FURTHER RESULTS ON THE CONNECTIVITY OF PARSEVAL FRAME WAVELETS

G. GARRIGÓS, E. HERNÁNDEZ, H. ŠIKIĆ, AND F. SORIA

ABSTRACT. New ideas were introduced in [3] to treat the problem of connectivity of Parseval frames. With these ideas it was shown that a large set of Parseval frames is arcwise connected. In this article we exhibit a larger class of Parseval frames for which the arcwise connectivity is true. This larger class fails to include all Parseval frames.

1. INTRODUCTION AND MAIN RESULT

The connectivity on the set of orthonormal wavelets is an interesting problem which remains open many years after it was proposed (see for instance [1, 10, 9, 5]). The question has been extended to the larger class of Parseval frame wavelets (see [7] and [3]). Even in this extended setting, the problem seems very difficult and is still open. We recall that a function $\psi \in L^2(\mathbb{R})$ is said to be a **Parseval frame wavelet** (or a normalized tight frame wavelet), if the collection $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ forms a tight frame (with constant 1) for $L^2(\mathbb{R})$. When this is the case we shall write $\psi \in \mathbf{PFW}$. It is well-known (see [4, Ch.7]) that these functions are characterized by the two equations:

$$(A1) \quad \sum_{j \in \mathbb{Z}} |\widehat{\psi}(2^j \xi)|^2 = 1 \qquad a.e.\, \xi \in \mathbb{R}$$

$$(A2) \quad t_q(\xi; \psi) := \sum_{j=0}^{\infty} \widehat{\psi}(2^j \xi) \overline{\widehat{\psi}(2^j(\xi + 2q\pi))} = 0 \qquad a.e.\, \xi \in \mathbb{R}, \quad \forall q \in 2\mathbb{Z}+1.$$

This characterization is used in [3] to give a new approach to the connectivity of the set $\mathbf{PFW}$. In particular, in that paper we were able to construct explicit paths for two large subclasses $\mathbf{K}_\tau$ and $\mathbf{K}_d$ of $\mathbf{PFW}$, defined by size conditions on the spectrum (see §3 below). Since each subclass in some sense takes care of one of the basic transformations (translations and dilations), one may conjecture that finding a condition that encompasses both would settle the problem completely. Unfortunately, as we shall prove in this article, this is not necessarily so. We provide a new condition along that direction which defines an even larger class of connected Parseval frame wavelets, but that it does not include the entire $\mathbf{PFW}$.

We shall say that $J \subset \mathbb{R}$ is a *Calderón set* if the family $\{2^j J\}_{j \in \mathbb{Z}}$ forms an *a.e.* partition of $\mathbb{R}\setminus\{0\}$. A particular example is the *Shannon set* $S = [-2\pi, -\pi) \cup (\pi, 2\pi]$.

Observe that J is a Calderón set if and only if the function $\hat{\psi} = \chi_J$ satisfies (A1). By (A2), if in addition $J \subset [-\pi, \pi]$, then $\psi \in \mathbf{PFW}$. For sets $A, K \subset \mathbb{R}$ we shall also use the following notation from [3]:

$$[A] = \bigcup_{j \in \mathbb{Z}} 2^{-j} A, \qquad [A]_- = \bigcup_{j=0}^{\infty} 2^{-j} A, \quad \text{and} \quad \tau_K(A) = \bigcup_{\substack{k \in \mathbb{Z} \\ k \neq 0}} (A + 2k\pi) \cap K.$$

Definition 1.1. Let J_0 be a bounded Calderón set and denote $J_j = 2^{-j} J_0$, $j \in \mathbb{Z}$. We say that *a measurable subset $K \subset \mathbb{R}$ is J_0-admissible or that it belongs to the class $\mathcal{K}(J_0)$* whenever there exists $M \in \mathbb{Z}$ such that the set correspondence:

$$I \subset [J_M]_- = \bigcup_{\ell=M}^{\infty} 2^{-\ell} J_0 \longmapsto \mathcal{L}(I) = \mathcal{L}_K(I) = [\tau_K(I)] \cap S$$

is continuous in the sense of the measure algebras. That is, for each $\eta > 0$ there exists $\delta > 0$, such that $\forall \, I \subset [J_M]_-$ with $|I| < \delta$, we have $|\mathcal{L}(I)| < \eta$.

It is easy to see that bounded sets belong to the class $\mathcal{K}(J_0)$, for all bounded Calderón sets J_0. Indeed, in such a case only finite unions of translates and dilates of I are involved in the definition of $\mathcal{L}_K(I)$, whenever $I \subset (-\pi, \pi)$. The continuity then follows easily by subadditivity. Moreover, below we shall show that all sets K with a certain sparseness condition at infinity belong also to this class. In particular, this will be the case for sets K satisfying the $\mathcal{K}_\tau$ or $\mathcal{K}_d$ conditions introduced in [3] (see §3 below). We shall also show, however, that $\mathbb{R}$ does not belong to $\mathcal{K}(J_0)$ for any J_0.

Concerning the connectivity of $\mathbf{PFW}$, the main result in this paper is the following.

Theorem 1.2. *Let $J_0 \subset [-\pi, \pi]$ be a Calderón set. Then, for every $\psi \in \mathbf{PFW}$ such that $\mathrm{Supp}\, \widehat{\psi}$ belongs to the class $\mathcal{K}(J_0)$, there exists a continuous path $\{\psi_t\}_{0 \leq t \leq 1}$ inside $\mathbf{PFW}$ such that:*

$$\psi_0 = \psi, \quad \psi_1 = (\chi_{J_0})^{\vee} \quad \text{and} \quad \mathrm{Supp}\, \widehat{\psi}_t \subset \mathrm{Supp}\, \widehat{\psi} \cup [J_0]_- \,.$$

In particular, the set of all $\psi \in \mathbf{PFW}$ with $\mathrm{Supp}\, \widehat{\psi} \in \mathcal{K}(J_0)$ is arc-wise connected in $L^2(\mathbb{R})$.

The paper is structured as follows. The proof of Theorem 1.2 is given in section 2. In section 3 we show the relation between the class $\mathcal{K}(J_0)$ and the classes $\mathcal{K}_\tau$ and $\mathcal{K}_d$ from the previous paper [3]. Finally, in sections 4 and 5 we show that $\mathbb{R} \notin \mathcal{K}(J_0)$ for every Calderón set J_0, as well as the connections of this result with number theory.

2. The proof of the theorem

The following elementary properties about the class $\mathcal{K}(J_0)$ are easily verified:

(1) For any $j \in \mathbb{Z}$, $\mathcal{K}(J_0) = \mathcal{K}(2^j J_0)$. Thus, there is no loss in assuming $J_0 \subset [-\pi, \pi]$.

(2) If K_1, K_2 belong to $\mathcal{K}(J_0)$, then so do $K_1 \cup K_2$ and any subset of K_1 or K_2.

(3) In Definition 1.1, the Shannon set S can be replaced by any Calderón set H, provided it has finite measure. More precisely, given any fixed set E in $\mathbb{R}$ the map

(2.1) $$I \subset E \longmapsto \mathcal{L}^{(H)}(I) := [\tau_K(I)] \cap H$$

is continuous if and only if $I \subset E \mapsto \mathcal{L}^{(S)}(I)$ is continuous. This is an elementary consequence of the following lemma.

Lemma 2.1. *Let H_1, H_2 be two Calderón sets with finite Lebesgue measure. Then, $\forall \, \eta > 0$, $\exists \, \delta > 0$ such that $\forall \, A \subset \mathbb{R} : \; |[A] \cap H_1| < \delta \implies |[A] \cap H_2| < \eta$.*

PROOF: Define the set function: $\mu(F) = |[F] \cap H_2|, \quad F \subset H_1$. From the Calderón property of H_1 one easily checks that μ is a well-defined measure in H_1. Moreover, it is clear that μ is finite and absolutely continuous with respect to the Lebesgue measure $dx_{|_{H_1}}$. Then, the usual characterization of absolutely continuous measures applied to $F = [A] \cap H_1$ establishes the lemma (see [8, Th. 6.11]). $\qquad\square$

PROOF of Theorem 1.2: Let $\psi \in \mathbf{PFW}$ be so that its spectrum, $K = \mathrm{Supp}\, \widehat{\psi}$, is J_0-admissible. Let us choose a large positive integer $M = M(K, J_0)$ so that the map $I \subset [J_M]_- \mapsto \mathcal{L}_K(I)$ is continuous (by the $\mathcal{K}(J_0)$ condition). Since J_0 and $J_M = 2^{-M} J_0$ are trivially connected with the continuous path $\{tJ_0\}_{t \in [2^{-M}, 1]}$, we may assume through the rest of the proof that $M = 0$. Following the ideas in [3], we divide the proof in three parts.

Step 1.- Construction of new PFW's. As in [3], we present a general procedure to construct new PFW's starting from a given one ψ and an arbitrary interval I in J_0. The dynamics of the construction is the following: consider any sequence $\{k_j\}_{j=1}^{\infty} \subset \mathbb{N} = \{0, 1, 2, \ldots\}$, and suppose $I = I_0$ is an arbitrary subset of J_0, hence contained in $[-\pi, \pi]$. Construct by induction the following sequence of subsets of $[-\pi, \pi]$:

$$
\begin{aligned}
I_1 &= ([\tau_K(I_0)] \setminus [I_0]) \cap J_{k_1} \\
I_2 &= ([\tau_K(I_1)] \setminus [I_0 \cup I_1]) \cap J_{k_2} \\
&\;\;\vdots \qquad\qquad \vdots \\
I_{N+1} &= ([\tau_K(I_N)] \setminus [I_0 \cup \ldots \cup I_N]) \cap J_{k_{N+1}}.
\end{aligned}
$$

Define $E_I = \cup_{N=0}^{\infty} I_N$ and notice that by construction this union is disjoint (we shall denote this by the symbol $\uplus$). Moreover, using the fact

$$
[I_\ell] = \biguplus_{j \in \mathbb{Z}} 2^j I_\ell = [\tau_K(I_{\ell-1})] \setminus (\cup_{N=0}^{\ell-1}[I_N]),
$$

it is easy to verify that:

    (1) $[E_I] = \biguplus_{\ell=0}^{\infty}[I_\ell]$, with disjoint union;
    (2) $\{2^j E_I\}_{j \in \mathbb{Z}}$ is a disjoint family;
    (3) $[\tau_K(E_I)] = \cup_{N=0}^{\infty}[\tau_K(I_N)] \subset \cup_{N=0}^{\infty}(\cup_{\ell=0}^{N+1}[I_\ell]) = [E_I]$.

Since we also have $E_I \subset [-\pi, \pi]$, we conclude from Proposition 2.1 of [3] that $\psi_{E_I} \in \mathbf{PFW}$, where $\widehat{\psi}_{E_I}(\xi) = \widehat{\psi}(\xi)$ if $\xi \in \mathbb{R} \setminus [E_I]$, and $\widehat{\psi}_{E_I}(\xi) = \chi_{E_I}(\xi)$ if $\xi \in [E_I]$.

Step 2.- Construction of the path. Since J_0 is a Calderón set, there is a bijective measurable mapping: $\Gamma : \xi \in J_0 \mapsto 2^{j(\xi)}\xi \in S$. Define the following family of subsets of J_0: $I(t) = \Gamma^{-1}(\{\pi < |\xi| \leq (1+t)\pi\})$, $t \in [0, 1]$. Observe that $I(0) = \emptyset$ and $I(1) = J_0$. Consider the family $\psi_t = \psi_{E_{I(t)}}$ inside $\mathbf{PFW}$ obtained from the sets $E_{I(t)}$ as in the previous step. Observe that

(2.2)
$$
\widehat{\psi}_t = \widehat{\psi}_{E_{I(t)}} = \begin{cases} \widehat{\psi}, & \text{if } t = 0 \\ \chi_{J_0}, & \text{if } t = 1. \end{cases} \;,
$$

while Supp $\widehat{\psi}_t \subset \left(\text{Supp } \widehat{\psi} \setminus [E_{I(t)}]\right) \cup E_{I(t)} \subset \text{Supp } \widehat{\psi} \cup [J_0]_-$. Finally, since bounded sets always belong to $\mathcal{K}(J_0)$, we have $[J_0]_- \subset [-\pi, \pi] \in \mathcal{K}(J_0)$. Therefore, using Supp $\widehat{\psi} \in \mathcal{K}(J_0)$ and the properties at the beginning of §2, we conclude that Supp $\widehat{\psi}_t \in \mathcal{K}(J_0)$. Thus, we have constructed a path $t \in [0, 1] \mapsto \psi_t \in \mathbf{PFW}$ with the properties required in the theorem, and only remains to show the $L^2(\mathbb{R})$-continuity of such path.

Step 3.- Continuity of the path. By Proposition 2.2 of [3], a path of the form $\{\psi_{E(t)}\}_{0 \le t \le 1}$ inside $\mathbf{PFW}$ is continuous provided we can show

$$\lim_{\substack{t \to t_0 \\ t \in [0,1]}} |E(t) \Delta E(t_0)| = \lim_{\substack{t \to t_0 \\ t \in [0,1]}} |\widetilde{E}(t) \Delta \widetilde{E}(t_0)| = 0, \quad \forall t_0 \in [0, 1],$$

where $\widetilde{E} = [E] \cap S$, $E \subset \mathbb{R} \setminus \{0\}$. Now, observe that when $t \searrow t_0$ we have:

$$|I(t) \Delta I(t_0)| = \left|\Gamma^{-1}\left(\{(1 + t_0)\pi < |\xi| \le (1 + t)\pi\}\right)\right| \longrightarrow |\Gamma^{-1}(\emptyset)| = 0,$$

and when $t \nearrow t_0$

$$|I(t) \Delta I(t_0)| = \left|\Gamma^{-1}\left(\{(1 + t)\pi < |\xi| \le (1 + t_0)\pi\}\right)\right| \to \left|\Gamma^{-1}\left(\{|\xi| = (1 + t_0)\pi\}\right)\right| = 0,$$

in this last case since Γ^{-1} preserve null sets. Thus, for the continuity of our path $t \to \psi_t$ it suffices to show that

$$(2.3) \qquad |E_I \Delta E_{I'}| + |\widetilde{E}_I \Delta \widetilde{E}_{I'}| \longrightarrow 0, \quad \text{as } |I \Delta I'| \to 0.$$

We shall deduce this from our key assumption: $K = \text{Supp } \widehat{\psi} \in \mathcal{K}(J_0)$. That is, the set mapping $I \subset [J_0]_- \mapsto \mathcal{L}(I)$ is continuous in the sense of measure algebras. Since we have freedom to choose the sequence $\{k_j\}$ in step 1, we shall assume that $\lim_{j \to \infty} k_j = +\infty$. Now, by continuity, given $\eta > 0$ there exists an integer $M_0 = M_0(\eta) \ge 0$ such that

$$\left|\left[\tau_K\left([J_0]_- \cap \left(-\tfrac{\pi}{2^{M_0}}, \tfrac{\pi}{2^{M_0}}\right)\right)\right] \cap S\right| < \eta.$$

We may assume that M_0 is large enough so that $\left|\left(-\tfrac{\pi}{2^{M_0}}, \tfrac{\pi}{2^{M_0}}\right)\right| = \tfrac{2\pi}{2^{M_0}} < \eta$. Also, since $\lim_{j \to \infty} k_j = +\infty$, there exists $M_1 = M_1(\eta) \ge 0$ such that $k_j \ge M_0$ for all $j \ge M_1$. Therefore, for all $M \ge M_1$, and for all $I \subset J_0$ we have

$$\bigcup_{N=M}^{\infty} I_N \subset [J_0]_- \cap \left(-\tfrac{\pi}{2^{M_0}}, \tfrac{\pi}{2^{M_0}}\right) \qquad \text{and}$$

$$\bigcup_{N=M+1}^{\infty} \widetilde{I}_N = \bigcup_{N=M}^{\infty} [I_{N+1}] \cap S \subset \bigcup_{N=M}^{\infty} [\tau_K(I_N)] \cap S = [\tau_K(\bigcup_{N=M}^{\infty} I_N)] \cap S = \mathcal{L}(\bigcup_{N=M}^{\infty} I_N).$$

Thus,

$$(2.4) \qquad \left|\bigcup_{N=M+1}^{\infty} I_N\right| + \left|\bigcup_{N=M+1}^{\infty} \widetilde{I}_N\right| < 2\eta, \quad \forall M \ge M_1.$$

Let us now show that the second term in the sum (2.3) converges to 0. Observe that, for all $M \ge 0$

$$\widetilde{E}_I \Delta \widetilde{E}_{I'} = \left(\bigcup_{N=0}^{\infty} \widetilde{I}_N\right) \Delta \left(\bigcup_{N=0}^{\infty} \widetilde{I}'_N\right)$$

$$(2.5) \qquad \subset \bigcup_{N=0}^{M} (\widetilde{I}_N \Delta \widetilde{I}'_N) \cup \Big(\bigcup_{N=M+1}^{\infty} \widetilde{I}_N \Big) \cup \Big(\bigcup_{N=M+1}^{\infty} \widetilde{I}'_N \Big).$$

Therefore, if we are able to show that for $I, I' \subset J_0$

$$(2.6) \qquad |\widetilde{I}_N \Delta \widetilde{I}'_N| \longrightarrow 0, \text{ as } |I \Delta I'| \to 0, \quad \forall N = 0, 1, 2, \dots$$

then, combining (2.5) with (2.4) it will follow that $|\widetilde{E}_I \Delta \widetilde{E}_{I'}| \to 0$, as $|I \Delta I'| \to 0$. Now, to show (2.6) we first consider the case $N = 0$: since $I, I' \subset J_0$ we have

$$(2.7) \qquad \widetilde{I} \Delta \widetilde{I}' = ([I] \Delta [I']) \cap S = [I \Delta I'] \cap S.$$

Then, by Lemma 2.1 the measure of this set goes to 0 as $|I \Delta I'| = |[I \Delta I'] \cap J_0| \to 0$. For the general case we proceed as in Lemma 4.4 of [3]: write $\widetilde{I}_{N+1} = [I_{N+1}] \cap S = \big([\tau_K(I_N)] \setminus [\cup_{\ell=0}^{N} I_\ell] \big) \cap S$, and use standard Boolean algebra to obtain $\widetilde{I}_{N+1} \Delta \widetilde{I}'_{N+1} \subset \big([\tau_K(I_N \Delta I'_N)] \cup \big(\cup_{\ell=0}^{N} [I_\ell \Delta I'_\ell] \big) \big) \cap S$. Then, we proceed by induction: for $N = 0$ we have $|\widetilde{I}_1 \Delta \widetilde{I}'_1| \leq |[\tau_K(I \Delta I')] \cap S| + |[I \Delta I'] \cap S|$, for which the first summand goes to 0 by the continuity assumption (as $|I \Delta I'| \to 0$, $I, I' \subset J_0$), and the second one by (2.7). Suppose now that we have shown

$$|\widetilde{I}_\ell \Delta \widetilde{I}'_\ell| \longrightarrow 0, \text{ as } |I \Delta I'| \to 0, \quad \forall \ell = 0, \dots, N.$$

Again, by Lemma 2.1 we have that

$$(2.8) \qquad |I_\ell \Delta I'_\ell| = |[I_\ell \Delta I'_\ell] \cap J_{k_\ell}| \longrightarrow 0, \text{ as } |[I_\ell \Delta I'_\ell] \cap S| = |\widetilde{I}_\ell \Delta \widetilde{I}'_\ell| \to 0.$$

Then, using the induction hypothesis and the continuity assumption we obtain

$$|\widetilde{I}_{N+1} \Delta \widetilde{I}'_{N+1}| \leq |[\tau_K(I_N \Delta I'_N)] \cap S| + \sum_{\ell=0}^{N} |\widetilde{I}_\ell \Delta \widetilde{I}'_\ell| \longrightarrow 0, \text{ as } |I \Delta I'| \to 0,$$

establishing our claim, and concluding the proof of (2.6). To establish completely (2.3), it remains to show that $|E_I \Delta E_{I'}| \to 0$ as $|I \Delta I'| \to 0$. This follows from our previous arguments since

$$|E_I \Delta E_{I'}| \leq \sum_{N=0}^{M} |I_N \Delta I'_N| + \Big| \bigcup_{N=M+1}^{\infty} I_N \Big| + \Big| \bigcup_{N=M+1}^{\infty} I'_N \Big|,$$

so we just need to combine (2.4) with (2.8). $\qquad \square$

3. Sufficient conditions

3.1. **The $\mathcal{K}_\tau$ condition.** We recall from [3] that a set $K \subset \mathbb{R}$ is said to belong to the class $\mathcal{K}_\tau$ whenever there exists $\varepsilon \in (0, \pi]$ such that

$$(3.1) \qquad \sum_{|n| \geq 1} \frac{1}{|n|} \big| K \cap (2n\pi + (-\varepsilon, \varepsilon)) \big| < \infty.$$

Proposition 3.1. $\mathcal{K}_\tau \subset \mathcal{K}(S)$.

PROOF: Let $K \in \mathcal{K}_\tau$ and choose ε as in (3.1). Given $H = [-\varepsilon, -\varepsilon/2] \cup (\varepsilon/2, \varepsilon]$, it suffices to show that $I \subset (-\varepsilon, \varepsilon) \mapsto \mathcal{L}^{(H)}(I)$ (as in (2.1)) is continuous in measure. By Lemma 4.3 in [3], there exists a constant $C > 0$ such that for each such I

$$\big| [\tau_K(I)] \cap H \big| \leq C \varepsilon \sum_{|n| \geq 1} \frac{1}{|n|} |K \cap (2n\pi + I)| = C \varepsilon \int_I \sum_{n \geq 1} \frac{1}{|n|} \chi_{K+2n\pi}(\xi) \, d\xi.$$

Since $K \in \mathcal{K}_\tau$ the right hand side of this equalities is finite. The result follows by an application of the Lebesgue dominated convergence theorem. □

3.2. **The $\mathcal{K}_d$ condition.** Next we turn to the condition $\mathcal{K}_d$ in [3]. We say that a set K belongs to such class whenever there exists $\varepsilon \in (0, \pi]$ such that

$$(3.2) \qquad \Big| \limsup_{n \to \infty} \frac{1}{2^n} (K \cap 2^n S^\varepsilon) \Big| = 0 \,, \quad \text{where} \quad S^\varepsilon := [-\varepsilon, -\varepsilon/2) \cup (\varepsilon/2, \varepsilon].$$

If we define $L_m = L_m^{(\varepsilon)} = \bigcup_{n=m}^\infty \frac{1}{2^n}(K \cap 2^n S^\varepsilon)$, condition (3.2) is the same as saying that $|L_m| \to 0$ as $m \to \infty$. By simple Boolean algebra, $L_m = [K \cap (\cup_{n=m}^\infty 2^n S^\varepsilon)] \cap S^\varepsilon$. Since $\cup_{n=m}^\infty 2^n S^\varepsilon = \pm(\frac{2^m \varepsilon}{2}, \infty)$, we see that (3.2) is actually equivalent to

$$(3.3) \qquad \big| [K \cap (-\infty, -R) \cup (R, \infty)] \cap S^\varepsilon \big| \to 0 \qquad \text{as} \qquad R \to \infty \,.$$

Proposition 3.2. $\mathcal{K}_d \subset \mathcal{K}(S)$.

PROOF: Let $K \in \mathcal{K}_d$ and ε as in (3.2). Take any $\eta > 0$ and $I \subset (-\varepsilon, \varepsilon)$. By (3.3), there exists $R_0 > 0$ such that

$$(3.4) \qquad\qquad |[K \cap (-\infty, -R_0) \cup (R_0, \infty)] \cap S^\varepsilon| < \frac{\eta}{2} \,.$$

Let k_0 be such that $2k_0 \pi > R_0 + 2\pi$, so that if $|n| > k_0$ we have $(2n\pi + I) \cap K \subset K \cap (-\infty, -R_0) \cup (R_0, \infty)$. Then,

$$[\tau_K(I)] \cap S^\varepsilon = \Big[\bigcup_{n \in \mathbb{Z},\, n \neq 0} (2n\pi + I) \cap K \Big] \cap S^\varepsilon$$

$$(3.5) \quad \subset \Big\{ \bigcup_{0 < |n| \leq k_0} [(2n\pi + I) \cap K] \cap S^\varepsilon \Big\} \cup \Big\{ [K \cap (-\infty, -R_0) \cup (R_0, \infty))] \cap S^\varepsilon \Big\} \,.$$

Thus, by (3.4) and Lemma 5.6 in [3]

$$|\tau_K(I) \cap S^\varepsilon| < \frac{\eta}{2} + \sum_{0 < |n| \leq k_0} \big| [(2n\pi + I) \cap K] \cap S^\varepsilon \big| \leq \frac{\eta}{2} + k_0 |I| \,.$$

Choosing $\delta < \frac{\eta}{2k_0}$ we obtain $|\tau_K(I) \cap S^\varepsilon| < \frac{\eta}{2} + \frac{\eta}{2}$ if $|I| < \delta$. This shows that $\mathcal{L}_K^{(S^\varepsilon)}$ is continuous and hence that $K \in \mathcal{K}(S)$. □

3.3. **Other examples.** The following examples show that $\mathcal{K}_\tau \cup \mathcal{K}_d \subsetneq \mathcal{K}(S)$, that is, there exist sets K which are admissible according to Definition 1.1, but are not contained in the setting of our previous paper [3]. In fact, it will suffice to consider $K = K_1 \cup K_2$, where $K_1 \in \mathcal{K}_\tau \setminus \mathcal{K}_d$ and $K_2 \in \mathcal{K}_d \setminus \mathcal{K}_\tau$. Indeed, when this is the case we necessarily have that $K \notin \mathcal{K}_\tau \cup \mathcal{K}_d$, while from the second observation at the beginning of §2 (and Propositions 3.1 and 3.2) we see that $K \in \mathcal{K}(S)$.

Example 3.3. Let $K_1 = \cup_{|n| \geq 1}(\frac{1}{2}S + 2\pi n)$. Then $K_1 \in \mathcal{K}_\tau \setminus \mathcal{K}_d$.

PROOF: From the definition we see that K_1 is disjoint with $2\pi m + (-\frac{\pi}{2}, \frac{\pi}{2})$, for every $m \in \mathbb{Z}$. This trivially implies that $K_1 \in \mathcal{K}_\tau$. Next we see that $K_1 \notin \mathcal{K}_d$. Given any $\varepsilon > 0$ and any large integer m we have

$$\big| L_m^{(\varepsilon)} \big| \geq \Big| \cup_{\ell=m}^\infty \frac{1}{2^\ell} \big\{ K_1 \cap 2^\ell [\tfrac{\varepsilon}{2}, \varepsilon) \big\} \Big| \geq 2^{-m} \big| K_1 \cap 2^m [\tfrac{\varepsilon}{2}, \varepsilon) \big|.$$

Now, if m is large enough we have

$$\big| K_1 \cap 2^m [\tfrac{\varepsilon}{2}, \varepsilon) \big| \geq \sum_{\substack{n \in \mathbb{Z} \\ 2^{m-1}\varepsilon < 2\pi n < 2^m \varepsilon - \pi}} \big| 2\pi n + [\tfrac{\pi}{2}, \pi) \big| \geq c\, 2^m \varepsilon,$$

for some constant $c > 0$. Thus, $\lim_{m \to \infty} |L_m^{(\varepsilon)}| \geq c\varepsilon > 0$, showing that $K_1 \notin \mathcal{K}_d$. $\square$

Example 3.4. *Let $K_2 = \cup_{n \geq 1} \left[2^n \pi, 2^n \pi (1 + \frac{1}{n}) \right)$. Then $K_2 \in \mathcal{K}_d \setminus \mathcal{K}_\tau$.*

PROOF: This time we have, for all $m \geq 1$,

$$\left| L_{m+1}^{(\pi)} \right| \;=\; \left| \cup_{\ell=m}^{\infty} \tfrac{1}{2^{\ell+1}} \left\{ K_2 \cap 2^{\ell+1} [\tfrac{\pi}{2}, \pi) \right\} \right| \;=\; \left| \cup_{\ell=m}^{\infty} \tfrac{1}{2^\ell} \left[2^\ell \pi, 2^\ell \pi (1 + \tfrac{1}{\ell}) \right) \right| \;=\; \frac{\pi}{m}.$$

Since this number tends to 0 as $m \to \infty$, we have established that $K_2 \in \mathcal{K}_d$. On the other hand, for any $\varepsilon > 0$, and for integers $m = 2^n + r$ (with n large enough) we have $\left| K_2 \cap \{ 2\pi(2^n + r) + (-\varepsilon, \varepsilon) \} \right| = 2\varepsilon$, at least when $0 < r < \frac{c2^n}{n}$, for some universal constant $c > 0$. This implies

$$
\sum_{m \geq 1} \frac{1}{m} \left| K_2 \cap \{ 2\pi m + (-\varepsilon, \varepsilon) \} \right| \;\geq\; \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{0 \leq r < 2^n} \left| K_2 \cap \{ 2\pi(2^n + r) + (-\varepsilon, \varepsilon) \} \right|
$$

$$
\geq\; c \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \frac{2^n}{n} \varepsilon \;=\; \infty,
$$

and therefore K_2 cannot belong to $\mathcal{K}_\tau$. $\square$

4. Non admissibility of $\mathbb{R}$

In this section we show that the classes $\mathcal{K}(J_0)$ do not contain the set $\mathbb{R}$, and as a consequence our results on connectivity do not apply to the full class of **PFW**'s. In fact, if $\psi \in \mathbf{PFW}$ and $\operatorname{supp} \widehat{\psi} = \mathbb{R}$, then from Theorem 4.1 below it follows that the "intermediate path" produced with our construction is of the form $\{ \widehat{\psi}_t = \chi_{E_{I(t)}} \}_{t \in (0,1]}$, which is clearly discontinuous at $t = 0$.

As we shall see the proof of Theorem 4.1 is not completely trivial, and has an unexpected connection with number theory which we shall make clear in the next section. For convenience, throughout these two sections we shall replace the number 2π by 1.

Theorem 4.1. *For all sets $A \subset \mathbb{R}$ of positive Lebesgue measure we have*

$$(4.1) \qquad \left| \mathbb{R} \setminus \bigcup_{\ell \in \mathbb{Z}} \bigcup_{k \neq 0} 2^{-\ell}(A + k) \right| \;=\; 0.$$

PROOF: Set $S = [-1, -1/2) \cup [1/2, 1)$. It suffices to show that

$$(4.2) \qquad \left| S \setminus \bigcup_{\ell \in \mathbb{Z}} \bigcup_{k \neq 0} 2^{-\ell}(A + k) \right| \;=\; 0.$$

for all sets $A \subset \mathbb{R}$ of positive Lebesgue measure. By symmetry first and then by rescaling, we may assume with no loss of generality that $A \subset [0, 1)$. Moreover, (4.2) will follow from the statement:

For all dyadic intervals $I \subset [0, 1)$

$$(4.3) \qquad \left| [1/2, 1) \setminus \bigcup_{\ell \geq L+1} \bigcup_{k \geq 1} 2^{-\ell}(I + k) \right| \;=\; 0, \quad \forall L = 0, 1, 2, \dots$$

To see this, given any set $A \subset [0, 1)$ of positive Lebesgue measure and $\delta > 0$, take a dyadic interval I such that

$$(4.4) \qquad 1 - \delta \leq \frac{|I \cap A|}{|I|} \leq 1 \,.$$

By the statement (4.3) with $L = 0$ and the nesting property of the family of dyadic intervals $I_{\ell,k} = 2^{-\ell}(I + k)$, we can write $[1/2, 1)$ as a disjoint union $\biguplus_{(\ell,k)\in\mathcal{D}} I_{\ell,k}$ (a.e.), for a certain index set $\mathcal{D} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$. If we denote $A_{\ell,k} \equiv 2^{-\ell}(A + k)$, we also have

$$\big|\,[1/2, 1) \setminus \bigcup_{\ell\geq 1} \bigcup_{k\geq 1} 2^{-\ell}(A + k)\,\big| \leq \sum_{(\ell,k)\in\mathcal{D}} |I_{\ell,k} - A_{\ell,k}| = \sum_{(\ell,k)\in\mathcal{D}} |I_{\ell,k}| - |I_{\ell,k} \cap A_{\ell,k}|.$$

Now, by a simple rescaling of (4.4), we see that $|I_{\ell,k}| - |I_{\ell,k} \cap A_{\ell,k}| \leq \delta|I_{\ell,k}|$ and, therefore, we obtain

$$\big|\,[1/2, 1) \setminus \bigcup_{\ell\geq 1} \bigcup_{k\geq 1} 2^{-\ell}(A + k)\,\big| \leq \sum_{(\ell,k)\in\mathcal{D}} \delta\,|I_{\ell,k}| = \delta\,\big|\,\biguplus_{(\ell,k)\in\mathcal{D}} I_{\ell,k}\,\big| = \frac{\delta}{2}.$$

Since δ is arbitrary, the left hand side equals 0. By symmetry,

$$\big|\,(-1, -1/2] \setminus \bigcup_{\ell\geq 1} \bigcup_{k\leq -1} 2^{-\ell}(A + k)\,\big| = 0,$$

which gives (4.2). It remains then to prove (4.3).

The result is clearly true for the particular case $I = [0, 1)$ since

$$2^{-(L+1)} \bigcup_{k=2^L}^{2^{L+1}-1} (I + k) = 2^{-(L+1)}[2^L, 2^{L+1}) = [1/2, 1).$$

The key step to finish the proof of Theorem 4.1 is contained in the following lemma.

Lemma 4.2. *If the inclusion in (4.3) is true for a given dyadic interval $I \subset [0, 1)$, then it is true for its two dyadic sons I^+ and I^-, $I = I^+ \cup I^-$, and therefore, for all its descendants.*

Proof. Take $L \in \mathbb{N}$. From the hypothesis, there exist positive integers k_j, ℓ_j, with $\ell_j > L$, so that the intervals $I_j = 2^{-\ell_j}(I + k_j)$ are mutually disjoint, for $j = 1, 2, \ldots$, and $[1/2, 1) = \biguplus_{j\geq 1} I_j$, a.e. If we set $I_j^{\pm} = 2^{-\ell_j}(I^{\pm} + k_j)$, then $|I_j^+| = \frac{1}{2}|I_j|$ and so the sequence $\{I_j^+\}_j$ "covers" $\frac{1}{2}$ of the size of $[1/2, 1)$. Fix now $j \in \mathbb{N}$ and take $L_j = \ell_j + 1 > L$ so that $2^{-L_j}|I| = |I_j^-|$. Then, from the hypothesis again we can cover $[1/2, 1)$ by disjoint intervals of the form $2^{-\ell}(I + k)$ with $\ell > L_j$. In particular, those intersecting I_j^- are necessarily contained in I_j^- (since they are smaller). Thus, for this j, there are positive integers $k_{j,i}, \ell_{j,i}$, with $\ell_{j,i} > L_j$, so that the intervals $I_{j,i} = 2^{-\ell_{j,i}}(I + k_{j,i})$ are mutually disjoint, for $i = 1, 2, \ldots$, and $I_j^- = \biguplus_{i\geq 1} I_{j,i}$ a.e. Hence

$$[1/2, 1) = \left(\biguplus_{j\geq 1} \biguplus_{i\geq 1} I_{j,i}\right) \biguplus \left(\biguplus_{j\geq 1} I_j^+\right).$$

Call $I_{j,i}^+ = 2^{-\ell_{j,i}}(I^+ + k_{j,i})$. Then $\{I_j^+, I_{j,i}^+ : j = 1, 2, \ldots; i = 1, 2, \ldots\}$ covers $\frac{3}{4}$ of $[1/2, 1)$. By a straightforward induction process, we obtain the result for I^+ and, similarly, for I^-.

5. CONNECTIONS TO NUMBER THEORY

We begin with an elementary observation. The statement of Theorem 4.1 is equivalent to

(A) *There exists a null set $E \subset [0,1)$ so that for all $x \in [0,1) \setminus E$ the sequence of points $\{ 2^\ell x \pmod 1 : \ell \geq 1 \}$ is dense in $[0,1)$.*

Suppose (A) holds and let $L \geq 0$ be a fixed integer. If I is any dyadic interval in $[0,1)$ and $x \in [0,1) \setminus E$, by density of $\{ 2^\ell x \pmod 1 : \ell > L \}$ there must exist integers $\ell > L$, $k \in \mathbb{N}$ so that $2^\ell x - k \in I$. Thus $|[0,1) \setminus \cup_{\ell > L, k \geq 1} I_{\ell,k}| = 0$ which implies (4.3). For the converse, if Theorem 4.1 holds we also have

$$\Big| [0,1) \setminus \bigcup_{\ell \geq 1} \bigcup_{k \geq 1} 2^{-\ell}(I + k) \Big| = 0,$$

for every dyadic interval $I \subset [0,1)$. Since the number of dyadic intervals is countable, we can define a set of Lebesgue measure zero by

$$E = \bigcup_{I \text{ dyadic}} \Big\{ [0,1) \setminus \bigcup_{\ell \geq 1} \bigcup_{k \geq 1} 2^{-\ell}(I + k) \Big\}$$

Then, for every $x \in [0,1) \setminus E$ the sequence $\{ 2^\ell x \pmod 1 : \ell \geq 1 \}$ touches each dyadic interval I (in fact infinitely often), which implies clearly that this sequence must be dense.

The assertion (A) has a number theoretic flavour. In fact, it is actually a consequence of a stronger result related with "uniformly distributed sequences". We say that a sequence $\{a_\ell\}_{\ell=1}^\infty$ of real numbers in $[0,1)$ is *uniformly distributed in $[0,1)$* if for every interval $I \subset [0,1)$ we have

$$\lim_{n \to \infty} \frac{1}{n} \#\big\{ 1 \leq \ell \leq n : a_\ell \in I \big\} = |I|.$$

Of course, every uniformly distributed sequence must be dense in $[0,1)$. A classical theorem of H. Weyl et al. establishes that for all irrational numbers $x \in [0,1)$ the sequence $\{\ell x\}_{\ell=1}^\infty$ is uniformly distributed modulo 1 (see, e.g., [6, Th.6.3]). There exist several extensions of this result, based on Weyl's celebrated characterization of uniformly distributed sequences in terms of limits of exponential sums. One of these extensions establishes that *for every strictly increasing sequence $\{a_\ell\}_{\ell=1}^\infty \subset \mathbb{N}$ and for a.e. $x \in [0,1)$ the sequence $\{a_\ell x\}_{\ell=1}^\infty$ is uniformly distributed modulo 1* (see, e.g., [2, p. 138]). If as a particular case we let $a_\ell = 2^\ell$, then we obtain a stronger version of the result stated in (A).

There is another interpretation of (A) purely in terms of the intrinsic properties of numbers. Indeed if we represent the number $x \in [0,1)$ in its binary expansion, then the fact that the sequence $\{2^\ell x\}_{\ell=1}^\infty$ is dense modulo 1 in $[0,1)$ is the same as saying that any arbitrary block of $\{0,1\}$-digits "$b_1 \ldots b_k$" is contained infinitely often in the binary expansion of x. We shall call the numbers $x \in [0,1)$ with this property *weakly normal* (to base 2). Then, in view of the previous, the statement of Theorem 4.1 is equivalent to say that $a.e.\, x \in [0,1)$ is a weakly normal number.

In relation with this concept, a stronger result is also known. We say that a number $x \in [0,1)$ is *normal to base r* if any arbitrary block made of digits in $\{0,1,\ldots,r-1\}$ of length k, say "$B_k = b_1 b_2 \ldots b_k$", is contained in the decimal expansion of x in base r with frequency $1/r^k$. That is, if $N(B_k, n)$ counts the number of occurrences of the block B_k within the first n digits of the expansion of x, then we must have

$$\lim_{n \to \infty} \frac{N(B_k, n)}{n} = \frac{1}{r^k}, \qquad \forall\, B_k, \quad \forall\, k \geq 1$$

(see e.g. [6, Ch.8] for this and other equivalent definitions). In particular, every normal number to base 2 is weakly normal according to our previous definition. Then, a classical theorem of E. Borel (*Rend. Circ. Mat. Palermo* **27** (1909), 247–271) establishes that *a.e. $x \in [0, 1)$ is normal to every base r*. This gives another proof of (A), and hence of Theorem 4.1. In fact, the two notions we have introduced are related by a theorem due to D. Wall which asserts that *a number $x \in [0, 1)$ is normal to base r if and only if the sequence $\{r^\ell x\}_{\ell=1}^\infty$ is uniformly distributed modulo 1* (see [6, Th. 8.15]).

Finally we would like to point out that verifying one of the previous properties for a specific number x can be a very hard question. For instance it is not known whether such numbers as $\sqrt{2}$, e or π are normal to any base. In fact, the set of numbers which are *not* normal to any base are of course dense (since it contains the rationals) and can be shown to be uncountable [6, p.116]. As a simple example observe that the binary numbers of the form $x = .0b_1 0b_2 0b_3....$ (for $b_i = 0, 1$) are not normal, nor even weakly normal to base 2. Thus, the exceptional set in assertion (A) is a dense uncountable subset of $[0, 1)$.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID (SPAIN)
E-mail address: `gustavo.garrigos@uam.es, eugenio.hernandez@uam.es, fernando.soria@uam.es`

DEPT. OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 10000 ZAGREB (CROATIA)
E-mail address: `hsikic@math.hr`