=< \varphi, \psi >$ for all $g \in G$ and $\varphi, \psi \in H$, or equivalently $\|T_g\|_{H\to H} = 1$ for all $g \in G$.

Fix a Haar meeasure on $\widehat{G}$. A unitary representation T of a LCA group G on a Hilbert space H is said to be **dual integrable** if there exists a function, which we shall call **bracket**,

$$[\cdot, \cdot] : H \times H \longrightarrow L^1(\widehat{G}, d\alpha) \tag{2.3}$$

such that

$$< \varphi, T_g\psi >_H= \int_{\widehat{G}} [\varphi, \psi](\alpha)\overline{\alpha(g)}\, d\alpha\,, \qquad \text{for all } \varphi, \psi \in H \text{ and all } g \in G\,, \tag{2.4}$$

A closed linear subspace S of the Hilbert space H is said to be T-**invariant** if $T_g(S) \subset S$ for all $g \in G$. Given $\psi \in H \setminus \{0\}$, define the closed linear subspace of H generated by ψ to be

$$< \psi >= \overline{\text{ span } \{T_g\psi : g \in G\}}^H \tag{2.5}$$

The subspace $< \psi >$ is T-invariant and it is called the **cyclic** T-invariant subspace of H generated by ψ.

Example 2.1. The map $g \longrightarrow \mathcal{M}_g$ where $\mathcal{M}_g : L^2(\widehat{G}, d\alpha) \longrightarrow L^2(\widehat{G}, d\alpha)$ is given by $(\mathcal{M}_g\varphi)(\alpha) = \alpha(g)\varphi(\alpha)$ is a unitary representation of a LCA group G on the Hilbert space $L^2(\widehat{G}, d\alpha)$. This is called the **modulation representation** of G on $L^2(\widehat{G}, d\alpha)$. This representation is dual integrable: since for $\varphi, \psi \in L^2(\widehat{G}, d\alpha)$ we have

$$< \varphi, \mathcal{M}_g\psi >= \int_{\widehat{G}} \varphi(\alpha)\overline{\psi(\alpha)\alpha(g)}\, da\,,$$

we can set $[\varphi, \psi](\alpha) = \varphi(\alpha)\overline{\psi(\alpha)}$, which satisfies the conditions of dual integrability.

Example 2.2. For $h \in G$, the map $h \longrightarrow R_h$ where $R_h : L^2(G, dg) \longrightarrow L^2(G, dg)$ is defined by $(R_h\varphi)(g) = \varphi(g + h)$ is a unitary representation of the LCA group G on the Hilbert space $L^2(G, dg)$, since dg is translation invariant. This is called the **regular representation** of G on $L^2(G, dg)$. To see that it is dual integrable use the Fourier transform $\mathcal{F}_G\psi = \widehat{\psi}$ on $L^2(G, dg)$ to deduce

$$(R_h\psi)\widehat{\ }(\alpha) = \int_G \psi(g + h)\, \overline{\alpha(g)}\, dg = \int_G \psi(g + h)\, \overline{\alpha(g + h)}\, \alpha(h)\, dg = \alpha(h)\, \widehat{\psi}(\alpha).$$

Thus, for $\varphi, \psi \in L^2(G, dg)$ and $h \in G$, we use Plancherel's theorem to obtain

$$< \varphi, R_h\psi >= \int_G \varphi(g)\, \overline{R_h\psi(g)}\, dg = \int_{\widehat{G}} \widehat{\varphi}(\alpha)\overline{\widehat{R_h(\psi)}}(\alpha)\, d\alpha = \int_{\widehat{G}} \widehat{\varphi}(\alpha)\, \overline{\widehat{\psi}(\alpha)}\, \overline{\alpha(g)}\, d\alpha\,.$$

Thus, we can let $[\varphi, \psi](\alpha) = \widehat{\varphi}(\alpha)\overline{\widehat{\psi}(\alpha)}$, which satisfies the conditions of dual integrability.

Remark 2.1. The unitary representation $h \longrightarrow R_h$ of G on $L^2(G, dg)$ given in Example 2.2 is equivalent to the unitary representation $g \longrightarrow \mathcal{M}_g$ of G on $L^2(\widehat{G}, d\alpha)$ of Example 2.1 via de Fourier map $\mathcal{F}_G$. To see this observe that

$$\mathcal{F}_G R_h \psi(\alpha) = \int_G \psi(g+h)\overline{\alpha(g)}\, dg = \alpha(h) \int_G \psi(g+h)\overline{\alpha(g+h)}\, dg = \mathcal{M}_h \mathcal{F}_G \psi(\alpha).$$

2.2. **Properties of the bracket.** In order to deduce some properties of the bracket we recall the following theorem (see [13], page 147 or Theorem 4.44 in [2], page 105).

Theorem 2.2. *Let T be a unitary representation of a LCA group G on a Hilbert space H.*

i) There is a regular measure P on $\widehat{G}$ with values in the set of self-adjoint projection operators on H such that

$$T_g = \int_{\widehat{G}} \alpha(g)\, dP(\alpha)\,, \qquad \textit{for all } g \in G\,.$$

ii) For each $\varphi, \psi \in H$, the function $\mu_{\varphi,\psi}(S) =< P(S)\varphi, \psi >=< \varphi, P(S)\psi >=< P(S)\varphi, P(S)\psi >$ defines a complex measure on $\widehat{G}$ such that

$$< T_{-g}\varphi, \psi >=< \varphi, T_g \psi >= \int_{\widehat{G}} \overline{\alpha(g)}\, d\mu_{\varphi,\psi}(\alpha)\,, \qquad \textit{for all } g \in G\,.$$

Corollary 2.3. *Let T be a unitary representation of a LCA group G on a Hilbert space H. The following are equivalent:*

i) T is dual integrable.

ii) For each $\varphi, \psi \in H$, the measure $\mu_{\varphi,\psi}$ defined in Theorem 2.2 is absolutely continuous with respect to $d\alpha$.

In this situation, the bracket $[\varphi, \psi]$ is the Radon-Nikodym derivative of $\mu_{\varphi,\psi}$.

Proof. ii) $\Rightarrow$ i). If $\mu_{\varphi,\psi}$ is absolutely continuous with respect to $d\alpha$ we write $d\mu_{\varphi,\psi}(\alpha) = [\varphi, \psi](\alpha)\, d\alpha$. The result follows from part *ii*) of Theorem 2.2.

i) $\Rightarrow$ ii). Let $d\nu_{\varphi,\psi}(\alpha) = d\mu_{\varphi,\psi}(\alpha) - [\varphi, \psi](\alpha)\, d\alpha$. It follows that $\nu_{\varphi,\psi}$ is a bounded regular measure whose Fourier transform satisfies

$$\begin{aligned} \mathcal{F}_{\widehat{G}}(\nu_{\varphi,\psi}(g) &= \int_{\widehat{G}} \overline{\alpha(g)}\, d\mu_{\varphi,\psi}(\alpha)\, d\alpha - \int_{\widehat{G}} \overline{\alpha(g)}\, [\varphi, \psi](\alpha)\, d\alpha \\ &= < \varphi, T_g \psi > - < \varphi, T_g \psi >= 0 \end{aligned}$$

for all $g \in G$. By the uniqueness theorem for the Fourier transform (see, for example, page 103 of [2]) we conclude that $\nu_{\varphi,\psi} = 0$ showing that $\mu_{\varphi,\psi}$ is absolutely continuous with respect to $d\alpha$. □

Corollary 2.4. *Suppose T is a dual integrable unitary representation of a LCA group G on a Hilbert space H. Then $[\varphi, \psi] : H \times H \to L^1(\widehat{G}, d\alpha)$ is a sesquilinear form, hermitian symmetric map, with the following properties:*

i) Positivity: $[\varphi, \varphi](\alpha) \geq 0$ a.e. $\alpha \in \widehat{G}$ for all $\varphi \in H$.

ii)(Cauchy-Schwartz): $|[\varphi, \psi](\alpha)| \leq ([\varphi, \varphi](\alpha))^{1/2}\, ([\psi, \psi](\alpha))^{1/2}$ a.e. $\alpha \in \widehat{G}$ for all $\varphi, \psi \in H$.

iii) $\|[\varphi, \psi]\|_{L^1(\widehat{G})} \leq \|\varphi\|_H\, \|\psi\|_H$, for all $\varphi, \psi \in H$.

Proof. For each measurable set $S \in \widehat{G}$, $\mu_{\varphi,\psi}(S)$ is linear in φ, conjugate-linear in ψ, and $\overline{\mu_{\varphi,\psi}(S)} = \mu_{\psi,\varphi}(S)$. It follows that $[\varphi, \psi]$ has the same properties.

To prove *i*) observe that $\mu_{\varphi,\varphi}(S) =< P(S)\varphi, P(S)\varphi >= \|P(S)\varphi\|^2 \geq 0$ by *ii*) of Theorem 2.2. Thus, $[\varphi, \varphi](\alpha) \geq 0$ a.e. $\alpha \in \widehat{G}$.

Again by *ii*) of Theorem 2.2, and the Cauchy-Schwartz inequality of the Hilbert space H, for a measurable set $S \in \widehat{G}$ we have $|\mu_{\varphi,\psi}(S)| = |< P(S)\varphi, P(S)\psi >| \leq \|P(S)\varphi\|\|P(S)\psi\| = (\mu_{\varphi,\varphi}(S))^{(1/2)}\,(\mu_{\psi,\psi}(S))^{(1/2)}$. Part *ii*) follows immediately.

Part *iii*) follows from part *ii*) together with the Cauchy-Schwartz inequality for functions in $L^2(\widehat{G})$ and the fact that $\int_{\widehat{G}}[\varphi, \varphi](\alpha)\, d\alpha = \|\varphi\|^2$ (Notice that this last equality follows from (2.4) with $g = 0$ and $\psi = \varphi$.) □

Corollary 2.5. *Suppose T is a dual integrable unitary representation of a LCA group G on a Hilbert space H.*

i) For $g \in G$ and $\varphi, \psi \in H$ we have

$$[T_g\varphi, \psi](\alpha) = \alpha(g)[\varphi, \psi](\alpha) = [\varphi, T_{-g}\psi](\alpha) \quad a.e.\ \widehat{G}.$$

ii) Let Γ be a finite subset of G and $\varphi, \psi \in H$. For $P_\Gamma(\alpha) = \sum_{g\in\Gamma} a_g\alpha(g)$ a trigonometric polynomial on $\widehat{G}$ and $P_\Gamma(T) = \sum_{g\in\Gamma} a_gT_g$ we have

$$[P_\Gamma(T)\varphi, \psi](\alpha) = P_\Gamma(\alpha)[\varphi, \psi] = [\varphi, \overline{P_\Gamma(T)}\psi](\alpha) \quad a.e.\ \widehat{G}\,,$$

and

$$[P_\Gamma(T)\psi, P_\Gamma(T)\psi](\alpha) = |P_\Gamma(\alpha)|^2[\psi, \psi](\alpha) \quad a.e.\ \widehat{G}.$$

Proof. For $g, k \in G$, using (2.4) we obtain

$$< T_g\varphi, T_k\psi >=< \varphi, T_{k-g}\psi >= \int_{\widehat{G}} [\varphi, \psi](\alpha)\, \overline{\alpha(k-g)}\, d\alpha = \alpha(g) \int_{\widehat{G}} [\varphi, \psi](\alpha)\, \overline{\alpha(k)}\, d\alpha\,.$$

Also using (2.4) we deduce

$$< T_g\varphi, T_k\psi >= \int_{\widehat{G}} [T_g\varphi, \psi](\alpha)\, \overline{\alpha(k)}\, d\alpha\,.$$

By the uniqueness theorem for the Fourier transform on $\widehat{G}$ we obtain $[T_g\varphi, \psi](\alpha) = \alpha(g)\,[\varphi, \psi](\alpha)$ a.e. $\alpha \in G$. The rest of the properties follow from the linearity and the sesquilinearity of the bracket. □

Lemma 2.6. *Let $\varphi, \psi \in H$. Then, $\varphi \perp< \psi >$ if and only if $[\varphi, \psi](\alpha) = 0$ a.e. $\alpha \in \widehat{G}$.*

Proof. The perpendicularity of φ and $< \psi >$ is equivalent to $< \varphi, T_g\psi >= 0$ for all $g \in G$. By (2.4) this is equivalent to $\int_{\widehat{G}}[\varphi, \psi](\alpha)\overline{\alpha(g)}\, d\alpha = 0$ for all $g \in G$. The uniqueness theorem for the Fourier transform shows that this is equivalent to $[\varphi, \psi] = 0$ in $L^2(\widehat{G}, d\alpha)$. □

Lemma 2.7. *Let $\Omega_\psi = \{\alpha \in \widehat{G} : [\psi, \psi](\alpha) > 0\}$ (well defined a.e. $\alpha \in \widehat{G}$). If $\varphi, \psi \in H$, then $[\varphi, \psi](\alpha) = 0$ a.e. $\alpha \in \Omega^c_\psi$.*

Proof. By *ii*) of Corollary 2.4

$$0 \leq \int_{\Omega^c_\psi} |\,[\varphi, \psi](\alpha)|\, d\alpha \leq \int_{\Omega^c_\psi} ([\varphi, \varphi](\alpha))^{1/2}\, ([\psi, \psi](\alpha))^{1/2}\, d\alpha = 0\,,$$

since $[\psi, \psi](\alpha) = 0$ a.e. on Ω^c_ψ. □

Remark 2.8. There are equivalent characterizations of dual integrability. This follows from Theorem 3.2 below. Let T be an unitary representation of a LCA group G on a separable Hilbert space H. Then, the following are equivalent:

i) T is dual integrable as defined in Section 2.1.

ii) T is unitarily equivalent to a subrepresentation of countably many copies of the regular representation R of G on $L^2(G)$.

iii) T is **square integrable** in the sense that for each $\psi \in H \setminus \{0\}$, there exist $\psi_0 \in <\psi>$ with $<\psi_0> = <\psi>$ for which the map $(W_{\psi_0}\varphi)(g) = <\varphi, T_g\psi_0>$ defines an isometry from $<\psi>$ into $L^2(G, dg)$.

In practice, it is very difficult to check directly whether or not a given unitary representation T of G is either square integrable or a subrepresentation of countably many copies of the regular representation R. As we shall see in sections 4 and 7 for concrete representations T on L^2 spaces, there is a natural way to check integrability of T by introducing appropriate bracket functions. In the special case when G is a countable group we can use the brackets to read off properties of the generating sets $\{T_g\psi : g \in G\}$ inside $<\psi>$ (see Sections 5 and 6).

3. An isometric isomorphism

This section is dedicated to show that there is a linear one-to-one isometry from the space $<\psi>$ onto the weighted space $L^2(\widehat{G}, [\psi,\psi](\alpha)\, d\alpha)$.

Theorem 3.1. *Let $g \longrightarrow T_g$ be a dual integrable unitary representation of a LCA group G on a Hilbert space H. For $\psi \in H \setminus \{0\}$ define $\Omega_\psi = \{\alpha \in \widehat{G} : [\psi,\psi](\alpha) > 0\}$ (well defined a.e. $\alpha \in \widehat{G}$).*

i) The map

$$S_\psi(\varphi) = \mathbf{1}_{\Omega_\psi} \frac{[\varphi,\psi]}{[\psi,\psi]}, \qquad \varphi \in H\,,$$

is a linear one-to-one isometry from $<\psi> = \overline{span\{T_g\psi : g \in G\}}^{\,H}$ onto the weighted space $L^2(\widehat{G}, [\psi,\psi](\alpha)\, d\alpha)$.

ii) For $g \in G$ and $\varphi, \psi \in H$

$$S_\psi(T_g\varphi)(\alpha) = \alpha(g)\, S_\psi(\varphi)(\alpha) \quad a.e.\ \alpha \in \widehat{G}\,.$$

Proof. $i)$ For $\varphi \in H$ using the definition of S_ψ and $ii)$ of Corollary 2.4 we have,

$$\begin{aligned}
&\int_{\widehat{G}} |S_\psi(\varphi)(\alpha)|^2\, [\psi,\psi](\alpha)\, d\alpha = \int_{\Omega_\psi} \left| \frac{[\varphi,\psi](\alpha)}{[\psi,\psi](\alpha)} \right|^2 [\psi,\psi](\alpha)\, d\alpha \\
&\le \int_{\Omega_\psi} \frac{[\varphi,\varphi](\alpha)\, [\psi,\psi](\alpha)}{([\psi,\psi](\alpha))^2}\, [\psi,\psi](\alpha)\, d\alpha = \int_{\Omega_\psi} [\varphi,\varphi](\alpha)\, d\alpha \\
&\le \int_{\widehat{G}} [\varphi,\varphi](\alpha)\, d\alpha = <\varphi,\varphi>_H = \|\varphi\|_H^2\,,
\end{aligned} \tag{3.1}$$

where the next to the last equality is due to the definition of the bracket given in (2.4). This shows that S_ψ maps H into $L^2(\widehat{G}, [\psi,\psi](\alpha)\, d\alpha)$.

We now show that S_ψ is an isometry when defined in $<\psi>$ (then the one-to-one property follows). For $\varphi = \sum_{h\in\Gamma} a_h T_h \psi$, $\Gamma \subset G$ finite, by part $ii)$ of Corollary 2.5 we deduce

$$\begin{aligned}\int_{\widehat{G}} |S_\psi(\varphi)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha &= \int_{\widehat{G}} |\sum_{h\in\Gamma} a_h\,\alpha(h)|^2\,[\psi,\psi](\alpha)\,d\alpha \\ &= \int_{\widehat{G}} [\varphi,\varphi](\alpha)\,d\alpha = \|\varphi\|_H^2\,, \qquad (3.2)\end{aligned}$$

where the last equality follows from (2.4). For a general $\varphi \in <\psi>$, take $\varepsilon > 0$ and choose $\varphi_\varepsilon = \sum_{h\in\Gamma_\varepsilon} a_h T_h\psi$, $\Gamma_\varepsilon \subset G$ finite, such that $\|\varphi - \varphi_\varepsilon\|_H \le \varepsilon$. Then we have

$$\|\varphi\|_H \le \|\varphi-\varphi_\varepsilon\|_H + \|\varphi_\varepsilon\|_H \le \varepsilon + \|\varphi_\varepsilon\|_H\,. \qquad (3.3)$$

Also, by (3.2) and the triangle inequality we obtain

$$\begin{aligned}&\|\varphi_\varepsilon\|_H = (\int_{\widehat{G}} |S_\psi(\varphi_\varepsilon)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha)^{1/2} \\ \le\ & (\int_{\widehat{G}} |S_\psi(\varphi_\varepsilon-\varphi)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha)^{1/2} + (\int_{\widehat{G}} |S_\psi(\varphi)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha)^{1/2} \\ \le\ & \|\varphi_\varepsilon-\varphi\|_H + (\int_{\widehat{G}} |S_\psi(\varphi)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha)^{1/2}\,,\end{aligned}$$

where the last inequality is due to (3.1). Use (3.3) to deduce

$$(\int_{\widehat{G}} |S_\psi(\varphi)(\alpha)|^2\,[\psi,\psi](\alpha)\,d\alpha)^{1/2} \ge \|\varphi_\varepsilon\|_H - \varepsilon \ge \|\varphi\|_H - 2\varepsilon\,.$$

Letting $\varepsilon \to 0$ we obtain $\|S_\psi(\varphi)\|_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} \ge \|\varphi\|_H$, while the reverse inequality is proved in (3.1).

It remains to show that the map S_ψ is onto. Suppose that we have the strict inclusion $S_\psi(<\psi>) \subsetneq L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)$, so that we can choose $m \neq 0$, $m \in L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)$ such that $m \perp S_\psi(<\psi>)$ on $L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)$. Thus, for all $h \in G$, $0 = \int_{\widehat{G}} m(\alpha)\,S_\psi(T_h\psi)(\alpha)\,[\psi,\psi](\alpha)\,d\alpha$. By part i) of Corollary 2.5

$$S_\psi(T_h\psi)(\alpha) = \mathbf{1}_{\Omega_\psi}(\alpha)\frac{[T_h\psi,\psi](\alpha)}{[\psi,\psi](\alpha)} = \mathbf{1}_{\Omega_\psi}(\alpha)\,\alpha(h)\,.$$

Thus, $0 = \int_{\widehat{G}} m(\alpha)\,\alpha(h)\,[\psi,\psi](\alpha)\,d\alpha$ for all $h \in G$. By the uniqueness theorem for the Fourier transform, $m(\alpha)\,[\psi,\psi](\alpha) = 0$ a.e. $\alpha \in \widehat{G}$. This implies $m = 0$ in $L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)$, contradicting our assumption.

ii) This property follows from part i) of Corollary 2.5. □

Another way to write this theorem without using the space $L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)$ is the following. Its proof is similar to the proof of Theorem 3.1.

Theorem 3.2. *Let $g \longrightarrow T_g$ be a dual integrable unitary representation of a LCA group G on a Hilbert space H. For $\psi \in H \setminus \{0\}$ define $\Omega_\psi = \{\alpha \in \widehat{G} : [\psi,\psi](\alpha) > 0\}$ (well defined for a.e. $\alpha \in \widehat{G}$).*

i) The map

$$J_\psi(\varphi) = \mathbf{1}_{\Omega_\psi}\frac{[\varphi,\psi]}{([\psi,\psi])^{1/2}}, \qquad \varphi \in H\,,$$

is a linear one-to-one isometry from $<\psi> = \overline{span\{T_g\psi : g \in G\}}^{H}$ onto the space $L^2(\Omega_\psi, d\alpha) \subset L^2(\widehat{G}, d\alpha)$.

ii) For $g \in G$ and $\varphi, \psi \in H$

$$J_\psi(T_g\varphi)(\alpha) = \alpha(g)\, J_\psi(\varphi)(\alpha) \quad (a.e.)\alpha \in \widehat{G}\,.$$

We illustrate Theorem 3.1 in the particular cases of Examples 2.1 and 2.2. For Example 2.1 the map S_ψ of Theorem (3.1) is given by

$$S_\psi(\varphi) = \mathbf{1}_{\Omega_\psi} \frac{\varphi\,\overline{\psi}}{\psi\,\overline{\psi}} = \mathbf{1}_{\Omega_\psi} \frac{\varphi}{\psi}, \qquad \varphi,\, \psi \in L^2(\widehat{G}, d\alpha)\,,$$

where $\Omega_\psi = \{\alpha : \widehat{G} : |\psi(\alpha)| > 0\}$. This map is a linear one-to-one isometry from $<\psi> = \overline{\mathrm{span}\{\mathcal{M}_g\psi : g \in G\}}^{L^2(\widehat{G},d\alpha)}$ onto the weighted space $L^2(\widehat{G}, |\psi(\alpha)|^2\, d\alpha)$. It is easy to compute the inverse map which is given by $S_\psi^{-1}(m) = m\,\psi$, $m \in L^2(\widehat{G}, |\psi(\alpha)|^2\, d\alpha)$.

For Example 2.2 the map S_ψ of Theorem (3.1) is given by

$$S_\psi(\varphi) = \mathbf{1}_{\Omega_\psi} \frac{\widehat{\varphi}\,\overline{\widehat{\psi}}}{\widehat{\psi}\,\overline{\widehat{\psi}}} = \mathbf{1}_{\Omega_\psi} \frac{\widehat{\varphi}}{\widehat{\psi}}, \qquad \varphi,\, \psi \in L^2(G, dg)\,,$$

where $\Omega_\psi = \{\alpha : \widehat{G} : |\widehat{\psi}(\alpha)| > 0\}$. This map is a linear one-to-one isometry from $<\psi> = \overline{\mathrm{span}\{R_g\psi : g \in G\}}^{L^2(G,dg)}$ onto the weighted space $L^2(\widehat{G}, |\widehat{\psi}(\alpha)|^2\, d\alpha)$. It is easy to compute the inverse map which is given by $S_\psi^{-1}(m) = (m\,\widehat{\psi})^\vee$, $m \in L^2(\widehat{G}, |\widehat{\psi}(\alpha)|^2\, d\alpha)$.

4. Translations and Gabor systems in $L^2(\mathbb{R}^n)$

4.1. **Integer translations in $L^2(\mathbb{R}^n)$.** The map $k \longrightarrow T_k$ given by $T_k f(x) = f(x+k)$ is a unitary representation of the LCA group $(\mathbb{Z}^n, +)$ on the Hilbert space $L^2(\mathbb{R}^n)$. We will show in this section that this unitary representation is dual integrable. To see this choose $\varphi\,, \psi \in L^2(\mathbb{R}^n)$ and use that $\{[0,1]^n + \ell : \ell \in \mathbb{Z}^n\}$ is an almost everywhere partition of $\mathbb{R}^n$ to deduce

$$\begin{aligned} <\varphi, T_k\psi>_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \varphi(x)\,\overline{\psi(x+k)}\,dx = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\,\overline{\widehat{\psi}(\xi)} e^{-2\pi i k\cdot\xi}\,d\xi \\ &= \sum_{\ell\in\mathbb{Z}^n} \int_{[0,1]^n+\ell} \widehat{\varphi}(\xi)\,\overline{\widehat{\psi}(\xi)} e^{-2\pi i k\cdot\xi}\,d\xi \\ &= \int_{[0,1]^n} \left\{ \sum_{\ell\in\mathbb{Z}^n} \widehat{\varphi}(\xi+\ell)\,\overline{\widehat{\psi}(\xi+\ell)} \right\} e^{-2\pi i k\cdot\xi}\,d\xi\,. \end{aligned} \tag{4.1}$$

Comparing (4.1) with (2.4) of the definition of dual integrability we must choose

$$[\varphi,\psi](\xi) = \sum_{\ell\in\mathbb{Z}^n} \widehat{\varphi}(\xi+\ell)\,\overline{\widehat{\psi}(\xi+\ell)}\,, \qquad \xi \in \mathbb{R}^n\,, \quad \varphi\,, \psi \in L^2(\mathbb{R}^n)\,. \tag{4.2}$$

The function $[\psi,\psi](\xi)$ is the function P_ψ used in [10] in the case $n = 1$.

Remark 4.1. Cyclic subspaces of this representation are usually called principal **shift** invariant subspaces of $L^2(\mathbb{R}^n)$.

In this case we can describe the inverse of the linear isometry S_ψ, $\psi \in L^2(\mathbb{R}^n)$, from $<\psi>$ onto $L^2([0,1]^n, [\psi,\psi](\xi)\, d\xi)$ (We have identified the dual of $\mathbb{Z}^n$ with $[0,1]^n$). In fact, if $m \in L^2([0,1]^n, [\psi,\psi](\xi)\, d\xi)$ we have

$$[(m\,\widehat{\psi})^\vee\,,\psi](\xi) = \sum_{\ell\in\mathbb{Z}^n} m(\xi+\ell)\,\widehat{\psi}(\xi+\ell)\,\overline{\widehat{\psi}(\xi+\ell)} = m(\xi)\,[\psi,\psi](\xi)\,, \qquad \xi\in\mathbb{R}^n\,. \quad (4.3)$$

Thus, if we want to find $\varphi \in L^2(\mathbb{R}^n)$ such that $S_\psi(\varphi) = m$ we must have (see the definition of S_ψ given in Theorem 3.1) $[\varphi,\psi](\xi) = [\psi,\psi](\xi)\, m(\xi) = [(m\,\widehat{\psi})^\vee\,,\psi](\xi)$ for all $\xi \in [0,1]$(by (4.3)). Thus $S_\psi^{-1}(m) = (m\,\widehat{\psi})^\vee$ is the inverse map. This is the map J_ψ used in [10] in the case $n = 1$.

4.2. **Gabor systems in $L^2(\mathbb{R}^n)$.** The Gabor representation of the (discrete) LCA group $(\mathbb{Z}^n \times \mathbb{Z}^n, +)$ on $L^2(\mathbb{R}^n)$ is given by

$$(k,l) \longrightarrow T_k\, M_\ell\, f(x) = f(x-k)\, e^{2\pi i\ell\cdot x}\,, \qquad x\in\mathbb{R}^n\,,\ f\in L^2(\mathbb{R}^n)\,. \quad (4.4)$$

Observe that since $k,\ell \in \mathbb{Z}^n$ the translations T_k and the modulations M_ℓ commute. It is clear that this is a unitary representation. We will show in this section that this representation is dual integrable in the sense defined in section 2. For $f, g \in L^2(\mathbb{R}^n)$, periodizing $\mathbb{R}^n$ with the integer translates of $[0,1]^n$, we obtain

$$< f,g >_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\,dx = \int_{[0,1]^n} \{\sum_{\ell\in\,\mathbb{Z}^n} f(x+\ell)\,\overline{g(x+\ell)}\}\,dx\,.$$

Since $\{e^{2\pi ik\xi} : k \in Z^n\}$ is an orthonormal system of $L^2([0,1]^n)$ we obtain

$$\begin{aligned} < f,g >_{L^2(\mathbb{R}^n)} &= \int_{[0,1]^n} \sum_{\ell\in\,\mathbb{Z}^n}\sum_{k\in\,\mathbb{Z}^n} f(x+\ell)\,\overline{g(x+k)}\,\left(\int_{[0,1]^n} e^{2\pi i(\ell-k)\xi}\,d\xi\right)dx \\ &= \int_{[0,1]^n}\int_{[0,1]^n} \left\{\sum_{\ell\in\,\mathbb{Z}^n} f(x+\ell)\,e^{2\pi i\ell\xi}\right\}\overline{\left\{\sum_{k\in\,\mathbb{Z}^n} g(x+k)\,e^{2\pi ik\xi}\right\}}\,dx\,d\xi \\ &= \int_{[0,1]^n}\int_{[0,1]^n} Zf(x,\xi)\,\overline{Zg(x,\xi)}\,dx\,d\xi\,. \qquad (4.5)\end{aligned}$$

where

$$Zf(x,\xi) = \sum_{k\in\mathbb{Z}^n} f(x+k)\,e^{2\pi ik\cdot\xi} \quad (4.6)$$

is the Zak transform of the function $f \in L^2(\mathbb{R}^n)$. The Zak transform has been used in several articles. It was studied by J. Zak in connection with solid state physics ([19]). Some of the most recent results about the Zak transform have been obtained by A.J.E.M. Janssen ([11]). A detailed study of the Zak transform and its properties can be found in [12] and Chapter 8 of [4].

Notice that (4.5) shows that the Zak transform is an isometry from $L^2(\mathbb{R}^n)$ into $L^2([0,1]^n \times [0,1]^n)$.

If $k, \ell \in \mathbb{Z}^n$, and $\psi \in L^2(\mathbb{R}^n)$ we have

$$Z(T_k\, M_l\psi)(x,\xi) \;=\; \sum_{j\in\mathbb{Z}^n} T_k\, M_l\psi(x+j)\,e^{2\pi ij\xi}$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}^n} e^{2\pi i\ell x}\psi(x+j-k)\, e^{2\pi i j\,\xi} \\
&= e^{2\pi i\ell x}\, e^{2\pi i k\,\xi}\, Z\psi(x,\xi)\,. \qquad (4.7)
\end{aligned}
$$

From (4.5) and (4.7) applied to $\varphi, \psi \in L^2(\mathbb{R}^n)$ we deduce

$$
\begin{aligned}
< \varphi, T_k\, M_l\psi >_{L^2(\mathbb{R}^n)} &= \int_{[0,1]^n}\int_{[0,1]^n} Z\varphi(x,\xi)\,\overline{Z(T_k\, M_l\psi)(x,\xi)}\, dx\, d\xi \\
&= \int_{[0,1]^n}\int_{[0,1]^n} Z\varphi(x,\xi)\,\overline{Z\psi(x,\xi)}\, e^{-2\pi i\ell x}\, e^{-2\pi i k\xi}\, dx\, d\xi \quad (4.8)
\end{aligned}
$$

obtaining

$$
[\varphi,\psi](x,\xi) = Z\varphi(x,\xi)\,\overline{Z\psi(x,\xi)}\,. \qquad (4.9)
$$

Property (2.3) of the bracket follows from the fact that Z is an isometry on $L^2(\mathbb{R}^n)$ (see (4.5)).

Remark 4.2. The expression (4.6) is a formal definition. It is obviously well defined for f continuous with compact support. To see that it is defined for every function $f \in L^2(\mathbb{R}^n)$ we observe that the collection $\{T_k\, M_\ell\, \mathbf{1}_{[0,1]^n} : k, \ell \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ and that

$$
Z(T_k\, M_\ell\, \mathbf{1}_{[0,1]^n})(x,\xi) = e^{2\pi i k\cdot\xi}\, e^{2\pi\ell\cdot x}
$$

is an orthonormal basis of $L^2([0,1]^n \times [0,1]^n)$. If follows from this observation that the Zak transform Z is well defined in $L^2(\mathbb{R}^n)$. Using (4.8) it is easy to see that the closed subspace $< \psi >$ generated by the collection $\{T_k\, M_\ell\,\psi : k, \ell \in \mathbb{Z}^n\}$ coincides with $L^2(\mathbb{R}^n)$ if and only if $|\ Z\psi(x,\xi)| > 0$ a. e. $(x,\xi) \in [0,1]^n \times [0,1]^n$.

Remark 4.3. The Zak transform is $\mathbb{Z}^n$-periodic in the variable $\xi \in \mathbb{R}^n$. But, for $\ell \in \mathbb{Z}^n$ we have

$$
Zf(x+\ell,\xi) = \sum_{k\in\,\mathbb{Z}^n} f(x+\ell+k)\, e^{2\pi i k\cdot\xi} = \sum_{s\in\,\mathbb{Z}^n} f(x+s)\, e^{2\pi i(s-\ell)\cdot\xi} = e^{-2\pi i\ell\cdot\xi}\, Zf(x,\xi)\,.
$$

Thus, although Zf is not $\mathbb{Z}^n \times \mathbb{Z}^n$-periodic, the function $[\varphi,\psi] = Z\varphi\,\cdot Z\psi$ defined in (4.9) is $\mathbb{Z}^n \times \mathbb{Z}^n$-periodic.

In the case under consideration we can find the inverse of the map S_ψ given in Theorem 3.1. In fact, the computations

$$
S_\psi\varphi(x,\xi) = \mathbf{1}_{\Omega_\psi}(x,\xi)\,\frac{[\varphi,\psi](x,\xi)}{[\varphi,\psi](x,\xi)} = \mathbf{1}_{\Omega_\psi}(x,\xi)\,\frac{Z\varphi(x,\xi)}{Z\psi(x,\xi)} = m(x,\xi)
$$

tells us that for $m \in L^2([0,1]^n \times [0,1]^n, |Z\psi|^2)$ we have $S_\psi^{-1}(m) = Z^{-1}(m\, Z\psi)$, which is a linear isometry from $L^2([0,1]^n \times [0,1]^n, |Z\psi|^2)$ onto the space $L^2(\mathbb{R}^n)$ when $|Z\psi(x,\xi)| > 0$ a.e. $(x,\xi) \in [0,1]^n \times [0,1]^n$.

5. Properties of the set of generators

In this section we consider a **countable** abelian group $(G, +)$ equipped with the discrete topology and a dual integrable unitary representation $k \to T_k$ of G on a Hilbert spaces H. For $\psi \in H \backslash \{0\}$ the cyclic T-invariant subspace $< \psi >= \overline{\text{span}\{T_k\psi : k \in G\}}$ has, by definition, the countable set $\{T_k\psi : k \in G\}$ as a set of generators. We seek to find characterizations of properties of this set in terms of the bracket $[\cdot, \cdot]$ defined in section 2.

One of these properties is the **orthonormality**, meaning that $< T_k\psi, T_\ell\psi >= \delta_{k,\ell}$ for all $k, \ell \in G$. Since each T_k is a unitary operator, this is equivalent to $< T, T_k\psi >= \delta_k$ for all $k \in G$. If this is the case, property (2.4) of the definition of integrability implies that all the Fourier coefficients of the bracket $[\psi, \psi]$, as a function in $L^1(\widehat{G}, d\alpha)$, are zero, except the one corresponding to $k = 0$. Hence, $[\psi, \psi](\alpha) = 1$ a.e. in $\widehat{G}$. Conversely, if $[\psi, \psi](\alpha) = 1$ a.e. in $\widehat{G}$ we have $< \psi, T_k\psi >= \int_{\widehat{G}} \overline{\alpha(k)}\, d\alpha = \delta_k$ for all $k \in G$ (for the last equality see [15], Chapter 1.)

We have proved:

Proposition 5.1. *Let $(G, +)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$. The collection $\{T_k\psi : k \in G\}$ is an orthonormal basis for $< \psi >$ if and only if $[\psi, \psi](\alpha) = 1$ a.e. $\alpha \in \widehat{G}$.*

Notice that by *ii*) of Theorem 3.1

$$S_\psi(T_k\psi) = \mathbf{1}_{\Omega_\psi} \frac{[T_k\psi, \psi]}{[\psi, \psi]} = \mathbf{1}_{\Omega_\psi}\, e_k\,, \qquad k \in G\,, \tag{5.1}$$

where $e_k(\alpha) = \alpha(k)$ for all $\alpha \in \widehat{G}$. Thus, the properties of the collection $\{T_k\psi : k \in G\}$ inside $< \psi > \subset H$ can be read from the properties of the collection $\{\mathbf{1}_{\Omega_\psi}\, e_k : k \in G\}$ inside $L^2(\widehat{G}, [\psi, \psi](\alpha)\, d\alpha)$. Furthermore, the properties of this collection are tied to the behaviour of the bracket $[\psi, \psi]$, defined in $\widehat{G}$. We take this point of view in the next propositions.

Proposition 5.2. *Let $(G, +)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$. The collection $\{T_k\psi : k \in G\}$ is a Riesz basis for $< \psi >$ with constants $A > 0$ and $B < \infty$ if and only if*

$$A \le [\psi, \psi](\alpha) \le B \text{ a.e. } \alpha \in \widehat{G}\,.$$

Proof. Recall that the collection $\{T_k\psi : k \in G\}$ is a Riesz basis for $< \psi >$ with constants A and B if is a basis and if for all sequences $\{a_k : k \in G\} \in \ell^2(G)$ we have

$$A \sum_{k\in G} |a_k|^2 \le \| \sum_{k\in G} a_k\, T_k\psi\|_H^2 \le B \sum_{k\in G} |a_k|^2\,. \tag{5.2}$$

Suppose $A \le [\psi, \psi](\alpha) \le B$ a.e. $\alpha \in \widehat{G}$. Then $L^2(\widehat{G}, [\psi, \psi](\alpha)\, d\alpha) \approx L^2(\widehat{G}, d\alpha)$. Since $\{e_k : k \in G\}$ is an orthonormal basis for $L^2(\widehat{G}, d\alpha)$, by (5.1) and the fact that S_ψ is an isometric isomorphism we conclude that $\{T_k\psi : k \in G\}$ is a basis for $< \psi >$.

To prove (5.2) take $\{a_k : k \in G\} \in \ell^2(G)$ and use Theorem 3.1 together with (5.1) to obtain

$$\|\sum_{k\in G} a_k\, T_k\psi\|_H^2 = \|S_\psi(\sum_{k\in G} a_k\, T_k\psi)\|^2_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} = \|\sum_{k\in G} a_k\, e_k\|^2_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} \tag{5.3}$$

since $\Omega_\psi = \widehat{G}$ a.e. by our assumption. Since $L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha) \approx L^2(\widehat{G}, d\alpha)$ in our situation, use that $\{e_k : k \in G\}$ is an orthonormal basis of $L^2(\widehat{G}, d\alpha)$ to deduce

$$\|\sum_{k\in G} a_k\, T_k\psi\|_H^2 \approx \int_{\widehat{G}} |\sum_{k\in G} a_k\, e_k(\alpha)|^2\, d\alpha = \sum_{k\in G} |a_k|^2\,,$$

where the equivalence in the above expressions is up to the Riesz basis constants A and B.

Suppose now that $\{T_k\psi : k \in G\}$ is a Riesz basis for $< \psi >$ with constants $A > 0$ and $B < \infty$. Use (5.1) and Theorem 3.1 to obtain (as in the proof of (5.3)):

$$\|\sum_{k\in G} a_k\, T_k\psi\|_H^2 = \int_{\widehat{G}} |\sum_{k\in G} a_k\, \mathbf{1}_{\Omega_\psi}(\alpha)\, e_k(\alpha)|^2\, [\psi,\psi](\alpha)\, d\alpha\,. \tag{5.4}$$

By (5.2) we have

$$A\sum_{k\in G}|a_k|^2 \le \int_{\widehat{G}} |\sum_{k\in G} a_k\, \mathbf{1}_{\Omega_\psi}(\alpha) e_k(\alpha)|^2\, [\psi,\psi](\alpha)\, d\alpha \le B\sum_{k\in G}|a_k|^2 \tag{5.5}$$

Suppose, contrary to what we want to prove, that $[\psi,\psi](\alpha) < A$ on a set $E \subset \widehat{G}$ with $0 < \int_{\widehat{G}} \mathbf{1}_E(\alpha)\, d\alpha < \infty$. Since $\mathbf{1}_E \in L^2(\widehat{G}, d\alpha)$ we have $\mathbf{1}_E = \sum_{k\in G} a_k\, e_k$ with $\int_{\widehat{G}} |\mathbf{1}_E|^2 = \sum_{k\in G}|a_k|^2$. Then,

$$\begin{aligned}\int_{\widehat{G}} |\sum_{k\in G} a_k\, \mathbf{1}_\Omega(\alpha)\, e_k(\alpha)|^2\, [\psi,\psi](\alpha)\, d\alpha &= \int_{\widehat{G}} |\mathbf{1}_E(\alpha)|^2\, [\psi,\psi](\alpha)\, d\alpha \\ = \int_E [\psi,\psi](\alpha)\, d\alpha < A\int_{\widehat{G}} |\mathbf{1}_E(\alpha)|^2\, d\alpha &= A\sum_{k\in G}|a_k|^2\,,\end{aligned}$$

contrary to (5.5). This proves the left hand side inequality of (5.2). To prove the right hand side inequality proceed similarly by assuming $[\psi,\psi](\alpha) > B$ on a set of finite positive measure, to obtain a contradiction. □

Remark 5.3. Following Chapter 2, section 11, of [16] we say that the collection $\{T_k\psi : k \in G\}$ has the **Besselian** property if, when $\sum_{k\in G} a_k\, T_k\psi$ converges in $< \psi >$, then $\sum_{k\in G}|a_k|^2 < \infty$. Theorem 11.1 in [16] shows that this property is equivalent to the existence of a constant $A > 0$ such that

$$A\sum_{k\in G}|a_k|^2 \le \|\sum_{k\in G} a_k\, T_k\psi\|_H^2$$

for all finite sequences $\{a_k : k \in G\}$. Arguing as in the proof of Proposition 5.2 it can be shown that $\{T_k\psi : k \in G\}$ has the Besselian property in $< \psi >$ if and only if $[\psi,\psi](\alpha) \ge A$ a.e. $\alpha \in \widehat{G}$.

Remark 5.4. Also according to Chapter 2, section 11, of [16] we say that the collection $\{T_k\psi : k \in G\}$ has the **Hilbertian** property if, when $\sum_{k\in G}|a_k|^2 < \infty$, then

$\sum_{k\in G} a_k\, T_k\psi$ converges in $< \psi >$. Theorem 11.1 in [16] shows that this property is equivalent to the existence of a constant $B < \infty$ such that

$$\| \sum_{k\in G} a_k\, T_k\psi\|_H^2 \leq B \sum_{k\in G} |a_k|^2$$

for all finite sequences $\{a_k : k \in G\}$. Arguing as in the proof of Proposition 5.2 it can be shown that $\{T_k\psi : k \in G\}$ has the Hilbertian property in $< \psi >$ if and only if $[\psi, \psi](\alpha) \leq B$ a.e. $\alpha \in \widehat{G}$.

Proposition 5.5. *Let $(G, +)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$. The collection $\{T_k\psi : k \in G\}$ is a frame for $< \psi >$ with constants $A > 0$ and $B < \infty$ if and only if*

$$A \leq [\psi, \psi](\alpha) \leq B \ \textit{a.e.}\ \alpha \in \Omega_\psi\,,$$

where $\Omega_\psi = \{\alpha : \widehat{G} : [\psi, \psi](\alpha) > 0\}$

Proof. Recall that $\{T_k\psi : k \in G\}$ is a frame for $< \psi >$ with constants $A > 0$ and $B < \infty$ if for every $\varphi \in < \psi >$ we have

$$A\|\varphi\|_H^2 \leq \sum_{k\in G} | < \varphi, T_k\psi > |^2 \leq B\|\varphi\|_H^2\,. \tag{5.6}$$

Suppose $A \leq [\psi, \psi](\alpha) \leq B$ a.e. $\alpha \in \Omega_\psi$. Then, by Theorem 3.1 and (5.1) we obtain

$$\begin{aligned}
\sum_{k\in G} | < \varphi, T_k\psi > |^2 &= \sum_{k\in G} | < S_\psi(\varphi), S_\psi(T_k\psi) >_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} |^2 \\
&= \sum_{k\in G} | < S_\psi(\varphi), \mathbf{1}_{\Omega_\psi}\, e_k >_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} |^2 \\
&= \sum_{k\in G} \left| \int_{\widehat{G}} S_\psi(\varphi)(\alpha)\, \mathbf{1}_{\Omega_\psi}(\alpha)\, e_k(\alpha)\, [\psi, \psi](\alpha)\, d\alpha \right|^2 .
\end{aligned}$$

Since $\{e_k : k \in G\}$ is an orthonormal basis of $\widehat{G}$, by Plancherel's Theorem we can write

$$\sum_{k\in G} | < \varphi, T_k\psi > |^2 = \int_{\widehat{G}} |S_\psi(\varphi)(\alpha)\,|^2\, \mathbf{1}_{\Omega_\psi}(\alpha)\, ([\psi, \psi](\alpha))^2\, d\alpha\,. \tag{5.7}$$

Use the hypothesis and Theorem 3.1 to conclude

$$\sum_{k\in G} | < \varphi, T_k\psi > |^2 \approx \int_{\widehat{G}} |S_\psi(\varphi)(\alpha)\,|^2\, [\psi, \psi](\alpha)\, d\alpha = \|\varphi\|_H^2\,,$$

where the equivalence is up to the constants A and B. This proves (5.6).

Suppose now that $\{T_k\psi : k \in G\}$ is a frame for $< \psi >$, so that (5.6) hold for all $\varphi \in < \psi >$. To argue by contradiction, suppose that $[\psi, \psi](\alpha) < A$ on a set $E \subset \Omega_\psi$ with $\int_{\widehat{G}} \mathbf{1}_E(\alpha)\, d\alpha > 0$. Observe that $\mathbf{1}_E \in L^2(\widehat{G}, [\psi, \psi](\alpha)\, d\alpha)$ since $[\psi, \psi] \in L^1(\widehat{G}, d\alpha)$ by definition of the bracket. Since S_ψ is onto, we can choose $\varphi_E \in < \psi >$ such that $S_\psi(\varphi_E) = \mathbf{1}_E$, and the isometry property of S_ψ gives

$$\|\mathbf{1}_E\|_{L^2(\widehat{G},[\psi,\psi](\alpha)\,d\alpha)} = \|\varphi_E\|_H\,. \tag{5.8}$$

A calculation similar to the one leading to (5.7) with φ replaced by φ_E gives

$$\sum_{k\in G} | < \varphi_E, T_k\psi > |^2 = \int_{\widehat{G}} \mathbf{1}_E(\alpha)\,([\psi,\psi](\alpha))^2\,d\alpha\,.$$

Since $[\psi,\psi](\alpha) < A$ on the set E, the last expression is smaller than $A\int_E [\psi,\psi](\alpha)\,d\alpha = A\|\varphi_E\|_H^2$, where the last equality is due to (5.8). Hence, the left hand side of (5.6) does not hold for $\varphi = \varphi_E$. This shows that $[\psi,\psi](\alpha) \geq A$ a.e. Ω_ψ. A similar argument, supposing $[\psi,\psi](\alpha) > B$ on a set $F \subset \Omega_\psi$ with $\int_{\widehat{G}} \mathbf{1}_F(\alpha)\,d\alpha > 0$ gives, again, a contradiction. Hence $A \leq [\psi,\psi] \leq B$ a.e. on Ω_ψ. □

Remark 5.6. When propositions 5.1, 5.2 and 5.5 are specified for the system of integer translations in $\mathbb{R}^n$ described in section 4.1 we obtain the results in [10] for $n = 1$. Recall that in this case $[\psi,\psi](\xi) = P_\psi(\xi) = \sum_{\ell\in\mathbb{Z}^n} |\widehat{\psi}(\xi+\ell)|^2\,,\ \xi \in [0,1]\,.$

Remark 5.7. When propositions 5.1, 5.2 and 5.5 are specified for the Gabor unitary representation on $\mathbb{R}^n$ given in section 4.2 we obtain the results stated in [8] (Theorem 4.3.3) and [7] (Theorem 2.8). Recall that in this case the bracket is related to the Zak transform $[\psi,\psi](x,\xi) = |Z\psi(x,\xi)|^2,\ (x,\xi) \in [0,1]^n \times [0,1]^n\,.$

6. Biorthogonality and minimality

In this section we consider a **countable** abelian group $(G,+)$ equipped with the discrete topology and a dual integrable unitary representation $k \to T_k$ of G on a Hilbert spaces H.

Given $\psi, \widetilde{\psi} \in H \setminus \{0\}$, the collections $\{T_k\psi : k \in G\}$ and $\{T_k\widetilde{\psi} : k \in G\}$ are said to be **biorthogonal** if $< T_k\psi, T_\ell\widetilde{\psi} >= \delta_{k,\ell}$ for all $k,\ell \in G$.

Given $\psi \in H \setminus \{0\}$, the collection $\{T_k\psi : k \in G\}$ is said to be **minimal** for $< \psi >$ if it does not exists $k_0 \in G$ such that $T_{k_0}\psi \notin \overline{\{T_k\psi : k \in G, k \neq k_0\}}^H$. Since each T_k is unitary in H it can be shown that $\{T_k\psi : k \in G\}$ is **minimal** for $< \psi >$ if and only if $\psi \notin \overline{\{T_k\psi : k \in G, k \neq 0\}}^H$.

Proposition 6.1. *Let $(G,+)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$.*

a) Suppose that associated with ψ there exists $\widetilde{\psi} \in < \psi >$ such that $\mathcal{G} = \{T_k\psi : k \in G\}$ and $\widetilde{\mathcal{G}} = \{T_k\widetilde{\psi} : k \in G\}$ are biorthogonal. Then, $\mathcal{G}$ is minimal.

b) Conversely, if $\mathcal{G} = \{T_k\psi : k \in G\}$ is minimal, there exits $\widetilde{\psi} \in < \psi >$ such that $\mathcal{G} = \{T_k\psi : k \in G\}$ and $\widetilde{\mathcal{G}} = \{T_k\widetilde{\psi} : k \in G\}$ are biorthogonal.

Proof. a) If $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are biorthogonal we have $< T_k\psi, \widetilde{\psi} >= 0$ for all $k \in G, k \neq 0$. Thus, $\widetilde{\psi} \perp \overline{\mathrm{span}\,\{T_k\psi : k \in G, k \neq 0\}}^H$. Since $< \psi, \widetilde{\psi} >= 1$ we must have $\psi \notin \overline{\mathrm{span}\,\{T_k\psi : k \in G, k \neq 0\}}^H$. Thus, $\mathcal{G}$ is minimal.

b) Assume that $\mathcal{G}$ is minimal. Then $\{T_k\psi : k \in G, k \neq 0\} \subsetneq < \psi >$. Thus, there exists $\widetilde{\psi} \in < \psi >, \widetilde{\psi} \neq 0$ such that

$$\widetilde{\psi} \perp \{T_k\psi : k \in G, k \neq 0\}\,. \tag{6.1}$$

Clearly, $< \psi, \widetilde{\psi} >\neq 0$, so that we can assume $< \psi, \widetilde{\psi} >= 1$. Then we conclude $< T_k\psi, T_k\widetilde{\psi} >=< \psi, \widetilde{\psi} >= 1$ and if $k \neq \ell, < T_k\psi, T_\ell\widetilde{\psi} >=< T_{k-l}\psi, \widetilde{\psi} >= 0$ by (6.1). □

Proposition 6.2. *Let $(G, +)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$. The collection $\mathcal{G} = \{T_k\psi : k \in G\}$ is biorthogonal to $\widetilde{\mathcal{G}} = \{T_k\widetilde{\psi} : k \in G\}$ with $\widetilde{\psi} \in <\psi>$ if and only if*

$$\frac{1}{[\psi, \psi]} \in L^1(\widehat{G}, d\alpha)\,.$$

Proof. Suppose $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are biorthogonal with $\widetilde{\psi} \in <\psi>$. By Theorem 3.1 $S_\psi(\widetilde{\psi}) \in L^2(\widehat{G}, [\psi, \psi](\alpha)d\alpha)$ and $\|S_\psi(\widetilde{\psi})\|_{L^2(\widehat{G},[\psi,\psi](\alpha)d\alpha)} = \|\widetilde{\psi}\|_H$. By Hölder's inequality it follows that $S_\psi(\widetilde{\psi})\,[\psi, \psi] \in L^1(\widehat{G}, d\alpha)$:

$$\begin{aligned}\int_{\widehat{G}} |S_\psi(\widetilde{\psi})(\alpha)|\,[\psi,\psi](\alpha)d\alpha &\le \left(\int_{\widehat{G}} |S_\psi(\widetilde{\psi})(\alpha)|^2\,[\psi,\psi](\alpha)d\alpha\right)^{1/2} \left(\int_{\widehat{G}} [\psi,\psi](\alpha)d\alpha\right)^{1/2} \\ &= \|\widetilde{\psi}\|_H\,\|\psi\|_H < \infty\,,\end{aligned}$$

where the equality $(\int_{\widehat{G}} [\psi, \psi](\alpha)d\alpha)^{1/2} = \|\psi\|_H$ is due to condition (2.4) of the definition of dual integrability. Moreover, by (5.1), Theorem 3.1, and the biorthogonality we deduce, for $k \in G$,

$$\begin{aligned}&\int_{\widehat{G}} S_\psi(\widetilde{\psi})(\alpha)\,[\psi,\psi](\alpha)\,\overline{\alpha(k)}\,d\alpha = \int_{\widehat{G}} S_\psi(\widetilde{\psi})(\alpha)\,\overline{S_\psi(T_k\psi)(\alpha)}\,[\psi,\psi](\alpha)\,d\alpha \\ &=< S_\psi(\widetilde{\psi}), S_\psi(T_k\psi) >_{L^2(\widehat{G},[\psi,\psi](\alpha)d\alpha)} =< \widetilde{\psi}, T_k\psi >_H = \delta_{k,0}\,.\end{aligned}$$

Thus, the Fourier coefficients of the $L^1(\widehat{G}, d\alpha)$ function $S_\psi(\widetilde{\psi})\,[\psi, \psi]$ are all zero except the one corresponding to $k = 0$, which is 1. By uniqueness, $S_\psi(\widetilde{\psi})(\alpha)\,[\psi, \psi](\alpha) = 1$ a.e. $\alpha \in \widehat{G}$. Thus, $[\psi, \psi] > 0$ a.e. Writing $|S_\psi(\widetilde{\psi})|^2\,[\psi, \psi] = \frac{1}{[\psi,\psi]}$ and using Theorem 3.1 we dedude

$$\int_{\widehat{G}} \frac{1}{[\psi,\psi]}\,d\alpha = \int_{\widehat{G}} |S_\psi(\widetilde{\psi})|^2\,[\psi,\psi]\,d\alpha = \|\widetilde{\psi}\|_H\,,$$

showing the desired result.

Suppose now that $\frac{1}{[\psi,\psi]} \in L^1(\widehat{G}, d\alpha)$. Then $\frac{1}{[\psi,\psi]} \in L^2(\widehat{G}, [\psi, \psi](\alpha)\,d\alpha)$. By Theorem 3.1 we can consider $\widetilde{\psi} \equiv S_\psi^{-1}(\frac{1}{[\psi,\psi]}) \in <\psi>$. Moreover, by Theorem 3.1, equality (5.1), and the fact that $\Omega_\psi = \widehat{G}$ in our situation, for $k \in G$ we have

$$\begin{aligned}< T_k\psi, \widetilde{\psi} >_H &= < T_k\psi, S_\psi^{-1}(\frac{1}{[\psi,\psi]}) >_H =< S_\psi(T_k\psi), \frac{1}{[\psi,\psi]} >_{L^2(\widehat{G},[\psi,\psi](\alpha)d\alpha)} \\ &= \int_{\widehat{G}} \alpha(k)\,\frac{1}{[\psi,\psi](\alpha)}\,[\psi,\psi](\alpha)\,d\alpha = \int_{\widehat{G}} e_k(\alpha)\,d\alpha = \delta_k\end{aligned}$$

by the orthogonality of $\{e_k : k \in G\}$ in $L^2(\widehat{G}, d\alpha)$. This shows that $\widetilde{\mathcal{G}} = \{T_k\widetilde{\psi} : k \in G\}$, with $\widetilde{\psi}$ defined as above, is biorthogonal to $\mathcal{G}$. □

Remark 6.3. It is not necessary to assume that $\widetilde{\mathcal{G}}$ is generated by a single element $\widetilde{\psi}$. One could start with a general system $\widetilde{\mathcal{G}} = \{\varphi_k : k \in G\}$ with $\varphi_k \in <\psi>$ and biorthogonal to $\mathcal{G} = \{T_k\psi : k \in G\}$ and show that $\widetilde{\mathcal{G}}$ has to be generated by a single function.

From Propositions 6.1 and 6.2 we deduce:

Corollary 6.4. *Let $(G,+)$ be a countable abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \setminus \{0\}$. The collection $\mathcal{G} = \{T_k\psi : k \in G\}$ is minimal if and only if*

$$\frac{1}{[\psi,\psi]} \in L^1(\widehat{G}, d\alpha)\,.$$

7. A general framework for representations on $L^2(\mathbb{R}^n)$

We have shown in Example 2.2 (section 2) and in section 4.1 that our theory can be applied to translations. These are the kind of operators used in the definition of the scaling spaces V_0 of a Multiresolution Analysis ([14]) or more general Frame Multiresolution Analysis ([1]). All of these spaces are translation invariant. In the theory of wavelets the resolution spaces W_j are not translation invariant, but invariant under dyadic dilations (or invariant under more general expansive dilations matrices). Also, in the theory of wavelets with composite dilations some of the spaces involved are invariant under operations with shear matrices ([6]). We describe in this section a general framework based on the action of a **discrete** countable abelian group on $\mathbb{R}^n$ that includes all the above examples.

Assume that $(G,+)$ is a **discrete** countable abelian group that acts on a σ-finite measure space $\mathcal{M}$ by $(k,x) \longrightarrow k \bullet x$. Thus we have a mapping from $G \times \mathcal{M}$ to $\mathcal{M}$ which we assume to have the following properties:

1. $k \bullet (\ell \bullet x) = (k+\ell) \bullet x\,,$ for all $k, \ell \in G\,, x \in \mathcal{M}\,,$
2. $0 \bullet x = x\,,$ for all $x \in \mathcal{M}\,,$
3. $d\lambda(k \bullet x) = J(k,x)\, d\lambda(x)\,,\ J(k,x) > 0\,,$ for all $k \in G\,, x \in \mathcal{M},$, where $d\lambda$ denotes the element of Lebesgue measure on $\mathcal{M}$ and $J(k,x)$, called the **Jacobian**, is a real function defined on $G \times \mathcal{M}$, and
4. There exists a measurable set $C \subset \mathcal{M}$ such that $\{k \bullet C : k \in G\}$ is an almost everywhere partition of $\mathcal{M}$.

Simple examples of actions of a group in $\mathbb{R}^n$ are the following:

A. Translations by elements of $\mathbb{Z}^n$: in this case $J(k,x) = 1$ for all $k \in \mathbb{Z}^n\,, x \in \mathbb{R}^n\,,$ and $C = [0,1]^n$.

B. Dilations by $2^j\,, j \in \mathbb{Z}$, that is $j \bullet x = 2^j x$ for $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$: in this case $J(j,x) = 2^j$ for all $j \in \mathbb{Z}\,, x \in \mathbb{R}^n\,,$ and $C = ([0,1]^n \setminus [-1/2,1/2]^n)$.

C. Multiplication by the shear matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k \in \mathbb{Z}$. That is $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ kx+y \end{pmatrix}$: in this case the abelian group G is $\left\{K = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} : k \in \mathbb{Z}\right\}$ with multiplication by matrices, $J(K,x) = 1\,, K \in G\,, x \in \mathbb{R}^2$, and $C = \{(x,y) \in \mathbb{R}^2 : 0 \le y < x\} \cup \{(x,y) \in \mathbb{R}^2 : x < y \le 0\}$.

Lemma 7.1. *The Jacobian of the action $\bullet$ of a group G on a σ-finite measures space $\mathcal{M}$ satisfies $J(k+\ell,x) = J(k,\ell \bullet x)\, J(\ell,x)$*

Proof. On the one hand, $d\lambda(k \bullet (\ell \bullet x)) = d\lambda(k+\ell) \bullet x) = J(k+\ell,x)\, d\lambda(x)$, by 1 and 3 of the definition of the action. On the other hand, $d\lambda(k \bullet (\ell \bullet x)) = J(k,\ell \bullet x)\, d\lambda(l \bullet x) = J(k,\ell \bullet x)\, J(\ell,x)\, d\lambda(x)$, by 3 of the definition of the action. This proves the result. $\square$

For $f \in L^2(\mathcal{M})$ and $k \in G$ define

$$(D_k\, f)(x) = J(k,x)^{1/2}\, f(k \bullet x)\,, \qquad x \in \mathcal{M}\,. \tag{7.1}$$

This gives a representation of the abelian group G on $L^2(\mathcal{M})$. To see this choose $k, \ell \in G$ and use property 3 of the definition of an action and Lemma 7.1 to obtain

$$\begin{aligned} D_k\, D_\ell\, f(x) &= J(k,x)^{1/2}\, D_\ell\, f(k \bullet x) = J(k,x)^{1/2}\, J(\ell, k \bullet x)^{1/2}\, f(\ell \bullet k \bullet x) \\ &= J(k+\ell, x)^{1/2}\, f((k+\ell) \bullet x) = D_{k+\ell}\, f(x)\,. \end{aligned}$$

Changing variables we observe that this representation is unitary in $L^2(\mathcal{M})$.

Lemma 7.2. *Let $\bullet$ define an action of a countable abelian group G on σ-finite measure space $\mathcal{M}$. For $f \in L^2(\mathcal{M})$ we have $\|f\|_2^2 = \sum_{k \in G} \int_C |(D_k\, f)(x)|^2\, d\lambda(x)$, where D_k is defined in (7.1) and C is the set that appears in property 4 of the definition of the action.*

Proof. Use that $\mathcal{M}$ coincides a.e. with the almost everywhere disjoint union of the sets $k \bullet C$, $k \in G$, and the obvious change of variables to obtain

$$\begin{aligned} \|f\|_2^2 &= \int_{\bigcup_{k\in G} k \bullet C} |f(x)|^2\, d\lambda(x) = \sum_{k \in G} \int_{k \bullet C} |f(x)|^2\, d\lambda(x) \\ &= \sum_{k\in G} \int_C |f(k \bullet y)|^2\, J(k,y)\, d\lambda(y) = \sum_{k \in G} \int_C |(D_k\, f)(x)|^2\, d\lambda(y)\,. \end{aligned}$$

□

We want to prove the dual integrability of the unitary representation given by (7.1). For $f, g \in L^2(\mathcal{M})$ and C as in property 4. of the definition of the action we obtain

$$< f, g >_{L^2(\mathcal{M})} = \int_{\mathcal{M}} f(x)\, \overline{g(x)}\, d\lambda(x) = \int_C \{\sum_{k \in G} f(k \bullet x)\, \overline{g(k \bullet x)}\, J(k,x)\}\, d\lambda(x)\,.$$

Use that $\{e_k : k \in G\}$ is an orthonormal system of $L^2(\widehat{G})$ to obtain

$$\begin{aligned} < f, g >_{L^2(\mathcal{M})} &= \int_C \sum_{k\in G} \sum_{\ell \in G} f(k \bullet x)\, \overline{g(\ell \bullet x)}\, J(k,x)^{1/2}\, J(\ell,x)^{1/2}\, d\lambda(x) \left(\int_{\widehat{G}} e_{\ell-k}(\alpha)\, d\alpha \right) \\ &= \int_{\widehat{G}} \int_C \left\{ \sum_{k\in G} (D_k f)(x)\, \overline{\alpha(k)} \right\} \overline{\left\{ \sum_{\ell \in G} (D_l g)(y)\, \overline{\alpha(\ell)} \right\}}\, d\lambda(x)\, d\alpha \\ &= \int_{\widehat{G}} \int_C Zf(x,\alpha)\, \overline{Zg(x,\alpha)}\, d\lambda(x)\, d\alpha\,, \end{aligned} \tag{7.2}$$

where, for $\psi \in L^2(\mathcal{M})$,

$$(Z\psi)(y,\alpha) = \sum_{l \in G} (D_l \psi)(y) \overline{\alpha(l)}\,, \quad y \in \mathcal{M}\,,\ \alpha \in \widehat{G}\,. \tag{7.3}$$

If $j \in \mathbb{Z}^n$ and $\psi \in L^2(\mathcal{M})$ we have

$$Z(D_j\psi)(x,\alpha) = \sum_{k \in G} D_k\, D_j \psi(x)\, \overline{\alpha(k)} = \sum_{k\in G} D_{k+j} \psi(x)\, \overline{\alpha(k)}$$

$$= \sum_{m\in G} D_m\psi(x)\,\overline{\alpha(m-j)} = \alpha(j)\,Z\psi(x,\alpha)\,. \tag{7.4}$$

From (7.2) and (7.4) applied to $\varphi, \psi \in L^2(\mathcal{M})$ we deduce

$$\begin{aligned} <\varphi, D_j\psi>_{L^2(\mathcal{M})} &= \int_{\widehat{G}}\int_C Z\varphi(x,\alpha)\,\overline{Z(D_j\psi)(x,\alpha)}\,d\lambda(x)\,d\alpha \\ &= \int_{\widehat{G}}\int_C Z\varphi(x,\alpha)\,\overline{Z\psi(x,\alpha)}\;\overline{\alpha(j)}\,d\lambda(x)\,d\alpha \end{aligned} \tag{7.5}$$

obtaining

$$[\varphi,\psi](\alpha) = \int_C Z\varphi(x,\alpha)\,\overline{Z\psi(x,\alpha)}\,d\lambda(x)\,.$$

This shows (2.4) of the definition of dual integrability. To show (2.3) notice that (7.2) shows that Z is an isometry from $L^2(\mathcal{M})$ into the space $L^2(C\times\widehat{G})$.

The object defined in (7.3) is a generalization of the Zak transform adapted to our situation. It coincides with the Zak transform when $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translations. When the action is dilation by 2 in the real line, the object defined in (7.3) is called the multiplicative Zak transform in [3]. It also appears in the work [17] and more generaly in [18] and [5]. We thank Professor Wojtek Czaja for pointing out some of these references to us.

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