

LEBESGUE-TYPE INEQUALITIES FOR QUASI-GREEDY BASES

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ABSTRACT. We show that for quasi-greedy bases in real or complex Banach spaces the error of the thresholding greedy algorithm of order N is bounded by the best N -term error of approximation times a function of N which depends on the democracy functions and the quasi-greedy constant of the basis. If the basis is democratic this function is bounded by $C \log N$. We show with two examples that this bound is attained for quasi-greedy democratic bases.

1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space (real or complex) and $\mathcal{B} = \{\mathbf{e}_j\}_{j=1}^{\infty}$ a countable normalized basis¹. Let $\Sigma_N, N = 1, 2, 3, \dots$ be the set of all $y \in \mathbb{X}$ with at most N non-null coefficients in the unique basis representation. For $x \in \mathbb{X}$, the **N -term error of approximation** with respect to \mathcal{B} is

$$\sigma_N(x) = \sigma_N(x; \mathcal{B}, \mathbb{X}) := \inf_{y \in \Sigma_N} \|x - y\|_{\mathbb{X}}, \quad N = 1, 2, 3, \dots$$

Given $x = \sum_{k=1}^{\infty} a_k(x) \mathbf{e}_k \in \mathbb{X}$, let π denote any bijection of \mathbb{N} such that

$$|a_{\pi(k)}(x)| \geq |a_{\pi(k+1)}(x)| \quad \text{for all } k \in \mathbb{N}. \quad (1.1)$$

The **thresholding greedy algorithm of order N** (TGA) is defined by

$$G_N(x) = G_N^{\pi}(x; \mathcal{B}, \mathbb{X}) := \sum_{k=1}^N a_{\pi(k)}(x) \mathbf{e}_{\pi(k)}.$$

It is not always true that $G_N(x) \rightarrow x$ (in \mathbb{X}) as $N \rightarrow \infty$. A basis \mathcal{B} is called **quasi-greedy** if $G_N(x) \rightarrow x$ as $N \rightarrow \infty$ for all $x \in \mathbb{X}$. This turns out to be equivalent (see [22, Theorem 1]) to the existence of some constant \tilde{K} such that

$$\sup_N \|G_N(x)\| \leq \tilde{K} \|x\| \quad \text{for all } x \in \mathbb{X}. \quad (1.2)$$

We define the **quasi-greedy constant K** of the basis \mathcal{B} to be the least \tilde{K} such that (1.2) holds for all permutations π satisfying (1.1).

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¹We assume normalization, $\|\mathbf{e}_j\| = 1$, for notational convenience; all the results are actually valid for seminormalized bases, perhaps after suitable modifications in the constants.

Given a basis \mathcal{B} in a Banach space \mathbb{X} , a **Lebesgue-type inequality** for the TGA is an inequality of the form

$$\|x - G_N(x)\| \leq C(N) \sigma_N(x), \quad x \in \mathbb{X},$$

where $C(N)$ is a nondecreasing function of N . For a survey on Lebesgue-type inequalities for the greedy algorithm see [18, 19] and the references therein. We specially mention the recent papers [20, 21], which deal with Lebesgue-type inequalities for quasi-greedy bases in L^p spaces (see also [5]).

The purpose of this paper is to study such inequalities for quasi-greedy bases in general Banach spaces, thus complementing and in some cases improving the results in [20, 21, 5]. Towards this end we define the sequence

$$C_N := \sup_{x \in \mathbb{X}} \frac{\|x - G_N(x)\|}{\sigma_N(x)}.$$

Following the notation in previous papers, we write

$$h_l(n) = \inf_{|A|=n} \left\| \sum_{j \in A} \mathbf{e}_j \right\|, \quad h_r(n) = \sup_{|A|=n} \left\| \sum_{j \in A} \mathbf{e}_j \right\| \quad \text{and} \quad \mu(N) = \sup_{n \leq N} \frac{h_r(n)}{h_l(n)}.$$

These functions are implicit in the first works on N -term approximation. For instance, $\mu(N)$ is defined in [22], and h_l, h_r appear explicitly in [11]. In [8, 9] the latter are called left and right democracy functions of the basis \mathcal{B} .

For $A \subset \mathbb{N}$, we denote by S_A the projection operator

$$x = \sum_{j=1}^{\infty} a_j \mathbf{e}_j \mapsto S_A(x) = \sum_{j \in A} a_j \mathbf{e}_j,$$

and consider also the sequence

$$k_N := \sup_{|A| \leq N} \|S_A\|.$$

Our main result is the following:

Theorem 1.1. *If $\{\mathbf{e}_j\}_{j=1}^{\infty}$ is a (normalized) quasi-greedy basis in a Banach space \mathbb{X} (real or complex), then*

$$C_N \approx \max\{\mu(N), k_N\}, \quad \forall N = 1, 2, \dots \quad (1.3)$$

Remarks:

- (1) When $\{\mathbf{e}_j\}$ is unconditional, then $k_N = O(1)$, so we obtain as a special case Theorem 4 in [22].
- (2) For quasi-greedy bases it can be shown that

$$k_N \leq c \log N. \quad (1.4)$$

This is essentially contained in [4, Lemma 8.2] (see also [5, Lemma 2.3]). Since this result is often used in the paper, we outline a proof in §5 below.

- (3) When $\{\mathbf{e}_j\}$ is quasi-greedy, some upper bounds for C_N have recently appeared in the literature: in [21, Theorem 2.1] it was shown that

$$C_N \lesssim \mu(N) k_N, \quad (1.5)$$

while in [10, Thm 1.1] it is proved that

$$C_N \lesssim \sum_{1 \leq k \leq N} \frac{\mu(k)}{k} \quad \left(\lesssim \mu(N) \log N \right). \quad (1.6)$$

Notice that (1.3), being an equivalence, improves strictly over these in some cases. For instance, if \mathbb{X} is such that $\mu(N) \approx (\log N)^\alpha$ and say $k_N \approx \log N$, then (1.5) and (1.6) would only give $C_N \lesssim (\log N)^{\alpha+1}$, while Theorem 1.1 implies $C_N \approx (\log N)^{\min\{\alpha,1\}}$. For constructions of such examples, see (6.9) below.

- (4) When $\{\mathbf{e}_j\}$ is quasi-greedy and democratic (i.e. $\mu(N) = O(1)$), then (1.3) and (1.4) give

$$C_N \approx k_N \lesssim \log N. \quad (1.7)$$

We show in section 6 that this logarithmic bound can actually be attained, answering a question posed in [10]. One such example is given by the Haar basis in $BV(\mathbb{R}^d)$, $d > 1$ (see §6). This is in contrast with the Hilbert space case, where it was recently noticed by Wojtaszczyk that k_N cannot attain $\log N$ ([25]; see also §9 below).

Let also denote by $\tilde{\sigma}_N(x)$ the *expansional best approximation* to x , that is if $x = \sum_{k=1}^{\infty} a_k \mathbf{e}_k$, then

$$\tilde{\sigma}_N(x) = \tilde{\sigma}_N(x; \mathcal{B}, \mathbb{X}) := \inf_{A, |A|=N} \left\{ \left\| x - \sum_{k \in A} a_k \mathbf{e}_k \right\| \right\}.$$

In this case it is known that, for quasi-greedy bases,

$$\tilde{C}_N := \sup_{x \in \mathbb{X}} \frac{\|x - G_N(x)\|}{\tilde{\sigma}_N(x)} \approx \mu(N); \quad (1.8)$$

see [21, Theorem 2.2] for the upper bound (the lower bound was essentially in [22]; see also Proposition 3.1 below). In [10] it was asked whether one could prove bounds for C_N using (1.8) and suitable bounds on the sequence

$$D_N := \sup_{x \in \mathbb{X}} \frac{\tilde{\sigma}_N(x)}{\sigma_N(x)} \geq 1.$$

Here we prove the following

Theorem 1.2. *For any (normalized, not necessarily quasi-greedy) basis $\{\mathbf{e}_j\}$ we have*

$$\frac{k_N}{4} \leq D_N \leq 2k_N, \quad \forall N = 1, 2, \dots \quad (1.9)$$

Remark: The right hand side of (1.9) together with (1.4) gives $\tilde{\sigma}_N(x) \lesssim (\log N)\sigma_N(x)$ for quasi-greedy bases. This was noticed in [5, Lemma 2.4], answering a question from [10]. The left hand side of (1.9) seems to be new.

Our last result is the following theorem, which answers a question of Wojtaszczyk (personal communication to the second author on November, 2011).

Theorem 1.3. *If $\{\mathbf{e}_j\}_{j=1}^{\infty}$ is a quasi-greedy basis in \mathbb{X} , then there exists $c > 0$ such that for all $N, k = 1, 2, \dots$*

$$\|x - G_{N+k}(x)\| \leq c \left(1 + \frac{h_r(N)}{h_l(k)} \right) \sigma_N(x), \quad \forall x \in \mathbb{X}.$$

Results of this type have appeared before in the literature. For unconditional bases, this theorem was proved in [11, Thm 5]; see also [23, Thm 4]. For quasi-greedy democratic bases it is essentially contained in [6]. Here we extend its validity to general quasi-greedy bases.

A slightly weaker version of Theorem 1.3 can also be found in [13]; namely, given N and k there exists a set A of cardinality not exceeding $N + k$ such that

$$\|x - S_A(x)\| \leq c \left(1 + \frac{h_r(N)}{h_l(k)}\right) \sigma_N(x), \quad \forall x \in \mathbb{X}.$$

The improvement in Theorem 1.3 consists in showing that the set A can be obtained by running the greedy algorithm.

We finally remark that the proofs of Theorems 1.1, 1.2 and 1.3 combine ideas present in various of the above quoted references, but whose main lines essentially stem from the original work of Konyagin and Temlyakov [12].

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2. PROOF OF THEOREM 1.1: UPPER BOUNDS

The proof follows the strategy developed in [12], together with two known estimates for quasi-greedy bases. First, as mentioned in §1, there exists a (smallest) constant K such that

$$\|G_N(x)\| \leq K \|x\|, \quad \forall x \in \mathbb{X}, \quad \forall N = 1, 2, \dots, \quad (2.1)$$

see [22, Th 1]. Also, there exist $c_1, c_2 > 0$ such that

$$c_1 \left(\min_{k \in A} |a_k|\right) \left\| \sum_{k \in A} \mathbf{e}_k \right\| \leq \left\| \sum_{k \in A} a_k \mathbf{e}_k \right\| \leq c_2 \left(\max_{k \in A} |a_k|\right) \left\| \sum_{k \in A} \mathbf{e}_k \right\|. \quad (2.2)$$

These inequalities are proved in [6, Lemmas 2.1 and 2.2] for real scalars a_k , setting $c_1 = 1/(4K^2)$ and $c_2 = 2K$. For completeness, in the appendix (§10) we outline the proof also for complex scalars a_k , in which case one can let $c_1 = 1/(8\sqrt{2}K^2)$ and $c_2 = 4\sqrt{2}K$.

We shall write $a_k^*(x)$ for the decreasing rearrangement of the basis coefficients of x ; that is, if $x = \sum_{j=1}^{\infty} a_j \mathbf{e}_j$, we set $a_k^*(x) = |a_{\pi(k)}|$ when π is any permutation of \mathbb{N} such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq |a_{\pi(3)}| \geq \dots$. As in [10], we shall use the following simple (but crucial) observation.

Lemma 2.1. *For all $A \subset \mathbb{N}$ and $x \in \mathbb{X}$ we have $a_k^*(S_A(x)) \leq a_k^*(x)$.*

We now prove the theorem. Fix $N \geq 1$ and $x \in \mathbb{X}$. Take any $y = \sum_{j \in A} y_j \mathbf{e}_j$ with $|A| = N$. We shall show that

$$\|x - G_N(x)\| \leq c \max\{\mu(N), k_N\} \|x - y\|. \quad (2.3)$$

Then, taking the infimum of the right hand side over all $y \in \Sigma_N$, we obtain the upper estimate for C_N in (1.3).

Write $G_N(x) = S_\Gamma(x)$ with $|\Gamma| = N$. Then

$$\begin{aligned} \|x - G_N(x)\| &= \|x - S_A(x) + S_A(x) - S_\Gamma(x)\| \\ &\leq \|x - S_A(x)\| + \|S_{A \setminus \Gamma}(x)\| + \|S_{\Gamma \setminus A}(x)\|. \end{aligned}$$

The first and third terms are easily bound by $c k_N \|x - y\|$; namely,

$$\|x - S_A(x)\| \leq \|x - y\| + \|S_A(y) - S_A(x)\| \leq (1 + k_N) \|x - y\|,$$

while

$$\|S_{\Gamma \setminus A}(x)\| = \|S_{\Gamma \setminus A}(x - y)\| \leq k_N \|x - y\|.$$

We next show that $\|S_{A \setminus \Gamma}(x)\|$ can be controlled by $c \mu(N) \|x - y\|$. First notice that

$$\|S_{A \setminus \Gamma}(x)\| \leq c_2 \left(\max_{k \in A \setminus \Gamma} |a_k| \right) \left\| \sum_{k \in A \setminus \Gamma} \mathbf{e}_k \right\| \leq c_2 \mu(N) \left(\min_{k \in \Gamma \setminus A} |a_k| \right) \left\| \sum_{k \in \tilde{\Gamma}} \mathbf{e}_k \right\| \quad (2.4)$$

where we choose as $\tilde{\Gamma}$ any set of cardinality $|\tilde{\Gamma}| = |A \setminus \Gamma| = |\Gamma \setminus A|$ in which $x - y$ attains the largest coefficients, i.e. $G_{|\Gamma \setminus A|}(x - y) = S_{\tilde{\Gamma}}(x - y)$. From Lemma 2.1 one easily sees that

$$\begin{aligned} \min_{k \in \Gamma \setminus A} |a_k(x)| &= \min_{k \in \Gamma \setminus A} |a_k(S_{\Gamma \setminus A}(x - y))| = a_{|\Gamma \setminus A|}^*(S_{\Gamma \setminus A}(x - y)) \\ &\leq a_{|\Gamma \setminus A|}^*(x - y) = \min_{k \in \tilde{\Gamma}} |a_k(x - y)|. \end{aligned}$$

Thus, using again (2.2), one can bound the right side of (2.4) by a constant times

$$\mu(N) \|S_{\tilde{\Gamma}}(x - y)\| = \mu(N) \|G_{|\Gamma \setminus A|}(x - y)\| \leq K \mu(N) \|x - y\|,$$

as we wished to prove. Notice that the final multiplicative constant involved in this process is of order $K c_2 / c_1 = O(K^4)$. \square

3. PROOF OF THEOREM 1.1: LOWER BOUNDS

The bound $C_N \gtrsim \mu(N)$ was proved by Wojtaszczyk when $\{\mathbf{e}_j\}$ is an unconditional basis; [22, Thm 4]. As pointed out in [10], these arguments can easily be adapted to the more general setting of quasi-greedy bases; we include the proof for completeness. Recall that K is our notation for the quasi-greedy constant defined in (2.1).

Proposition 3.1. *If $\{\mathbf{e}_j\}$ is quasi greedy then $C_N \geq \tilde{C}_N \geq \frac{1}{3K} \mu(N)$.*

We shall use the following lemma. Here we denote $\mathbf{1}_A = \sum_{j \in A} \mathbf{e}_j$.

Lemma 3.2. *If $\{\mathbf{e}_j\}$ is quasi greedy with constant K , then for every N there exist disjoint sets A, B such that*

$$|A| = |B| \leq N \quad \text{and} \quad \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \geq \frac{1}{3K} \mu(N).$$

Proof. We may assume that $\mu(N) > 3K$ (otherwise choose $|A| = |B| = 1$). Then there exist A, B (not necessarily disjoint) with $|A| = |B| \leq N$ and

$$\max\left\{\frac{1}{2}\mu(N), 3K\right\} < \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|}.$$

The quasi-greedy condition implies that $\|\mathbf{1}_{A \cap B}\| \leq K\|\mathbf{1}_B\|$, which inserted above gives

$$3K < \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \leq K \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_{A \cap B}\|}$$

and therefore $\|\mathbf{1}_{A \cap B}\| \leq \frac{1}{3}\|\mathbf{1}_A\|$. Thus

$$\frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \leq \frac{\|\mathbf{1}_{A \cap B}\| + \|\mathbf{1}_{A \setminus B}\|}{\|\mathbf{1}_B\|} \leq \frac{1}{3} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} + \frac{\|\mathbf{1}_{A \setminus B}\|}{\|\mathbf{1}_B\|},$$

which can be rewritten as

$$\frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \leq \frac{3}{2} \frac{\|\mathbf{1}_{A \setminus B}\|}{\|\mathbf{1}_B\|}.$$

Now set $\tilde{A} := (A \setminus B) \cup C$, for any C , disjoint with B , such that $|\tilde{A}| = |B|$. Then

$$\frac{1}{2}\mu(N) \leq \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \leq \frac{3}{2} \frac{\|\mathbf{1}_{A \setminus B}\|}{\|\mathbf{1}_B\|} \leq \frac{3}{2}K \frac{\|\mathbf{1}_{\tilde{A}}\|}{\|\mathbf{1}_B\|},$$

which gives the desired result since $\tilde{A} \cap B = \emptyset$. \square

PROOF of Proposition 3.1: Consider sets A and B as in the lemma, and take any set C , disjoint with $A \cup B$, such that $|C| = N - |A| = N - |B|$. Choosing $x = (1 + 2\varepsilon)\mathbf{1}_B + (1 + \varepsilon)\mathbf{1}_C + \mathbf{1}_A$ we have

$$\|x - G_N(x)\| = \|\mathbf{1}_A\| \geq \frac{1}{3K} \mu(N) \|\mathbf{1}_B\| \geq \frac{1}{(1+2\varepsilon)3K} \mu(N) \tilde{\sigma}_N(x),$$

which proves the result when $\varepsilon \rightarrow 0$. \square

To establish the lower bound in Theorem 1.1 it remains to show the following.

Proposition 3.3. *For any basis (not necessarily quasi-greedy) we have*

$$C_N \geq k_N/4. \quad (3.1)$$

Proof. Assume that $k_N \geq 4$ (otherwise (3.1) is trivial). For fixed N find $A \subset \mathbb{N}$ with $|A| \leq N$ and $x \in \mathbb{X}$ such that $\|S_A(x)\| > (k_N/2)\|x\|$. We may assume that $x = \sum_{j \in B} x_j \mathbf{e}_j$ with B finite and $\|x\| = 1$. Note that $\|x - S_A(x)\| \geq \|S_A(x)\| - \|x\| \geq k_N/4$.

Take any number $r > \max |x_j|$, and set $y = x - S_A(x) + r\mathbf{1}_{\tilde{A}}$. Here \tilde{A} is any set of cardinality N containing A and (if necessary) some indices in B^c . Then

$$\|y - G_N(y)\| = \|x - S_A(x)\| \geq k_N/4.$$

On the other hand, since $r\mathbf{1}_{\tilde{A}} - S_A(x) \in \Sigma_N$ we have

$$\sigma_N(y) \leq \|y - (r\mathbf{1}_{\tilde{A}} - S_A(x))\| = \|x\| = 1,$$

which gives (3.1). \square

4. PROOF OF THEOREM 1.2

The upper bound $D_N \leq 2k_N$ is elementary. Indeed, let $x \in \mathbb{X}$ and $p = \sum_{k \in P} b_k \mathbf{e}_k \in \Sigma_N$ with $|P| = N$. Then,

$$\tilde{\sigma}_N(x) \leq \|x - S_P(x)\| \leq \|x - p\| + \|S_P(x - p)\| \leq (1 + k_N)\|x - p\|.$$

Taking the infimum over all $p \in \Sigma_N$ we obtain

$$\tilde{\sigma}_N(x) \leq (1 + k_N)\sigma_N(x),$$

which proves $D_N \leq 2k_N$.

For the converse we argue as in the proof of Proposition 3.3. That is, we choose an $x = \sum_{j \in B} x_j \mathbf{e}_j$ with $\|x\| = 1$, and a set A so that $\|x - S_A(x)\| \geq k_N/4$, and we let $y = x - S_A(x) + r\mathbf{1}_{\tilde{A}}$ as before. This time we shall choose $r > (2 + k_N + k_{2N})2\mathbf{c}$, where \mathbf{c} is the basis constant, and we shall prove that, with this choice

$$\tilde{\sigma}_N(y) \geq \frac{k_N}{4} \sigma_N(y),$$

which clearly implies $D_N \geq k_N/4$.

As shown before, $\sigma_N(y) \leq 1$, so we need to prove that

$$\tilde{\sigma}_N(y) = \inf_{|C| \leq N} \|y - \sum_{(B \setminus A) \cap C} x_j \mathbf{e}_j - r\mathbf{1}_{\tilde{A} \cap C}\| \geq k_N/4. \quad (4.1)$$

Suppose we are given one such set C which is not equal to \tilde{A} . Then there must be some $j_0 \in \tilde{A} \setminus C$, and we would have

$$\begin{aligned} \|y - \sum_{(B \setminus A) \cap C} x_j \mathbf{e}_j - r\mathbf{1}_{\tilde{A} \cap C}\| &= \left\| \sum_{B \setminus (A \cup C)} x_j \mathbf{e}_j + r\mathbf{1}_{\tilde{A} \setminus C} \right\| \\ &\geq r \|\mathbf{1}_{\tilde{A} \setminus C}\| - \left\| \sum_{B \setminus (A \cup C)} x_j \mathbf{e}_j \right\| \\ &\geq \frac{r}{2\mathbf{c}} \|\mathbf{e}_{j_0}\| - \|x - S_{A \cup C}(x)\| \geq \frac{r}{2\mathbf{c}} - (1 + k_{2N}) > 1 + k_N, \end{aligned}$$

where in the third line we have used that $e_{j_0} = P_{j_0}(\mathbf{1}_{\tilde{A} \setminus C}) - P_{j_0-1}(\mathbf{1}_{\tilde{A} \setminus C})$, and the partial sums operators P_j have norm bounded by \mathbf{c} . On the other hand, if we use $C = \tilde{A}$ we obtain a better estimate

$$\|y - r\mathbf{1}_{\tilde{A}}\| = \|x - S_A(x)\| \leq 1 + k_N.$$

Therefore,

$$\tilde{\sigma}_N(y) = \|y - r\mathbf{1}_{\tilde{A}}\| = \|x - S_A(x)\| \geq k_N/4,$$

proving (4.1).

5. AN UPPER BOUND FOR k_N

We prove a bound for the constants k_N when $\{\mathbf{e}_j\}$ is a quasi-greedy basis.

Theorem 5.1. *If the basis is quasi-greedy, there exists $c > 0$ such that*

$$k_N \leq c \log N, \quad \forall N = 2, 3, \dots \quad (5.1)$$

This was essentially shown in [4, Lemma 8.2] (see also [5, Lemma 2.3]), but we include a self-contained proof for completeness. We need two easy lemmas.

Lemma 5.2. *Let $(\mathbb{X}, \{\mathbf{e}_j\})$ be quasi-greedy. Consider $x = \sum_i a_i \mathbf{e}_i \in \mathbb{X}$ and $0 \leq \alpha < \beta < \infty$. Let $F = \{i : |a_i| \in (\alpha, \beta]\}$. Then $\|S_F(x)\| \leq 2K\|x\|$.*

Proof. Let $G = \{i : |a_i| > \alpha\}$ and $H = \{i : |a_i| > \beta\}$. By the definition of quasi-greediness, $\max\{\|S_G(x)\|, \|S_H(x)\|\} \leq K\|x\|$. However, $S_F(x) = S_G(x) - S_H(x)$. Apply the triangle inequality to finish the proof. \square

Lemma 5.3. *Let $(\mathbb{X}, \{\mathbf{e}_j\})$ be quasi-greedy. Consider $x = \sum_i a_i \mathbf{e}_i \in \mathbb{X}$ and $0 < \alpha < \beta < \infty$. Then, for any $P \subset F = \{i : |a_i| \in (\alpha, \beta]\}$, we have*

$$\|S_P(x)\| \leq K \frac{c_2}{c_1} \frac{\beta}{\alpha} \|S_F(x)\|,$$

where c_1, c_2 are as in (2.2).

Proof. We use (2.2) (see [6] or Proposition 10.5 below). We have

$$\|S_P(x)\| = \left\| \sum_{i \in P} a_i \mathbf{e}_i \right\| \leq c_2 \beta \left\| \sum_{i \in P} \mathbf{e}_i \right\|.$$

By quasi-greediness

$$\left\| \sum_{i \in P} \mathbf{e}_i \right\| \leq K \left\| \sum_{i \in F} \mathbf{e}_i \right\|.$$

Finally,

$$\left\| \sum_{i \in F} \mathbf{e}_i \right\| \leq \frac{1}{c_1 \alpha} \left\| \sum_{i \in F} a_i \mathbf{e}_i \right\| = \frac{1}{c_1 \alpha} \|S_F(x)\|.$$

\square

We now prove Theorem 5.1. Take $|A| = N \geq 2$. Let $x = \sum_i a_i \mathbf{e}_i$. By scaling we may assume $\max_i |a_i| = 1$. Under this assumption

$$\|x\| \geq (1/K). \tag{5.2}$$

In fact, if $\max_i |a_i| = |a_{i_0}| = 1$ for some i_0 , then $1 = |a_{i_0}| \|\mathbf{e}_{i_0}\| = \|G_1(x)\| \leq K\|x\|$, proving (5.2).

Let $\ell \in \mathbb{N}$ so that $2^{-\ell} \leq \frac{1}{N} < 2^{1-\ell}$. Represent A as a disjoint union of sets $\cup_{k=1}^{\ell} A_k$, where $A_k = \{i \in A : 2^{-k} < |a_i| \leq 2^{1-k}\}$, $1 \leq k \leq \ell - 1$, and $A_\ell = \{i \in A : |a_i| \leq 2^{1-\ell}\}$. Then, by (5.2),

$$\|S_{A_\ell} x\| \leq \sum_{i \in A_\ell} |a_i| \|\mathbf{e}_i\| \leq 2^{1-\ell} |A_\ell| \leq \frac{2|A_\ell|}{N} \leq 2 \leq 2K\|x\|.$$

For $1 \leq k \leq \ell - 1$, let $F_k = \{i \in \mathbb{N} : 2^{-k} < |a_i| \leq 2^{1-k}\}$. By Lemmas 5.3 and 5.2

$$\|S_{A_k}(x)\| \leq c' \|S_{F_k}(x)\| \leq c'' \|x\|,$$

with $c'' = 4K^2 c_2 / c_1$. Therefore,

$$\|S_A(x)\| \leq \sum_{k=1}^{\ell} \|S_{A_k}(x)\| \leq (2K + c''(\ell - 1))\|x\|.$$

As $\ell - 1 \leq \log_2 N$, we have shown (5.1) with c of the order $K^2 c_2 / c_1 = O(K^5)$.

6. EXAMPLES

We compute (asymptotically) the Lebesgue-type constants C_N for some explicit examples of quasi-greedy democratic bases. Notice that, in view of Theorem 1.1, for such bases we have

$$C_N \approx k_N = \sup_{|A| \leq N} \|S_A\|.$$

Example 1: *the Lindenstrauss basis.* Consider the system of vectors in ℓ^1 defined by

$$\mathbf{x}_n = \mathbf{e}_n - \frac{1}{2}(\mathbf{e}_{2n} + \mathbf{e}_{2n+1}), \quad n = 1, 2, \dots$$

where \mathbf{e}_n denotes the canonical basis. It is known that $\{\mathbf{x}_n\}_{n=1}^\infty$ is a monotone basic sequence in ℓ^1 , and a conditional basis in its closed linear span D ; see e.g. [15, p. 27] or [16, p. 455]. The space D was introduced by J. Lindenstrauss in [14] and has other interesting properties in functional analysis. In particular, it was shown by Dilworth and Mitra [7] that $\{\mathbf{x}_n\}_{n=1}^\infty$ is a quasi-greedy basis in D .

Here we show that

$$k_N \approx \log N,$$

which in particular gives a direct proof that the Lindenstrauss basis is not unconditional. By Theorem 5.1, it suffices to show the lower bound. We first notice that

$$\left\| \sum_{k=1}^n b_k \mathbf{x}_k \right\|_{\ell^1} = |b_1| + \sum_{k=2}^n \left| b_k - \frac{1}{2} b_{\lfloor \frac{k}{2} \rfloor} \right| + \frac{1}{2} \sum_{k=n+1}^{2n+1} |b_{\lfloor \frac{k}{2} \rfloor}|.$$

Now consider

$$\mathbf{x} = \sum_{j=0}^{n-1} \sum_{2^j \leq k < 2^{j+1}} 2^{-j} \mathbf{x}_k.$$

Clearly,

$$\|\mathbf{x}\|_{\ell^1} = 1 + \sum_{j=1}^{n-1} \sum_{2^j \leq k < 2^{j+1}} \left| 2^{-j} - \frac{1}{2} 2^{-(j-1)} \right| + \sum_{2^n \leq k < 2^{n+1}} 2^{-n} = 2.$$

Now choose $A = \cup_{\substack{0 \leq j < n \\ j \text{ even}}} [2^j, 2^{j+1}) \cap \mathbb{N}$, so that $N := |A| \approx 2^n$. Then, if say n is odd,

$$\|S_A \mathbf{x}\|_{\ell^1} = 1 + \sum_{j=1}^n \sum_{2^j \leq k < 2^{j+1}} 2^{-j} = n + 1 \approx \log N.$$

Thus $k_N \geq \|S_A\| \gtrsim \log N$, proving our claim.

Example 2. An important example of quasi-greedy basis arises in the context of $BV(\mathbb{R}^d)$, $d > 1$. This space is not separable, so we consider the closed linear span \mathbb{X} of the d -dimensional (non-homogeneous) Haar system in the BV -norm

$$\|f\|_{BV(\mathbb{R}^d)} = \|f\|_{L^1(\mathbb{R}^d)} + |f|_{BV(\mathbb{R}^d)},$$

where $|f|_{BV}$ is the total variation of the distributional gradient ∇f (as defined e.g. in [2, (1.1)]). It follows from the results in [3, 24] that the Haar system is a quasi-greedy

democratic basis in \mathbb{X} (see e.g. [24, Thm 10])². We claim that in this case

$$k_N \approx \log N.$$

It suffices to show the lower bound. For this we will argue as in [1], to find functions $f_N \in \Sigma_{2N}$ with $\|f_N\|_{BV(\mathbb{R}^d)} = O(1)$, and sets A_N with $|A_N| = N$ such that $|S_{A_N}(f_N)|_{BV} \geq c \log N$.

To do this carefully we first set some notation. The Haar functions are defined by

$$h_{j,\mathbf{k}}^{\mathbf{e}}(x) = 2^{j(d-1)} \prod_{\ell=1}^d h^{\mathbf{e}\ell}(2^j x_\ell - k_\ell), \quad j \geq 0, \mathbf{k} \in \mathbb{Z}^d, \mathbf{e} \in \{0, 1\}^d, x \in \mathbb{R}^d, \quad (6.1)$$

where $h^0 = \chi_{[0,1]}$ and $h^1 = \chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1]}$. With this definition the Haar system is semi-normalized, i.e. $c_1 < \|h_{j,\mathbf{k}}^{\mathbf{e}}\|_{BV(\mathbb{R}^d)} < c_2$. The (non-homogeneous) Haar system is obtained restricting to indices $\lambda = (j, \mathbf{k}, \mathbf{e})$ with $\mathbf{e} \neq \mathbf{0}$ when $j > 0$. We sometimes write it $\mathcal{H} = \{h_\lambda\}_{\lambda \in \Lambda}$. As explained above, it is a quasi-greedy democratic basis in \mathbb{X} , the $\|\cdot\|_{BV}$ -closure of its linear span.

Following [1] we consider the function $f = \chi_{[0, \frac{1}{3}] \times [0, 1]^{d-1}}$ and $f_n = P_{2n}f$, where P_J denotes the projection onto $V_J = \text{span}\{h_{j,\mathbf{k}}^{\mathbf{e}} \mid j \leq J\}$. The Haar coefficients of f are easily computed, leading to the expression

$$f_n = \frac{1}{3} \chi_{[0,1]^d} + \frac{1}{3} \sum_{j=0}^{2n} 2^{-j(d-1)} \sum_{\substack{k_2, \dots, k_d \\ 0 \leq k_\ell < 2^j}} h_{j, (k_1(j), k_2, \dots, k_d)}^{(1, 0, \dots, 0)} \quad (6.2)$$

where $k_1(j)$ denotes the only integer such that $\frac{1}{3} \in (\frac{k}{2^j}, \frac{k+1}{2^j})$, explicitly given by

$$k_1(j) = \begin{cases} \frac{2^j-1}{3} & \text{if } j = \text{even} \\ \frac{2^j-2}{3} & \text{if } j = \text{odd.} \end{cases} \quad (6.3)$$

Using for instance [24, Corollary 12] one justifies that $\|f_n\|_{BV} = \|P_{2n}f\|_{BV} = O(1)$. Note also that $f_n \in \Sigma_{2N}$ with $N = O(2^{2n(d-1)})$.

Consider now the set A_n consisting only of the indices in (6.2) with j even, so that $|A_n| = N$ and

$$S_{A_n}(f_n) = \frac{1}{3} \sum_{\substack{j=0 \\ j \text{ even}}}^{2n} 2^{-j(d-1)} \sum_{\substack{k_2, \dots, k_d \\ 0 \leq k_\ell < 2^j}} h_{j, (k_1(j), k_2, \dots, k_d)}^{(1, 0, \dots, 0)}.$$

To estimate $|S_{A_n}(f_n)|_{BV}$ from below we shall use the following linear functional

$$u \in BV \mapsto \Phi(u) = \int_{[\frac{1}{3}, \infty) \times \mathbb{R}^{d-1}} \partial_{x_1} u.$$

This is bounded in BV since $\partial_{x_1} u$ defines a finite measure. Thus,

$$|S_{A_n}(f_n)|_{BV} \geq |\Phi(S_{A_n}(f_n))|. \quad (6.4)$$

²Democracy is not explicitly stated, but follows easily from the inclusions $\ell^1 \hookrightarrow BV \hookrightarrow \ell^{1, \infty}$ as in [2, p. 239]. The fact that the Haar system is a basic sequence in BV (hence a basis in its closed linear span \mathbb{X}), is a consequence of the uniform boundedness of the projections, see [24, Corollary 12]. Finally, it is a seminormalized system with the normalization in (6.1); see [2, (1.6)].

On the other hand, when $(j, \mathbf{k}, \mathbf{e}) \in A_n$ we can compute explicitly

$$\begin{aligned}\Phi(h_{j,\mathbf{k}}^{\mathbf{e}}) &= \int_{[\frac{1}{3}, \infty)} 2^j (h^1)'(2^j x_1 - k_1(j)) dx_1 \\ &= (h_1)' \left[\frac{2^j}{3} - k_1(j), \infty \right) \\ &= (\delta_0 - 2\delta_{1/2} + \delta_1) \left[\frac{2^j}{3} - k_1(j), \infty \right) = -1,\end{aligned}$$

where in the last step we have used (6.3) for $j = \text{even}$. Thus

$$|\Phi(S_{A_n}(f_n))| = n/3 \geq c \log N$$

which together with (6.4) proves our assertion.

Example 3. We now show that $C_N \approx k_N$ may be strictly smaller than $\log N$. Modifying an example in [12], for $1 < p < \infty$ we let \mathbb{X}_p be the closure of $\text{span}\{\mathbf{e}_j\}$ with the norm

$$\|(x_j)\| := \max \left\{ \|(x_j)\|_{\ell^p}, \sup_{m \geq 1} \left| \sum_{n=1}^m \frac{x_n}{n^{1/p'}} \right| \right\}. \quad (6.5)$$

A simple generalization of the arguments in [12] shows that the canonical basis is quasi-greedy and democratic in \mathbb{X}_p . We claim that, in this example,

$$C_N \approx k_N \approx (\log N)^{1/p'}. \quad (6.6)$$

Clearly, for $x = \sum_{n=1}^{\infty} \mathbf{e}_n \in \mathbb{X}_p$

$$\|S_A x\|_{\ell^p} \leq \|x\|_{\ell^p} \leq \|x\|.$$

Also, if for simplicity we write $\|x\|_{b^p} := \sup_{m \geq 1} \left| \sum_{n=1}^m \frac{x_n}{n^{1/p'}} \right|$, using Hölder's inequality we have

$$\begin{aligned}\|S_A x\|_{b^p} &= \sup_{m \geq 1} \left| \sum_{\substack{n=1 \\ n \in A}}^m \frac{x_n}{n^{1/p'}} \right| \\ &\leq \|x\|_{\ell^p} \sup_{m \geq 1} \left(\sum_{\substack{n=1 \\ n \in A}}^m \frac{1}{n} \right)^{1/p'} \lesssim \|x\| (\log |A|)^{1/p'}.\end{aligned}$$

These two inequalities give the upper bound in (6.6).

On the other hand, testing with $x = \sum_{n=1}^{2N} \frac{(-1)^n \mathbf{e}_n}{n^{1/p}}$ and $A = \{1, \dots, 2N\} \cap 2\mathbb{Z}$ one easily sees that

$$\|x\| \approx (\log N)^{1/p} \quad \text{and} \quad \|S_A(x)\| \approx \log N.$$

This gives $k_N \gtrsim (\log N)^{1/p'}$, establishing (6.6).

Example 4. Above, we considered examples of quasi-greedy bases. We provide an example of a non quasi-greedy basis where

$$k_n, C_n, D_n \gtrsim n, \quad n = 1, 2, \dots$$

Consider the sequence space ℓ^1 with the difference basis

$$\mathbf{x}_1 = \mathbf{e}_1, \quad \mathbf{x}_n = \mathbf{e}_n - \mathbf{e}_{n-1}, \quad n = 2, 3, \dots$$

Clearly, for finitely supported scalars (b_n) , one has

$$\left\| \sum_n b_n \mathbf{x}_n \right\| = |b_1| + \sum_{n=1}^{\infty} |b_{n+1} - b_n|.$$

In particular, this basis is normalized with $\|\mathbf{x}_n\| = 2$.

Let $y = \sum_{n=1}^{2N} \mathbf{x}_n$, so that $\|y\| = 1$. Taking $A = \{2, 4, \dots, 2N\}$ we obtain

$$\|S_A(y)\| = \left\| \sum_{n=1}^N \mathbf{x}_{2n} \right\| = 2N.$$

Thus $k_N \geq \|S_A(y)\|/\|y\| \geq 2N$. By Proposition 3.3 and Theorem 1.2, we then conclude that $C_N \gtrsim k_N \approx D_N \gtrsim N$.

Example 5. The last example consists of a general procedure showing that k_N and $\mu(N)$ may essentially be arbitrary.

Let \mathbb{X} and \mathbb{Y} be Banach spaces with respective (normalized) bases $\{\mathbf{e}_j\}$ and $\{\mathbf{f}_j\}$. We consider the direct sum space $\mathbb{X} \oplus \mathbb{Y}$, consisting on pairs $(x, y) \in \mathbb{X} \times \mathbb{Y}$ with norm given by $\|x\|_{\mathbb{X}} + \|y\|_{\mathbb{Y}}$. Clearly, the system³ $\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2, \mathbf{f}_2, \dots\}$ is a basis of $\mathbb{X} \oplus \mathbb{Y}$. Moreover, we have the following.

Proposition 6.1. *If $\{\mathbf{e}_j\}_{j=1}^{\infty}$ is quasi-greedy in \mathbb{X} and $\{\mathbf{f}_j\}_{j=1}^{\infty}$ quasi-greedy in \mathbb{Y} , then $\{\mathbf{e}_j, \mathbf{f}_j\}_{j=1}^{\infty}$ is quasi-greedy in $\mathbb{X} \oplus \mathbb{Y}$. Moreover,*

- (a) $k_N^{\mathbb{X} \oplus \mathbb{Y}} = \max\{k_N^{\mathbb{X}}, k_N^{\mathbb{Y}}\}$
- (b) $h_r^{\mathbb{X} \oplus \mathbb{Y}}(N) \approx \max\{h_r^{\mathbb{X}}(N), h_r^{\mathbb{Y}}(N)\}$
- (c) $\min\{h_\ell^{\mathbb{X}}(N/2), h_\ell^{\mathbb{Y}}(N/2)\} \lesssim h_\ell^{\mathbb{X} \oplus \mathbb{Y}}(N) \leq \min\{h_\ell^{\mathbb{X}}(N), h_\ell^{\mathbb{Y}}(N)\}$.

Proof. The proof is elementary. Quasi-greediness follows from

$$\|G_N(x + y)\|_{\mathbb{X} \oplus \mathbb{Y}} \leq \max_{0 \leq k \leq N} (\|G_k(x)\|_{\mathbb{X}} + \|G_{N-k}(y)\|_{\mathbb{Y}}) \lesssim \|x\|_{\mathbb{X}} + \|y\|_{\mathbb{Y}}.$$

The statement (a) is an easy consequence of the identity

$$k_N^{\mathbb{X} \oplus \mathbb{Y}} = \sup_{|A_1| + |A_2| \leq N} \sup_{\substack{x \in \mathbb{X}, y \in \mathbb{Y} \\ \|x\|_{\mathbb{X}} + \|y\|_{\mathbb{Y}} \neq 0}} \frac{\|S_{A_1}x\|_{\mathbb{X}} + \|S_{A_2}y\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}} + \|y\|_{\mathbb{Y}}}.$$

Similarly, (b) follows from

$$h_r^{\mathbb{X} \oplus \mathbb{Y}}(N) = \sup_{|A_1| + |A_2| = N} (\left\| \sum_{i \in A_1} \mathbf{e}_i \right\|_{\mathbb{X}} + \left\| \sum_{j \in A_2} \mathbf{f}_j \right\|_{\mathbb{Y}}).$$

For (c) one uses

$$h_\ell^{\mathbb{X} \oplus \mathbb{Y}}(N) = \inf_{|A_1| + |A_2| = N} (\left\| \sum_{i \in A_1} \mathbf{e}_i \right\|_{\mathbb{X}} + \left\| \sum_{j \in A_2} \mathbf{f}_j \right\|_{\mathbb{Y}}) \leq \min\{h_\ell^{\mathbb{X}}(N), h_\ell^{\mathbb{Y}}(N)\}.$$

For the lower bound notice that

$$h_\ell^{\mathbb{X} \oplus \mathbb{Y}}(N) \geq \min_{1 \leq k \leq N} \{h_\ell^{\mathbb{X}}(k) + h_\ell^{\mathbb{Y}}(N - k)\} \gtrsim \min\{h_\ell^{\mathbb{X}}(N/2), h_\ell^{\mathbb{Y}}(N/2)\},$$

where in the last step one splits the cases $k \leq N/2$ and $k > N/2$, and uses that $h_\ell^{\mathbb{X}}$ and $h_\ell^{\mathbb{Y}}$ are almost increasing (by quasi-greediness; see (10.1)). \square

³As usual, in $\mathbb{X} \oplus \mathbb{Y}$ one just writes x in place of $(x, 0)$, and y in place of $(0, y)$.

As a particular case, consider \mathbb{X} as in Example 1, so that

$$k_N^{\mathbb{X}} \approx \log N \quad \text{and} \quad h_\ell^{\mathbb{X}}(N) \approx h_r^{\mathbb{X}}(N) \approx N. \quad (6.7)$$

Consider also the space \mathbb{Y} given by the closure of c_{00} with the norm

$$\|(y_j)\|_{\mathbb{Y}} = \sup_A \sum_{j \in A} |y_j| / (1 + \log |A|)^\alpha < \infty, \quad (6.8)$$

where $\alpha > 0$ is fixed. One easily checks that

$$k_N^{\mathbb{Y}} = 1 \quad \text{and} \quad h_\ell^{\mathbb{Y}}(N) = h_r^{\mathbb{Y}}(N) = \frac{N}{(1 + \log N)^\alpha}.$$

Combining (6.7), (6.8) and Proposition 6.1 we see that $\mathbb{X} \oplus \mathbb{Y}$ has

$$k_N^{\mathbb{X} \oplus \mathbb{Y}} \approx \log N \quad \text{and} \quad \mu^{\mathbb{X} \oplus \mathbb{Y}}(N) \approx (\log N)^\alpha. \quad (6.9)$$

To show possible applications of our results, construct a quasi-greedy basis with $k_N \approx \mu(N) \approx \log N$. Theorem 1.1 shows that $C_N \lesssim \log N$. This is an improvement over previously known estimates: both [21, Theorem 2.1] and [10, Thm 1.1] only yield $C_N \lesssim (\log N)^2$.

7. LIMITATIONS

One could use Theorem 5.1 to show that a given basis is not quasi-greedy, by establishing that its k_N constants grow faster than $c \log N$ for any $c > 0$. We also know that $k_N = O(1)$ characterizes unconditional bases. It is then fair to ask whether the slow growth $k_N \leq c \log N$ could characterize quasi-greedy bases. Below we show that it is not the case.

Proposition 7.1. *Suppose a sequence $1 \leq c_1 \leq c_2 \leq \dots$ increases without a bound (perhaps very slowly). Then there exists a Banach space \mathbb{X} with a normalized basis (\mathbf{e}_i) such that $k_{2N} \leq c_N$, and (\mathbf{e}_i) is not quasi-greedy.*

Proof. Without loss of generality, we may assume $c_n \leq n$. Furthermore, passing to the sequence

$$c'_j = \min \left\{ c_j, \min_{i < j} c_i j / i \right\}$$

if necessary, we may assume that the sequence (c_j/j) is non-increasing.

For $j \in \mathbb{N}$, let $S_j = \{5^j + 1, 5^j + 2, \dots, 5^j + 2j\}$. Define a norm on c_{00} by setting, for $x = (x_i)$,

$$\|x\| = \max \left\{ \sup_i |x_i|, \sup_n \sup_j \frac{c_j}{2j} \left| \sum_{\substack{i \in S_j \\ i \leq n}} (-1)^i x_i \right| \right\},$$

and let \mathbb{X} be the completion of c_{00} in this norm. Denote the canonical basis in \mathbb{X} by (\mathbf{e}_i) , which is clearly a monotone basis.

Note that (\mathbf{e}_i) is not unconditional with constant coefficients, hence not quasi-greedy. Indeed, for $j \in \mathbb{N}$, let $S'_j = \{5^j + 1, 5^j + 3, \dots, 5^j + 2j - 1\}$. Then

$$\left\| \sum_{i \in S_j} \mathbf{e}_i \right\| = 1, \quad \text{while} \quad \left\| \sum_{i \in S'_j} \mathbf{e}_i \right\| = \frac{c_j}{2}.$$

It remains to show that $\|S_B x\| \leq c_N$ whenever $|B| \leq 2N$, and $\|x\| \leq 1$. Write $x = \sum_i x_i \mathbf{e}_i$, with $\sup_i |x_i| \leq 1$. Let $A_j = S_j \cap B$. Then $\|S_B x\| \leq \max\{1, C\}$, where

$$C = \sup_n \sup_j \frac{c_j}{2^j} \left| \sum_{\substack{i \in A_j \\ i \leq n}} (-1)^i x_i \right| \quad (7.1)$$

$$= \sup_n \max \left\{ \max_{j \leq N} \frac{c_j}{2^j} \left| \sum_{\substack{i \in A_j \\ i \leq n}} (-1)^i x_i \right|, \sup_{j > N} \frac{c_j}{2^j} \left| \sum_{\substack{i \in A_j \\ i \leq n}} (-1)^i x_i \right| \right\} \quad (7.2)$$

$$\leq \max \left\{ \max_{j \leq N} c_j, \sup_{j > N} \frac{N c_j}{j} \right\} = c_N. \quad (7.3)$$

□

8. PROOF OF THEOREM 1.3

We must show that

$$\|x - G_{N+k}(x)\| \leq c \left(1 + \frac{h_r(N)}{h_l(k)}\right) \sigma_N(x), \quad \forall x \in \mathbb{X}. \quad (8.1)$$

Observe that this quantifies how many iterations of the greedy algorithm may be necessary to reach $\sigma_N(x)$. As mentioned in §1 estimates of this sort were obtained in [13, 6, 23, 11], with its roots going back to the work of Konyagin and Temlyakov [12]. Our proof is a suitable combination of these ideas, plus the argument we used in Theorem 1.1 to control the term $\|S_{\Gamma \setminus A}(x)\|$.

More precisely, take any $p \in \Sigma_N$, say with $\text{supp } p \subset P$ and $|P| = N$. We shall compare $G_{N+k}(x)$ with $p + G_k(x - p) \in \Sigma_{N+k}$. Let $\Gamma = \text{supp } G_{N+k}(x)$ and notice that $B = \text{supp } G_k(x - p)$ can be chosen⁴ such that $B \setminus P \subset \Gamma$.

Then

$$\begin{aligned} \|x - G_{N+k}(x)\| &= \|x - S_\Gamma x\| \\ &\leq \|x - S_{P \cup B}(x)\| + \|S_{P \cup B}(x) - S_\Gamma x\| \\ &\leq \|x - S_{P \cup B}(x)\| + \|S_{(P \cup B) \setminus \Gamma}(x)\| + \|S_{\Gamma \setminus (P \cup B)}(x)\| = I_1 + I_2 + I_3. \end{aligned}$$

The third term can be written as

$$I_3 = \|G_{|\Gamma \setminus (P \cup B)|}(x - S_{P \cup B} x)\| \leq K \|x - S_{P \cup B}(x)\|,$$

so it suffices to estimate the first two terms.

We begin with I_2 . Since $B \setminus P \subset \Gamma$, we have $(P \cup B) \setminus \Gamma = P \setminus \Gamma$. Use (2.2) and the definition of h_r to obtain

$$I_2 = \|S_{P \setminus \Gamma}(x)\| \leq c_2 \max_{P \setminus \Gamma} |a_k(x)| \left\| \sum_{P \setminus \Gamma} \mathbf{e}_k \right\| \leq c_2 \max_{P \setminus \Gamma} |a_k(x)| h_r(|P \setminus \Gamma|).$$

Now using Lemma 2.1

$$\begin{aligned} \max_{P \setminus \Gamma} |a_k(x)| &\leq \min_{\Gamma \setminus P} |a_k(x)| = \min_{\Gamma \setminus P} |a_k(S_{\Gamma \setminus P}(x - p))| \\ &\leq a_{|\Gamma \setminus P|}^*(x - p) = \min_{\Gamma \setminus P} |a_k(x - p)|. \end{aligned}$$

⁴Different choices may appear in case of ties in the size of coefficients.

Thus, by (2.2) again

$$\min_{\Gamma \setminus P} |a_k(x-p)| \leq \frac{1}{c_1} \frac{\|S_{\Gamma \setminus P}(x-p)\|}{\|\sum_{\Gamma \setminus P} \mathbf{e}_k\|} = \frac{1}{c_1} \frac{\|G_{|\Gamma \setminus P|}(x-p)\|}{\|\sum_{\Gamma \setminus P} \mathbf{e}_k\|} \leq \frac{K}{c_1} \frac{\|x-p\|}{h_l(|\Gamma \setminus P|)}.$$

Combining these inequalities we obtain

$$I_2 \leq \frac{c_2}{c_1} K \frac{h_r(|P \setminus \Gamma|)}{h_l(|\Gamma \setminus P|)} \|x-p\|.$$

Observe that since the basis is quasi-greedy, if $A \subset B$ we have $\|\mathbf{1}_A\| \leq K \|\mathbf{1}_B\|$. Hence, $h_r(|P \setminus \Gamma|) \leq K h_r(N)$ since $|P \setminus \Gamma| \leq N$. Similarly, $h_l(|\Gamma \setminus P|) \geq \frac{1}{K} h_l(k)$ since $k \leq |\Gamma \setminus P|$. Thus,

$$I_2 \leq \frac{c_2}{c_1} K^3 \frac{h_r(N)}{h_l(k)} \|x-p\|.$$

We now estimate I_1 , following the approach in [6]; namely,

$$\begin{aligned} I_1 &= \|x - S_{P \cup B}(x)\| = \|x - p - S_{P \cup B}(x-p)\| \\ &\leq \|x - p - S_B(x-p)\| + \|S_{P \setminus B}(x-p)\| = J_1 + J_2. \end{aligned}$$

Clearly

$$J_1 = \|x - p - G_k(x-p)\| \leq (1+K) \|x-p\|.$$

To estimate J_2 use (2.2) and the quasi-greediness of the basis to obtain

$$\begin{aligned} J_2 &\leq c_2 \max_{P \setminus B} |a_j(x-p)| h_r(|P \setminus B|) \leq c_2 \min_B |a_j(x-p)| h_r(|P \setminus B|) \\ &\leq \frac{c_2 h_r(|P \setminus B|)}{c_1 h_l(|B|)} \|G_{|B|}(x-p)\| \leq \frac{c_2}{c_1} K \frac{h_r(|P \setminus B|)}{h_l(k)} \|x-p\|. \end{aligned}$$

As before, $h_r(|P \setminus B|) \leq K h_r(N)$, so we deduce

$$J_2 \leq \frac{c_2}{c_1} K^2 \frac{h_r(N)}{h_l(k)} \|x-p\|.$$

Thus, putting together all cases we obtain

$$\|x - G_{N+k}(x)\| \leq c \left(1 + \frac{h_r(N)}{h_l(k)}\right) \|x-p\|, \quad \forall p \in \Sigma_N,$$

with the constant c of the order $K^3 c_2 / c_1 = O(K^6)$.

Remarks:

- As pointed out in [11], (8.1) improves over (1.3) in some situations. For instance, assume $h_l(N) = N^\alpha$ and $h_r(N) = N^\beta$ with $0 < \alpha < \beta \leq 1$. If x is such that $\sigma_N(x) = O(N^{-r})$ then (1.3) gives

$$\|x - G_M(x)\| \lesssim M^{\beta-\alpha} \sigma_M(x) \lesssim M^{-[r-(\beta-\alpha)]}$$

while (8.1) gives, when $M = k + N$ with $k \approx M \approx N^{\beta/\alpha}$,

$$\|x - G_M(x)\| \lesssim \sigma_{cM^{\alpha/\beta}}(x) \lesssim M^{-r\alpha/\beta}.$$

When $r < \beta$, the second estimate improves over the first (for large M). In the language of approximation spaces (see e.g. [9]), these estimates can also be read as

$$A_\infty^r(\mathbb{X}) \hookrightarrow \mathcal{G}_\infty^{\max\{\frac{r\alpha}{\beta}, r-(\beta-\alpha)\}}(\mathbb{X}).$$

- The estimate (8.1) is only interesting when $\lim_{k \rightarrow \infty} h_l(k) = \infty$ (so that $h_l(k)$ can reach $h_r(N)$), and cannot be improved when h_l is just bounded and $h_r(N) \rightarrow \infty$. To see the latter, arguing as in Lemma 3.2 one can find disjoint sets A, B with $|A| = N + k$, $|B| = N$ and $\|\mathbf{1}_B\|/\|\mathbf{1}_A\| \gtrsim h_r(N)$. Setting $x = 2\mathbf{1}_A + \mathbf{1}_B$ one sees that $\|x - G_{N+k}(x)\|/\sigma_N(x) \geq \|\mathbf{1}_B\|/\|\mathbf{1}_A\| \gtrsim h_r(N) \rightarrow \infty$.

9. SOME QUESTIONS

Quasi-greedy bases in $L^p(\mathbb{T}^d)$ were studied in [20, 21]. In these cases one always has $\mu(N) \lesssim N^{|\frac{1}{p} - \frac{1}{2}|}$, from the type and cotype properties of $L^p(\mathbb{T}^d)$, $1 < p < \infty$. Hence, using Theorem 1.1 (and (1.4)) one obtains that $C_N \lesssim N^{|\frac{1}{p} - \frac{1}{2}|}$, when $p \neq 2$, a result which was proved in [21]. When $p = 2$, this argument only gives $C_N \lesssim \log N$, a result which goes back to [22].

QUESTION 1. (Asked in [20, 5]). *Investigate whether, for quasi-greedy bases in a Hilbert space, the inequality $C_N \lesssim \log N$ can be replaced by a slower growing factor.*

Recently, P. Wojtaszczyk [25] has showed us that, for quasi-greedy bases in L^2 , say with constant K , there exists $\alpha = \alpha(K) < 1$, such that $C_N \lesssim (\log N)^\alpha$. Also, it can be deduced from the results in [5] that $C_N \lesssim (\log N)^{1/2}$ for all quasi-greedy *besselian*⁵ bases in L^2 . However, no examples where these bounds are attained seem to be known.

Consider now the trigonometric system $\mathcal{T}^d = \{e^{ikx} : k \in \mathbb{Z}^d\}$ in $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$ (understood as $C(\mathbb{T}^d)$ for $p = \infty$). Notice that \mathcal{T}^d is not quasi-greedy in L^p , $p \neq 2$. It was proved in [17, Theorem 2.1] that one also has

$$C_N = C_N(\mathcal{T}^d, L^p(\mathbb{T}^d)) \lesssim N^{|\frac{1}{p} - \frac{1}{2}|}, \quad 1 \leq p \leq \infty.$$

QUESTION 2. (Asked by V. N. Temlyakov at the *Concentration week on greedy algorithms in Banach spaces and compressed sensing* held on July 18-22, 2011 at Texas A&M University.)

- a) Characterize those systems \mathcal{B} in $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, such that

$$C_N(\mathcal{B}, L^p(\mathbb{T}^d)) \lesssim N^{|\frac{1}{p} - \frac{1}{2}|}, \quad N = 1, 2, \dots$$

Notice that if $1 < p \neq 2 < \infty$, the characterization must be satisfied by \mathcal{T}^d as well as any quasi-greedy basis.

More generally, let $v(N)$ be an increasing function of N .

- b) Characterize, in a Banach space \mathbb{X} , those systems \mathcal{B} (not necessarily quasi-greedy) for which $C_N(\mathcal{B}, \mathbb{X}) \lesssim v(N)$.

10. APPENDIX: PROOF OF (2.2)

The proof suggested in [6] for the inequalities in (2.2) is only valid for *real* scalars $a_k \in \mathbb{R}$; we give below a minor modification of their argument that establishes (2.2) also for *complex* scalars a_k . Below K denotes the quasi-greedy constant in \mathbb{X} .

The first two lemmas are similar to [22, Prop 2].

⁵Here *besselian* means $\|(a_k)\|_{\ell^2} \lesssim \|\sum_k a_k \mathbf{e}_k\|_2$, for all finitely scalars (a_k) .

Lemma 10.1. *Let $\{\mathbf{e}_j\}_{j=1}^\infty$ is a quasi-greedy basis in a Banach space \mathbb{X} . For all $\beta_j \in \mathbb{C}$ with $|\beta_j| = 1$, and all finite sets $A_1 \subset A$, it holds*

$$\left\| \sum_{j \in A_1} \beta_j \mathbf{e}_j \right\| \leq K \left\| \sum_{j \in A} \beta_j \mathbf{e}_j \right\|. \quad (10.1)$$

Proof. Call $A_2 = A \setminus A_1$. For $\varepsilon > 0$, define $x = \sum_{j \in A_1} (1 + \varepsilon) \beta_j \mathbf{e}_j + \sum_{j \in A_2} \beta_j \mathbf{e}_j$. Then

$$\|G_{|A_1|}(x)\| = (1 + \varepsilon) \left\| \sum_{j \in A_1} \beta_j \mathbf{e}_j \right\| \leq K \|x\| = \left\| (1 + \varepsilon) \sum_{j \in A_1} \beta_j \mathbf{e}_j + \sum_{j \in A_2} \beta_j \mathbf{e}_j \right\|.$$

Letting $\varepsilon \rightarrow 0$ we obtain (10.1). \square

Lemma 10.2. *Let $\{\mathbf{e}_j\}_{j=1}^\infty$ is a quasi-greedy basis in a Banach space \mathbb{X} . For all $\varepsilon_j \in \{\pm 1, \pm i\}$, and all finite sets A it holds*

$$\frac{1}{4K} \left\| \sum_{j \in A} \mathbf{e}_j \right\| \leq \left\| \sum_{j \in A} \varepsilon_j \mathbf{e}_j \right\| \leq 4K \left\| \sum_{j \in A} \mathbf{e}_j \right\|. \quad (10.2)$$

Proof. Call $A_k = \{j \in A : \varepsilon_j = i^k\}$, $k = 1, \dots, 4$. Then, the triangle inequality and (10.1) (with all $\beta_j = 1$) give

$$\left\| \sum_{j \in A} \varepsilon_j \mathbf{e}_j \right\| \leq \sum_{k=1}^4 \left\| \sum_{j \in A_k} \mathbf{e}_j \right\| \leq 4K \left\| \sum_{j \in A} \mathbf{e}_j \right\|,$$

establishing the right hand side of (10.2). Arguing similarly,

$$\left\| \sum_{j \in A} \mathbf{e}_j \right\| \leq \sum_{k=1}^4 \left\| \sum_{j \in A_k} \mathbf{e}_j \right\| \leq 4K \left\| \sum_{j \in A} \varepsilon_j \mathbf{e}_j \right\|$$

where we have now used (10.1) with $\beta_j = \varepsilon_j$. \square

Lemma 10.3. *For all complex $\beta = a + ib$ with $|a| + |b| \leq 1$, and for all $x, y \in \mathbb{X}$ it holds*

$$\|x + \beta y\| \leq \max \{ \|x \pm y\|, \|x \pm iy\| \}. \quad (10.3)$$

Proof. We may assume that $a \in [0, 1)$. Then

$$\begin{aligned} \|x + \beta y\| &\leq \|ax + ay\| + \|(1 - a)x + iby\| \\ &= a\|x + y\| + (1 - a)\|x + i\gamma y\|, \end{aligned} \quad (10.4)$$

where we have set $\gamma = b/(1 - a)$, which is a real number with $|\gamma| \leq 1$. Now

$$\begin{aligned} \|x + i\gamma y\| &= \left\| \frac{1-\gamma}{2}(x - iy) + \frac{1+\gamma}{2}(x + iy) \right\| \\ &\leq \frac{1-\gamma}{2} \|x - iy\| + \frac{1+\gamma}{2} \|x + iy\| \leq \max \|x \pm iy\|, \end{aligned}$$

where we have used that $-1 \leq \gamma \leq 1$. Inserting this into (10.4) easily leads to (10.3). \square

We now justify the right hand bound in (2.2). For a complex number $\alpha = a + ib$ we shall denote $|\alpha|_1 = |a| + |b|$. Then, iterating the previous lemma we obtain

$$\begin{aligned} \left\| \sum_{j \in A} \alpha_j \mathbf{e}_j \right\| &\leq \max_{j \in A} |\alpha_j|_1 \max_{\varepsilon_j \in \{\pm 1, \pm i\}} \left\| \sum_{j \in A} \varepsilon_j \mathbf{e}_j \right\| \\ &\leq 4\sqrt{2} K \max_{j \in A} |\alpha_j| \left\| \sum_{j \in A} \mathbf{e}_j \right\|, \end{aligned} \quad (10.5)$$

where in the last step we have used Lemma 10.2 and the trivial estimate $|\alpha|_1 \leq \sqrt{2}|\alpha|$.

We can now state a slightly more general version of Lemma 10.2.

Lemma 10.4. *Let $\{\mathbf{e}_j\}_{j=1}^\infty$ is a quasi-greedy basis in a Banach space \mathbb{X} . For all $\varepsilon_j \in \mathbb{C}$ with $|\varepsilon_j| = 1$, and all finite sets A it holds*

$$\frac{1}{4\sqrt{2}K} \left\| \sum_{j \in A} \mathbf{e}_j \right\| \leq \left\| \sum_{j \in A} \varepsilon_j \mathbf{e}_j \right\| \leq 4\sqrt{2}K \left\| \sum_{j \in A} \mathbf{e}_j \right\|. \quad (10.6)$$

Proof. The right hand side is a special case of (10.5). To obtain the left hand side, we consider the system $\{\tilde{\mathbf{e}}_j := \varepsilon_j \mathbf{e}_j\}$, which is also a quasi-greedy basis in \mathbb{X} with the same constant K . Thus, (10.5) for this system (with $\alpha_j = \bar{\varepsilon}_j$) gives

$$\left\| \sum_{j \in A} \bar{\varepsilon}_j \tilde{\mathbf{e}}_j \right\| \leq 4\sqrt{2}K \left\| \sum_{j \in A} \tilde{\mathbf{e}}_j \right\|,$$

but this is the same as the left hand side of (10.6). \square

We turn now to the left hand inequality in (2.2), for which we follow the arguments in [6, p. 579]. We shall prove that, if A is finite then

$$\left\| \sum_{j \in A} \alpha_j \mathbf{e}_j \right\| \geq \frac{1}{8\sqrt{2}K^2} \min_{j \in A} |\alpha_j| \left\| \sum_{j \in A} \mathbf{e}_j \right\|. \quad (10.7)$$

Write each scalar $\alpha_j = \varepsilon_j |\alpha_j|$, with $\varepsilon_j \in \mathbb{C}$ such that $|\varepsilon_j| = 1$, and consider a permutation $\{j_1, \dots, j_N\}$ of A such that $|\alpha_{j_1}| \geq |\alpha_{j_2}| \geq \dots \geq |\alpha_{j_N}|$. Let $x = \sum_{j \in A} \alpha_j \mathbf{e}_j$ and set $G_0(x) = 0$. Then

$$\begin{aligned} |\alpha_{j_N}| \left\| \sum_{\ell=1}^N \varepsilon_{j_\ell} \mathbf{e}_{j_\ell} \right\| &= |\alpha_{j_N}| \left\| \sum_{\ell=1}^N \frac{1}{|\alpha_{j_\ell}|} (G_\ell(x) - G_{\ell-1}(x)) \right\| \\ &= |\alpha_{j_N}| \left\| \sum_{\ell=1}^{N-1} \left(\frac{1}{|\alpha_{j_\ell}|} - \frac{1}{|\alpha_{j_{\ell+1}}|} \right) G_\ell(x) + \frac{1}{|\alpha_{j_N}|} G_N(x) \right\| \\ &\leq |\alpha_{j_N}| \left[\frac{1}{|\alpha_{j_N}|} + \sum_{\ell=1}^{N-1} \left(\frac{1}{|\alpha_{j_{\ell+1}}|} - \frac{1}{|\alpha_{j_\ell}|} \right) \right] K \|x\| \leq 2K \|x\|. \end{aligned} \quad (10.8)$$

On the other hand, by Lemma 10.4, the expression on the left of (10.8) can be estimated from below by $|\alpha_{j_N}| \left\| \sum_{j \in A} \mathbf{e}_j \right\| / 4\sqrt{2}K$, from which (10.7) follows.

Thus, putting together (10.5) and (10.7) we have shown

Proposition 10.5. *Let $\{\mathbf{e}_j\}_{j=1}^\infty$ is a quasi-greedy basis in a Banach space \mathbb{X} . If A is finite and $\alpha_j \in \mathbb{C}$ then*

$$\frac{1}{8\sqrt{2}K^2} \min_{j \in A} |\alpha_j| \left\| \sum_{j \in A} \mathbf{e}_j \right\| \leq \left\| \sum_{j \in A} \alpha_j \mathbf{e}_j \right\| \leq 4\sqrt{2}K \max_{j \in A} |\alpha_j| \left\| \sum_{j \in A} \mathbf{e}_j \right\|.$$

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