

DEMOCRACY FUNCTIONS AND OPTIMAL EMBEDDINGS FOR APPROXIMATION SPACES

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ABSTRACT. We prove optimal embeddings for nonlinear approximation spaces \mathcal{A}_q^α , in terms of weighted Lorentz sequence spaces, with the weights depending on the democracy functions of the basis. As applications we recover known embeddings for N -term wavelet approximation in L^p , Orlicz, and Lorentz norms. We also study the “greedy classes” \mathcal{G}_q^α introduced by Gribonval and Nielsen, obtaining new counterexamples which show that $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$ for most non democratic unconditional bases.

1. INTRODUCTION

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a quasi-Banach space with a countable **unconditional** basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$. A main question in **Approximation Theory** consists in finding a characterization (if possible) or at least suitable embeddings for the non-linear approximation spaces $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, $\alpha > 0$, $0 < q \leq \infty$, defined using the **N-term error of approximation** $\sigma_N(x, \mathbb{B})$ (see sections 2.2 and 2.3 for definitions). Such characterizations or inclusions are often given in terms of “smoothness classes” of the sort

$$\mathbf{b}(\mathcal{B}; \mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} : \{\|c_j e_j\|_{\mathbb{B}}\}_{j=1}^{\infty} \in \mathbf{b} \right\},$$

where \mathbf{b} is a suitable sequence space whose elements decay at infinity, such as ℓ^τ or more generally the discrete Lorentz classes $\ell^{\tau, q}$.

The simplest result in this direction appears when \mathcal{B} is an orthonormal basis in a Hilbert space \mathbb{H} , and was first proved by Stechkin when $\alpha = 1/2$ and $q = 1$ (see [30] or [8] for general α, q).

Theorem 1.1. ([30, 8]). *Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in a Hilbert space \mathbb{H} , and $\alpha > 0$, $0 < q \leq \infty$. Then*

$$\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{H}) = \ell^{\tau, q}(\mathcal{B}; \mathbb{H})$$

where τ is defined by $\frac{1}{\tau} = \alpha + \frac{1}{2}$.

Many results have been published in the literature similar to Theorem 1.1 when \mathbb{H} is replaced by a particular space (say, L^p) and the basis \mathcal{B} is a particular one (for example, a wavelet basis). We refer to the survey articles [5] and [34] for detailed statements and references.

Date: November 23, 2009.

2000 Mathematics Subject Classification. 41A17, 42C40.

Key words and phrases. Non-linear approximation, greedy algorithm, democratic bases, Jackson and Bernstein inequalities, discrete Lorentz spaces, wavelets.

Research supported by Grant MTM2007-60952 of Spain. The research of M. de Natividade supported by Instituto Nacional de Bolsas de Estudos de Angola, INABE.

There are also a number of results for general pairs $(\mathbb{B}, \mathcal{B})$ (even with the weaker notion of quasi-greedy basis [13, 9, 19]). We recall two of them in the setting of unconditional bases which we consider here. For simplicity, in all the statements we assume that the basis is *normalized*, meaning $\|e_j\|_{\mathbb{B}} = 1, \forall j \in \mathbb{N}$. The first result can be found in [20] (see also [11]).

Theorem 1.2. ([20, Th 1], [11, Th 6.1]). *Let \mathbb{B} be a quasi-Banach space and $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ a (normalized) unconditional basis satisfying the following property: there exists $p \in (0, \infty)$ and a constant $C > 0$ such that*

$$\frac{1}{C}|\Gamma|^{1/p} \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C|\Gamma|^{1/p} \quad (1.1)$$

for all finite $\Gamma \subset \mathbb{N}$. Then, for $\alpha > 0$ and $0 < q \leq \infty$ we have

$$\mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B}) = \ell^{\tau, q}(\mathcal{B}; \mathbb{B})$$

when τ is defined by $\frac{1}{\tau} = \alpha + \frac{1}{p}$.

Condition (1.1) is sometimes referred as \mathcal{B} having the p -Temlyakov property [19], or as \mathbb{B} being a p -space [16, 11]. For instance, wavelet bases in L^p satisfy this property [32]. The second result we quote is proved in [13] (see also [20]).

Theorem 1.3. ([13, Th 3.1]). *Let \mathbb{B} be a Banach space and $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ a (normalized) unconditional basis with the following property: there exist $1 \leq p \leq q \leq \infty$ and constants $A, B > 0$ such that when $x = \sum_{j \in \mathbb{N}} c_j e_j \in \mathbb{B}$ we have*

$$A \|\{c_j\}\|_{\ell^{q, \infty}} \leq \|x\|_{\mathbb{B}} \leq B \|\{c_j\}\|_{\ell^{p, 1}}. \quad (1.2)$$

Then, for $\alpha > 0$ and $0 < s \leq \infty$ we have

$$\ell^{\tau_p, s}(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{A}_s^{\alpha}(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell^{\tau_q, s}(\mathcal{B}; \mathbb{B}) \quad (1.3)$$

where $\frac{1}{\tau_p} = \alpha + \frac{1}{p}$ and $\frac{1}{\tau_q} = \alpha + \frac{1}{q}$. Moreover, the inclusions given in (1.3) are best possible in the sense described in section 4 of [13].

Condition (1.2) is referred in [13] as $(\mathbb{B}, \mathcal{B})$ having the (p, q) **sandwich property**, and it is shown to be equivalent to

$$A'|\Gamma|^{1/q} \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq B'|\Gamma|^{1/p} \quad (1.4)$$

for all $\Gamma \subset \mathbb{N}$ finite. Observe that (1.4) coincides with (1.1) when $p = q$.

The purpose of this article is to obtain optimal embeddings for $\mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B})$ as in (1.3) when no condition such as (1.4) is imposed. More precisely, we define the **right** and **left democracy functions** associated with a basis \mathcal{B} in \mathbb{B} by

$$h_r(N; \mathcal{B}, \mathbb{B}) \equiv \sup_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \right\|_{\mathbb{B}} \quad \text{and} \quad h_\ell(N; \mathcal{B}, \mathbb{B}) \equiv \inf_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} \frac{e_k}{\|e_k\|_{\mathbb{B}}} \right\|_{\mathbb{B}}$$

for $N = 1, 2, 3, \dots$. We refer to section 5 for various examples where $h_\ell(N)$ and $h_r(N)$ are computed explicitly (modulo multiplicative constants). As usual, when $h_\ell(N) \approx h_r(N)$ for all $N \in \mathbb{N}$ we say that \mathcal{B} is a democratic basis in \mathbb{B} [22].

The embeddings will be given in terms of **weighted discrete Lorentz spaces** ℓ_η^q , with quasi-norms defined by

$$\|\{c_k\}\|_{\ell_\eta^q} \equiv \left(\sum_{k=1}^{\infty} |\eta(k) c_k^*|^q \frac{1}{k} \right)^{\frac{1}{q}},$$

where $\{c_k^*\}$ denotes the decreasing rearrangement of $\{|c_k|\}$ and the *weight* $\eta = \{\eta(k)\}_{k=1}^{\infty}$ is a suitable sequence increasing to infinity and satisfying the doubling property (see section 2.4 for precise definitions and references). In the special case $\eta(k) = k^{1/\tau}$ we recover the classical definition $\ell_\eta^q = \ell^{\tau,q}$.

Theorem 1.4. *Let \mathbb{B} be a quasi-Banach space and \mathcal{B} an unconditional basis. Assume that $h_\ell(N)$ is doubling. Then if $\alpha > 0$ and $0 < q \leq \infty$ we have the continuous embeddings*

$$\ell_{k^\alpha h_r(k)}^q(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^\alpha h_\ell(k)}^q(\mathcal{B}; \mathbb{B}). \quad (1.5)$$

Moreover, for fixed α and q these inclusions are best possible in the scale of weighted discrete Lorentz spaces ℓ_η^q , in the sense explained in sections 3, 4 and 6.

Observe that this theorem generalizes Theorems 1.2 and 1.3. In Theorem 1.2 we have $h_r(N) \approx h_\ell(N) \approx N^{1/p}$ and in Theorem 1.3, $h_r(N) \lesssim N^{1/p}$ and $h_\ell(N) \gtrsim N^{1/q}$. When \mathcal{B} is democratic in \mathbb{B} , Theorem 1.4 shows that $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \approx \ell_{k^\alpha h(k)}^q(\mathcal{B}; \mathbb{B})$ with $h(k) = h_r(k) \approx h_\ell(k)$. Compare this result with Corollary 1 in [13, §6].

Theorem 1.4 is a consequence of the results proved in sections 3 and 4. Section 3 deals with the lower embedding in (1.5) and shows the relation to Jackson type inequalities. Section 4 deals with the upper embedding of (1.5) and its relation to Bernstein type inequalities. Section 5 contains various examples of democracy functions and embeddings with precise references; these are all special cases of Theorem 1.4. In section 6 we apply Theorem 1.4 to estimate the democracy functions h_ℓ and h_r of the approximation space \mathcal{A}_q^α .

Finally, the last section of the paper is dedicated to study the “greedy classes” $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ introduced by Gribonval and Nielsen in [13], and their relations with the approximation spaces $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$. The classes \mathcal{G}_q^α are defined similarly to the approximation spaces, but with the error of approximation $\sigma_N(x)$ replaced by the quantity $\|x - G_N(x)\|_{\mathbb{B}}$ (see section 2.3 for details). It is easy to see that $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \subset \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, and when \mathcal{B} is democratic, $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) = \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$. One may conjecture that for unconditional bases \mathcal{B} the converse is true, that is $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) = \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ implies \mathcal{B} democratic. We do not know how to show this, but we can exhibit a fairly general class of non democratic pairs $(\mathcal{B}, \mathbb{B})$ for which $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \neq \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ for all $\alpha > 0$ and $q \in (0, \infty]$. These include wavelet bases in the non democratic settings of $L^{p,q}$ and $L^p(\log L)^\alpha$. We also illustrate how irregular the classes $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ can be when \mathcal{B} is not democratic, showing in simple situations that they are not even linear spaces.

2. GENERAL SETTING

2.1. Bases. Since we work in the setting of quasi-Banach spaces $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$, we shall often use the ρ -power triangle inequality

$$\|x + y\|_{\mathbb{B}}^\rho \leq \|x\|_{\mathbb{B}}^\rho + \|y\|_{\mathbb{B}}^\rho, \quad (2.1)$$

which holds for a sufficiently small $\rho = \rho_{\mathbb{B}} \in (0, 1]$ (and hence for all $\mu \leq \rho_{\mathbb{B}}$); see [3, Lemma 3.10.1]. The case $\rho_{\mathbb{B}} = 1$ gives a Banach space.

A sequence of vectors $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ is a basis of \mathbb{B} if every $x \in \mathbb{B}$ can be uniquely represented as $x = \sum_{j=1}^{\infty} c_j e_j$ for some scalars c_j , with convergence in $\|\cdot\|_{\mathbb{B}}$. The basis \mathcal{B} is **unconditional** if the series converges unconditionally, or equivalently if there is some $K > 0$ such that

$$\left\| \sum_{j=1}^{\infty} \lambda_j c_j e_j \right\|_{\mathbb{B}} \leq K \left\| \sum_{j=1}^{\infty} c_j e_j \right\|_{\mathbb{B}} \quad (2.2)$$

for every sequence of scalars $\{\lambda_j\}_{j=1}^{\infty}$ with $|\lambda_j| \leq 1$ (see eg [15, Chapter 5]).

For simplicity in the statements, throughout the paper we shall assume that \mathcal{B} is a **normalized** basis, meaning $\|e_j\|_{\mathbb{B}} = 1$ for all $j \in \mathbb{N}$. We can also assume that the unconditionality constant in (2.2) is $K = 1$. To see so, one can introduce an equivalent quasi-norm in \mathbb{B}

$$\|x\|_{\mathbb{B}} = \sup_{\Gamma \text{ finite}, |\lambda_j| \leq 1} \left\| \sum_{j \in \Gamma} \lambda_j x_j e_j \right\|_{\mathbb{B}}, \quad \text{if } x = \sum_{j=1}^{\infty} x_j e_j.$$

Observe that with this renorming we still have $\|e_j\|_{\mathbb{B}} = 1$.

With the above assumptions, the following **lattice property** holds: if $|y_k| \leq |x_k|$ for all $k \in \mathbb{N}$ and $x = \sum_{k=1}^{\infty} x_k e_k \in \mathbb{B}$, then the series $y = \sum_{k=1}^{\infty} y_k e_k$ converges in \mathbb{B} and $\|y\|_{\mathbb{B}} \leq \|x\|_{\mathbb{B}}$. Also, using (2.2) with $K = 1$ we see that, for every $\Gamma \subset \mathbb{N}$ finite

$$\left(\inf_{j \in \Gamma} |c_j| \right) \left\| \sum_{j \in \Gamma} e_j \right\|_{\mathbb{B}} \leq \left\| \sum_{j \in \Gamma} c_j e_j \right\|_{\mathbb{B}} \leq \left(\sup_{j \in \Gamma} |c_j| \right) \left\| \sum_{j \in \Gamma} e_j \right\|_{\mathbb{B}}. \quad (2.3)$$

2.2. Non-Linear Approximation and Greedy Algorithm. Let $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ be a basis in \mathbb{B} . Let Σ_N , $N = 1, 2, 3, \dots$, be the set of all $y \in \mathbb{B}$ with at most N non-null coefficients in the unique basis representation. For $x \in \mathbb{B}$, the **N -term error of approximation** with respect to \mathcal{B} is defined as

$$\sigma_N(x) = \sigma_N(x; \mathcal{B}, \mathbb{B}) \equiv \inf_{y \in \Sigma_N} \|x - y\|_{\mathbb{B}}, \quad N = 1, 2, 3, \dots$$

We also set $\Sigma_0 = \{0\}$ so that $\sigma_0(x) = \|x\|_{\mathbb{B}}$. Using the lattice property mentioned in §2.1 it is easy to see that for $x = \sum_{j=1}^{\infty} c_j e_j$ we actually have

$$\sigma_N(x) = \inf_{|\Gamma|=N} \left\{ \left\| x - \sum_{\gamma \in \Gamma} c_{\gamma} e_{\gamma} \right\|_{\mathbb{B}} \right\}, \quad (2.4)$$

that is, only coefficients from x are relevant when computing $\sigma_N(x)$; see eg [11, (2.6)].

Given $x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B}$, let π denote any bijection of \mathbb{N} such that

$$\|c_{\pi(j)} e_{\pi(j)}\| \geq \|c_{\pi(j+1)} e_{\pi(j+1)}\|, \quad \text{for all } j \in \mathbb{N}. \quad (2.5)$$

Without loss of generality we may assume that the basis is normalized and then (2.5) becomes $|c_{\pi(j)}| \geq |c_{\pi(j+1)}|$, for all $j \in \mathbb{N}$. A **greedy algorithm of step N** is a correspondence assigning

$$x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} \mapsto G_N^{\pi}(x) \equiv \sum_{j=1}^N c_{\pi(j)} e_{\pi(j)}$$

for any π as in (2.5). The **error of greedy approximation** at step N is defined by

$$\gamma_N(x) = \gamma_N(x; \mathcal{B}, \mathbb{B}) \equiv \sup_{\pi} \|x - G_N^{\pi}(x)\|_{\mathbb{B}}. \quad (2.6)$$

Notice that $\sigma_N(x) \leq \gamma_N(x)$, but the reverse inequality may not be true in general. It is said that \mathcal{B} is a **greedy basis** in \mathbb{B} when there is a constant $c \geq 1$ such that

$$\gamma_N(x; \mathcal{B}, \mathbb{B}) \leq c \sigma_N(x; \mathcal{B}, \mathbb{B}), \quad \forall x \in \mathbb{B}, N = 1, 2, 3, \dots$$

A celebrated theorem of Konyagin and Temlyakov characterizes greedy bases as those which are unconditional and democratic [22].

2.3. Approximation Spaces and Greedy Classes. The classical non-linear approximation spaces $\mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B})$ are defined as follows: for $\alpha > 0$ and $0 < q < \infty$

$$\mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{A}_q^{\alpha}} \equiv \|x\|_{\mathbb{B}} + \left[\sum_{n=1}^{\infty} (N^{\alpha} \sigma_N(x; \mathcal{B}, \mathbb{B}))^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\}.$$

When $q = \infty$ the definition takes the form:

$$\mathcal{A}_{\infty}^{\alpha}(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{A}_{\infty}^{\alpha}} \equiv \|x\|_{\mathbb{B}} + \sup_{N \geq 1} N^{\alpha} \sigma_N(x) < \infty \right\}.$$

It is well known that $\mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B})$ are quasi-Banach spaces (see eg [28]). Also, equivalent quasi-norms can be obtained restricting to dyadic N 's:

$$\|x\|_{\mathcal{A}_q^{\alpha}} \approx \|x\|_{\mathbb{B}} + \left[\sum_{k=0}^{\infty} (2^{k\alpha} \sigma_{2^k}(x))^q \right]^{\frac{1}{q}}$$

and likewise for $q = \infty$. This is a simple consequence of the monotonicity of $\sigma_N(x)$ (see eg [28, Prop 2] or [7, (2.3)]).

The **greedy classes** $\mathcal{G}_q^{\alpha}(\mathcal{B}, \mathbb{B})$ are defined as before replacing the role of $\sigma_N(x)$ by the error of greedy approximation $\gamma_N(x)$ given in (2.6), that is

$$\mathcal{G}_q^{\alpha}(\mathcal{B}, \mathbb{B}) = \left\{ x \in \mathbb{B} : \|x\|_{\mathcal{G}_q^{\alpha}} \equiv \|x\|_{\mathbb{B}} + \left[\sum_{N=1}^{\infty} (N^{\alpha} \gamma_N(x; \mathcal{B}, \mathbb{B}))^q \frac{1}{N} \right]^{\frac{1}{q}} < \infty \right\} \quad (2.7)$$

(and similarly for $q = \infty$). We also have the equivalence

$$\|x\|_{\mathcal{G}_q^{\alpha}} \approx \|x\|_{\mathbb{B}} + \left[\sum_{k=0}^{\infty} (2^{k\alpha} \gamma_{2^k}(x))^q \right]^{\frac{1}{q}}, \quad (2.8)$$

since $\gamma_N(x)$ is non-increasing by the lattice property in §2.1.

Since $\sigma_N(x) \leq \gamma_N(x)$ for all $x \in \mathbb{B}$ it is clear that¹

$$\mathcal{G}_q^{\alpha}(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B}). \quad (2.9)$$

When \mathcal{B} is a greedy basis in \mathbb{B} it holds that $\mathcal{G}_q^{\alpha}(\mathcal{B}, \mathbb{B}) = \mathcal{A}_q^{\alpha}(\mathcal{B}, \mathbb{B})$ with equivalent quasi-norms. For non greedy bases, however, the inclusion may be strict, and the classes \mathcal{G}_q^{α} may not even be linear spaces (see section 7.1 below).

¹Here, as in the rest of the paper, $X \hookrightarrow Y$ means $X \subset Y$ and there exists $C > 0$ such that $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$. The equality of spaces $X = Y$ is interpreted as $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

2.4. Discrete Lorentz Spaces. Let $\eta = \{\eta(k)\}_{k=1}^{\infty}$ be a sequence so that

- (a) $0 < \eta(k) \leq \eta(k+1)$ for all $k = 1, 2, \dots$ and $\lim_{k \rightarrow \infty} \eta(k) = \infty$.
- (b) η is *doubling*, that is, $\eta(2k) \leq C\eta(k)$ for all $k = 1, 2, \dots$, and some $C > 0$.

We shall denote the set of all such sequences by \mathbb{W} . If $\eta \in \mathbb{W}$ and $0 < r \leq \infty$, the **weighted discrete Lorentz space** ℓ_{η}^r is defined as

$$\ell_{\eta}^r = \left\{ \mathbf{s} = \{s_k\}_{k=1}^{\infty} \in \mathfrak{c}_0 : \|\mathbf{s}\|_{\ell_{\eta}^r} \equiv \left[\sum_{k=1}^{\infty} (\eta(k)s_k^*)^r \frac{1}{k} \right]^{\frac{1}{r}} < \infty \right\}$$

(with $\|\mathbf{s}\|_{\ell_{\eta}^{\infty}} = \sup_{k \in \mathbb{N}} \eta(k)s_k^*$ when $r = \infty$). Here $\{s_k^*\}$ denotes the decreasing rearrangement of $\{|s_k|\}$, that is $s_k^* = |s_{\pi(k)}|$ where π is any bijection of \mathbb{N} such that $|s_{\pi(k)}| \geq |s_{\pi(k+1)}|$ for all $k = 1, 2, \dots$ (since we are assuming $\lim_{k \rightarrow \infty} s_k = 0$ such π 's always exist). When $\eta \in \mathbb{W}$ the set ℓ_{η}^r is a quasi-Banach space (see eg [4, §2.2]). Equivalent quasi-norms are given by

$$\|\mathbf{s}\|_{\ell_{\eta}^r} \approx \left[\sum_{j=0}^{\infty} (\eta(\kappa^j)s_{\kappa^j}^*)^r \right]^{1/r}, \quad (2.10)$$

for any fixed integer $\kappa > 1$. Particular examples are the classical Lorentz sequence spaces $\ell^{p,r}$ (with $\eta(k) = k^{1/p}$), and the Lorentz-Zygmund spaces $\ell^{p,r}(\log \ell)^{\gamma}$ (for which $\eta(k) = k^{1/p} \log^{\gamma}(k+1)$; see eg [2, p. 285]).

Occasionally we will need to assume a stronger condition on the weights η . For an increasing sequence η we define

$$M_{\eta}(m) = \sup_{k \in \mathbb{N}} \frac{\eta(k)}{\eta(mk)}, \quad m = 1, 2, 3, \dots$$

Observe that we always have $M_{\eta}(m) \leq 1$. We shall say that $\eta \in \mathbb{W}_+$ when $\eta \in \mathbb{W}$ and there exists some integer $\kappa > 1$ for which $M_{\eta}(\kappa) < 1$. This is equivalent to say that the “lower dilation index” $i_{\eta} > 0$, where we let

$$i_{\eta} \equiv \sup_{m \geq 1} \frac{\log M_{\eta}(m)}{-\log m}.$$

For example, $\eta = \{k^{\alpha} \log^{\beta}(k+1)\}$ has $i_{\eta} = \alpha$, and hence $\eta \in \mathbb{W}_+$ iff $\alpha > 0$. In general, if η is obtained from a increasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $\eta(k) = \phi(ak)$, for some fixed $a > 0$, then $i_{\eta} > 0$ iff $i_{\phi} > 0$, the latter denoting the standard lower dilation index of ϕ (see eg [23, p. 54] for the definition).

Below we will need the following result:

Lemma 2.1. *If $\eta \in \mathbb{W}_+$ then there exists a constant $C > 0$ such that*

$$\sum_{j=0}^n \eta(\kappa^j) \leq C\eta(\kappa^n), \quad \forall n \in \mathbb{N}, \quad (2.11)$$

where $\kappa > 1$ is an integer as in the definition of \mathbb{W}_+ .

Proof. Write $\delta = M_{\eta}(\kappa) < 1$. By definition $M_{\eta}(\kappa) \geq \eta(\kappa^j)/\eta(\kappa^{j+1})$, and therefore

$$\eta(\kappa^j) \leq \delta \eta(\kappa^{j+1}), \quad \forall j = 0, 1, 2, \dots \quad (2.12)$$

Iterating (2.12) we deduce that $\eta(\kappa^j) \leq \delta^{n-j}\eta(\kappa^n)$, for $j = 0, 1, 2, \dots, n$ and hence

$$\sum_{j=0}^n \eta(\kappa^j) \leq \eta(\kappa^n) \sum_{j=0}^n \delta^{n-j} \leq \eta(\kappa^n) \frac{1}{1-\delta}.$$

□

Remark 2.2. If η is increasing and doubling, then $\{k^\alpha \eta(k)\} \in \mathbb{W}_+$ for all $\alpha > 0$. Also, if $\eta \in \mathbb{W}_+$ then $\eta^r \in \mathbb{W}_+$, for all $r > 0$.

We now estimate the *fundamental function* of ℓ_η^r . We shall denote the indicator sequence of $\Gamma \subset \mathbb{N}$ by 1_Γ , that is the sequence with entries 1 for $j \in \Gamma$ and 0 otherwise.

Lemma 2.3. (a) If $\eta \in \mathbb{W}$ then

$$\|1_\Gamma\|_{\ell_\eta^\infty} = \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N}.$$

(b) If $\eta \in \mathbb{W}_+$ and $r \in (0, \infty)$ then

$$\|1_\Gamma\|_{\ell_\eta^r} \approx \eta(|\Gamma|), \quad \forall \text{ finite } \Gamma \subset \mathbb{N}$$

with the constants involved independent of Γ .

Proof. Part (a) is trivial since η is increasing. To prove (b) use (2.10) and the previous lemma. □

Finally, as mentioned in §1, given a (normalized) basis \mathcal{B} in \mathbb{B} we shall consider the following subspaces

$$\ell_\eta^q(\mathcal{B}, \mathbb{B}) := \left\{ x = \sum_{j=1}^{\infty} c_j e_j \in \mathbb{B} : \{c_j\}_{j=1}^{\infty} \in \ell_\eta^q \right\},$$

endowed with the quasi-norm $\|x\|_{\ell_\eta^q(\mathcal{B}, \mathbb{B})} := \|\{c_j\}\|_{\ell_\eta^q}$. These spaces are not necessarily complete, but they are when

$$\left\| \sum_j c_j e_j \right\|_{\mathbb{B}} \leq C \|\{c_j\}\|_{\ell_\eta^q}, \quad \forall \text{ finite } \{c_j\},$$

a property which holds in certain situations (see eg Remark 3.2). When this is the case, the space $\ell_\eta^q(\mathcal{B}, \mathbb{B})$ is just an isomorphic copy of ℓ_η^q inside \mathbb{B} .

2.5. Democracy Functions. Following [22], a (normalized) basis \mathcal{B} in a quasi-Banach space \mathbb{B} is said to be **democratic** if there exists $C > 0$ such that

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C \left\| \sum_{k \in \Gamma'} e_k \right\|_{\mathbb{B}},$$

for all finite sets $\Gamma, \Gamma' \subset \mathbb{N}$ with the same cardinality. This notion allows to characterize greedy bases as those which are both unconditional and democratic [22].

As we recall in §5, wavelet bases are well known examples of greedy bases for many function spaces, such as L^p , Sobolev, or more generally, the Triebel-Lizorkin spaces. However, they are not democratic in some other instances such as BMO , or the Orlicz L^Φ and Lorentz $L^{p,q}$ spaces (when these are different from L^p). In fact, it is proved in [37] that the Haar basis is democratic in a rearrangement invariant space \mathbb{X} in $[0, 1]$ if and only if $\mathbb{X} = L^p$ for some $p \in (1, \infty)$.

Thus, non-democratic bases are also common. To quantify the democracy of a (normalized) system $\mathcal{B} = \{e_j\}_{j=1}^\infty$ in \mathbb{B} one introduces the following concepts:

$$h_r(N; \mathcal{B}, \mathbb{B}) \equiv \sup_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \quad \text{and} \quad h_\ell(N; \mathcal{B}, \mathbb{B}) \equiv \inf_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}},$$

which we shall call the **right and left democracy functions of \mathcal{B}** (see also [9, 18, 12]). We shall omit \mathcal{B} or \mathbb{B} when these are understood from the context.

Some general properties of h_ℓ and h_r are proved in the next proposition.

Proposition 2.4. *Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in \mathbb{B} with the lattice property from §2.1. Then*

- (a) $1 \leq h_\ell(N) \leq h_r(N) \leq N^{1/\rho}$, $\forall N = 1, 2, \dots$, where $\rho = \rho_{\mathbb{B}}$ is as in (2.1).
- (b) $h_\ell(N)$ and $h_r(N)$ are non-decreasing in $N = 1, 2, 3, \dots$
- (c) $h_r(N)$ is doubling, that is, $\exists c > 0$ such that $h_r(2N) \leq c h_r(N)$, $\forall N \in \mathbb{N}$.
- (d) There exists $c \geq 1$ such that $h_\ell(N+1) \leq c h_\ell(N)$ for all $N = 1, 2, 3, \dots$

Proof. (a) and (b) follow immediately from the lattice property of \mathcal{B} and the ρ -triangular inequality.

(c) Given $N \in \mathbb{N}$, choose $\Gamma \subset \mathbb{N}$ with $|\Gamma| = 2N$ such that $\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \geq h_r(2N)/2$. Partitioning arbitrarily $\Gamma = \Gamma' \cup \Gamma''$ with $|\Gamma'| = |\Gamma''| = N$, and using the ρ -power triangle inequality, one easily obtains

$$\frac{1}{2} h_r(2N) \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} = \left\| \sum_{k \in \Gamma'} e_k + \sum_{k \in \Gamma''} e_k \right\|_{\mathbb{B}} \leq 2^{1/\rho} h_r(N).$$

(d) Given $N \in \mathbb{N}$, choose $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ such that $\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq 2 h_\ell(N)$. Let $\Gamma' = \Gamma \cup \{k_o\}$ for any $k_o \notin \Gamma$. Then

$$h_\ell(N+1) \leq \left\| \sum_{k \in \Gamma'} e_k \right\|_{\mathbb{B}} \leq \left(\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}}^\rho + 1 \right)^{1/\rho} \leq (2^\rho [h_\ell(N)]^\rho + 1)^{1/\rho}.$$

Thus, using (a) we obtain $h_\ell(N+1) \leq (2^\rho + 1)^{\frac{1}{\rho}} h_\ell(N) \leq 2 \cdot 2^{1/\rho} h_\ell(N)$. \square

Remark 2.5. We do not know whether property (d) can be improved to show that $h_\ell(N)$ is actually doubling. This seems however to be case in all the examples we have considered below (see §5).

3. RIGHT DEMOCRACY AND JACKSON TYPE INEQUALITIES

Our first result deals with inclusions for the greedy classes $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$.

Theorem 3.1. *Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty)$. Then, for any sequence η such that $\{k^\alpha \eta(k)\}_{k=1}^\infty \in \mathbb{W}_+$ the following statements are equivalent:*

1. There exists $C > 0$ such that for all $N = 1, 2, 3, \dots$

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \quad (3.1)$$

2. Jackson type inequality for $\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})$: $\exists C_\alpha > 0$ such that $\forall N = 0, 1, 2, \dots$

$$\gamma_N(x) \leq C_\alpha (N+1)^{-\alpha} \|x\|_{\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})}, \quad \forall x \in \ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B}). \quad (3.2)$$

3. $\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})$.
4. $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$.
5. Jackson type inequality for $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})$: $\exists C_{\alpha, q} > 0$ such that $\forall N = 0, 1, 2, \dots$

$$\gamma_N(x) \leq C_{\alpha, q} (N+1)^{-\alpha} \|x\|_{\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})}, \quad \forall x \in \ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}). \quad (3.3)$$

Proof. “1 \Rightarrow 2” Let $x = \sum_{k \in \mathbb{N}} c_k e_k \in \ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})$ and let π be a bijection of \mathbb{N} such that

$$|c_{\pi(k)}| \geq |c_{\pi(k+1)}|, \quad k = 1, 2, 3, \dots \quad (3.4)$$

For fixed $N = 0, 1, 2, \dots$, denote $\lambda_j = 2^j(N+1)$. Then, the ρ -power triangle inequality and (2.3) give

$$\begin{aligned} \|x - G_N^\pi(x)\|_{\mathbb{B}}^\rho &= \left\| \sum_{k=N+1}^{\infty} c_{\pi(k)} e_{\pi(k)} \right\|_{\mathbb{B}}^\rho \leq \sum_{j=0}^{\infty} \left\| \sum_{\lambda_j \leq k < \lambda_{j+1}} c_{\pi(k)} e_{\pi(k)} \right\|_{\mathbb{B}}^\rho \\ &\leq \sum_{j=0}^{\infty} |c_{\pi(\lambda_j)}|^\rho \left\| \sum_{\lambda_j \leq k < \lambda_{j+1}} e_{\pi(k)} \right\|_{\mathbb{B}}^\rho. \end{aligned}$$

There are exactly $\lambda_j = 2^j(N+1)$ elements in the interior sum, so using (3.1) we obtain

$$\begin{aligned} \|x - G_N^\pi(x)\|_{\mathbb{B}}^\rho &\leq C^\rho \sum_{j=0}^{\infty} (c_{\lambda_j}^* \eta(\lambda_j))^\rho = C^\rho \sum_{j=0}^{\infty} (\lambda_j^\alpha c_{\lambda_j}^* \eta(\lambda_j))^\rho \lambda_j^{-\alpha\rho} \\ &\leq C^\rho \|x\|_{\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})}^\rho (N+1)^{-\alpha\rho} \sum_{j=0}^{\infty} 2^{-j\alpha\rho} \\ &= C_{\alpha, \rho} (N+1)^{-\alpha\rho} \|x\|_{\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})}^\rho. \end{aligned}$$

The result follows taking the supremum over all bijections π satisfying (3.4).

Remark 3.2. The special case $N = 0$ in (3.2) says that

$$\|x\|_{\mathbb{B}} \leq C \|x\|_{\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})}, \quad (3.5)$$

which in particular implies $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathbb{B}$, for all $q \in (0, \infty]$.

“2 \Rightarrow 3” This is immediate from the definition of $\mathcal{G}_\infty^\alpha$ (and Remark 3.2), since

$$\|x\|_{\mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})} := \|x\|_{\mathbb{B}} + \sup_{N \geq 1} N^\alpha \gamma_N(x) \leq C_\alpha \|x\|_{\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B})}.$$

“3 \Rightarrow 1” Let $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$. Choose Γ' with $|\Gamma'| = N$ and so that $\Gamma \cap \Gamma' = \emptyset$, and consider $x = \sum_{k \in \Gamma} e_k + \sum_{k \in \Gamma'} 2e_k$. Then

$$\gamma_N(x) = \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}}, \quad (3.6)$$

and therefore

$$N^\alpha \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} = N^\alpha \gamma_N(x) \leq \|x\|_{\mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})}. \quad (3.7)$$

On the other hand, call $\omega(k) = k^\alpha \eta(k)$. By monotonicity, Lemma 2.3 and the doubling property of ω we have

$$\|x\|_{\ell_\infty(\mathcal{B}, \mathbb{B})} \leq 2 \left\| 1_{\Gamma \cup \Gamma'} \right\|_{\ell_\infty} = 2\omega(2N) \leq c\omega(N). \quad (3.8)$$

Combining (3.7) and (3.8) with the inclusion $\ell_{k^\alpha \eta(k)}^\infty(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})$ gives (3.1).

“5 \Rightarrow 1” Let $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$, and choose Γ' and x as in the proof of 3 \Rightarrow 1. As before call $\omega(k) = k^\alpha \eta(k)$. Then Lemma 2.3 and the assumption $\omega \in \mathbb{W}_+$ give

$$\|x\|_{\ell_\omega^q(\mathcal{B}, \mathbb{B})} \leq 2 \|1_{\Gamma \cup \Gamma'}\|_{\ell_\omega^q} \approx \omega(2N) \leq c\omega(N).$$

Since we are assuming 5 we can write (recall (3.6))

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} = \gamma_N(x) \leq C_{\alpha, \rho} (N+1)^{-\alpha} \|x\|_{\ell_\omega^q(\mathcal{B}, \mathbb{B})} \lesssim N^{-\alpha} \omega(N) = \eta(N),$$

which proves (3.1).

“1 \Rightarrow 4” The proof is similar to 1 \Rightarrow 2 with a few modifications we indicate next. Given $x \in \ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})$ and π as in (3.4) we write $x = \sum_{j=-1}^{\infty} \sum_{2^j < k \leq 2^{j+1}} c_{\pi(k)} e_{\pi(k)}$. Then arguing as before (with $N = 2^m$) we obtain

$$\|x - G_{2^m}^\pi(x)\|_{\mathbb{B}}^\mu \leq \sum_{j=m}^{\infty} |c_{\pi(2^j)}|^\mu \left\| \sum_{2^j < k \leq 2^{j+1}} e_{\pi(k)} \right\|_{\mathbb{B}}^\mu,$$

where we choose now any $\mu < \min\{q, \rho_{\mathbb{B}}\}$. Taking the supremum over all π 's and using (3.1) we obtain

$$\gamma_{2^m}(x; \mathcal{B}, \mathbb{B})^\mu \leq C^\mu \sum_{j=m}^{\infty} (c_{2^j}^* \eta(2^j))^\mu.$$

Therefore

$$\left[\sum_{m=0}^{\infty} (2^{m\alpha} \gamma_{2^m}(x))^q \right]^{\frac{1}{q}} \leq C \left[\sum_{m=0}^{\infty} 2^{m\alpha q} \left(\sum_{j=0}^{\infty} [c_{2^{j+m}}^* \eta(2^{j+m})]^\mu \right)^{q/\mu} \right]^{1/q}.$$

Since $q/\mu > 1$, we can use Minkowski's inequality on the right hand side to obtain

$$\begin{aligned} \left[\sum_{m=0}^{\infty} (2^{m\alpha} \gamma_{2^m}(x))^q \right]^{\frac{1}{q}} &\leq C \left[\sum_{j=0}^{\infty} \left(\sum_{m=0}^{\infty} 2^{m\alpha q} [c_{2^{j+m}}^* \eta(2^{j+m})]^\mu \right)^{\mu/q} \right]^{1/\mu} \\ &= C \left[\sum_{j=0}^{\infty} 2^{-j\alpha\mu} \left(\sum_{\ell=j}^{\infty} 2^{\ell\alpha q} [c_{2^\ell}^* \eta(2^\ell)]^\mu \right)^{\mu/q} \right]^{1/\mu} \leq C' \|\{c_k\}\|_{\ell_{k^\alpha \eta(k)}^q}. \end{aligned}$$

This implies the desired estimate

$$\|x\|_{\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})} \lesssim \|\{c_k\}\|_{\ell_{k^\alpha \eta(k)}^q},$$

using the dyadic expressions for the norms in (2.8) and (2.10) (and Remark 3.2).

“4 \Rightarrow 5” This is trivial since 4 implies $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})$, and this clearly gives (3.3). \square

Remark 3.3. The equivalences 1 to 3 remain true under the weaker assumption $\{k^\alpha \eta(k)\} \in \mathbb{W}$.

Remark 3.4. Observe that if any of the statements in 2 to 5 of Theorem 3.1 holds for one fixed $\alpha > 0$ and $q \in (0, \infty]$, then the assertions remain true for all α and q (as long as $\{k^\alpha \eta(k)\} \in \mathbb{W}_+$), since the statement in 1 is independent of these parameters.

Corollary 3.5. Optimal inclusions into \mathcal{G}_q^α .

Let \mathcal{B} be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty]$. Then

$$\ell_{k^\alpha h_r(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}). \quad (3.9)$$

Moreover, if $\omega \in \mathbb{W}_+$ then, $\ell_\omega^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ if and only if $\omega(k) \gtrsim k^\alpha h_r(k)$.

Proof. For $q < \infty$, the inclusion (3.9) is an application of 4 in the theorem with $\eta = h_r$ (after noticing that $\{k^\alpha h_r(k)\} \in \mathbb{W}_+$ by Proposition 2.4 and Remark 2.2). The second assertion is just a restatement of $1 \Leftrightarrow 4$ with $\eta(k) = \omega(k)/k^\alpha$. For $q = \infty$ use 3 instead of 4. \square

We now prove similar results for the approximation spaces $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$.

Theorem 3.6. Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty]$. Then, for any sequence $\eta \in \mathbb{W}_+$ the following are equivalent:

1. There exists $C > 0$ such that for all $N = 1, 2, 3, \dots$

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \quad (3.10)$$

2. $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$.

3. Jackson type inequality for $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})$: $\exists C_{\alpha, q} > 0$ such that $\forall N = 0, 1, 2, \dots$

$$\sigma_N(x) \leq C_{\alpha, q} (N+1)^{-\alpha} \|x\|_{\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})}, \quad \forall x \in \ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}). \quad (3.11)$$

Proof. $1 \Rightarrow 2$ follows directly from Theorem 3.1 and $\mathcal{G}_q^\alpha \hookrightarrow \mathcal{A}_q^\alpha$. Also, $2 \Rightarrow 3$ is trivial since $\mathcal{A}_q^\alpha \hookrightarrow \mathcal{A}_\infty^\alpha$, and 3 is equivalent to $\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_\infty^\alpha$.

We must show $3 \Rightarrow 1$. Let $\kappa > 1$ be a fixed integer as in the definition of the class \mathbb{W}_+ (and in particular satisfying (2.11)), and denote $1_\Delta = \sum_{k \in \Delta} e_k$ for a set $\Delta \subset \mathbb{N}$. For any $\Gamma_n \subset \mathbb{N}$ with $|\Gamma_n| = \kappa^n$, we can find a subset Γ_{n-1} with $|\Gamma_{n-1}| = \kappa^{n-1}$ such that

$$\|1_{\Gamma_n} - 1_{\Gamma_{n-1}}\|_{\mathbb{B}} \leq 2\sigma_{\kappa^{n-1}}(1_{\Gamma_n}).$$

Repeating this argument we choose $\Gamma_{j-1} \subset \Gamma_j$ with $|\Gamma_j| = \kappa^j$ and so that

$$\|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|_{\mathbb{B}} \leq 2\sigma_{\kappa^{j-1}}(1_{\Gamma_j}), \quad \text{for } j = 1, 2, \dots, n.$$

Setting $\Gamma_{-1} = \emptyset$, and using the ρ -power triangle inequality we see that

$$\|1_{\Gamma_n}\|_{\mathbb{B}}^\rho = \left\| \sum_{j=0}^n 1_{\Gamma_j} - 1_{\Gamma_{j-1}} \right\|_{\mathbb{B}}^\rho \leq \sum_{j=0}^n \|1_{\Gamma_j} - 1_{\Gamma_{j-1}}\|_{\mathbb{B}}^\rho \leq 2^\rho \sum_{j=0}^n \sigma_{\kappa^{j-1}}(1_{\Gamma_j})^\rho.$$

Now, the hypothesis (3.11) and Lemma 2.3 give

$$\sigma_{\kappa^{j-1}}(1_{\Gamma_j}) \lesssim \kappa^{-j\alpha} \|1_{\Gamma_j}\|_{\ell_{k^\alpha \eta(k)}^q(\mathcal{B}, \mathbb{B})} \approx \eta(\kappa^j).$$

Thus, combining these two expressions we obtain

$$\|1_{\Gamma_n}\|_{\mathbb{B}} \lesssim \left[\sum_{j=0}^n \eta(\kappa^j)^\rho \right]^{1/\rho} \leq C \eta(\kappa^n), \quad (3.12)$$

where the last inequality follows from the assumption $\eta \in \mathbb{W}_+$ and Lemma 2.1. This shows (3.10) when $N = \kappa^n$, $n = 1, 2, \dots$. The general case follows easily using the doubling property of η . \square

Remark 3.7. As before, if any of the statements in 2 or 3 holds for one fixed $\alpha > 0$ and $q \in (0, \infty]$, then the assertions remain true for all α and q , since 1 is independent of these parameters.

Remark 3.8. Observe also that $1 \Rightarrow 2 \Rightarrow 3$ hold with the weaker assumption $\{k^\alpha \eta(k)\} \in \mathbb{W}_+$ from Theorem 3.1 (and in particular hold for $\eta = h_r$ as stated in (1.5)). However, the stronger assumption $\eta \in \mathbb{W}_+$ is crucial to obtain $3 \Rightarrow 1$, and cannot be removed as shown in Example 5.6 below.

Corollary 3.9. Optimality of the inclusions into \mathcal{A}_q^α .

Let \mathcal{B} be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty]$. Then

$$\ell_{k^\alpha h_r(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}). \quad (3.13)$$

If for some $\omega \in \mathbb{W}_+$ we have $\ell_\omega^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, then necessarily $\omega(k) \gtrsim k^\alpha$. Moreover if $\omega(k) = k^\alpha \eta(k)$, with η increasing and doubling, then

(a) if $i_\eta > 0$, then necessarily $\eta(k) \gtrsim h_r(k)$, and hence $\ell_{k^\alpha \eta(k)}^q \hookrightarrow \ell_{k^\alpha h_r(k)}^q$.

(b) if $i_\eta = 0$, then $\eta(k) \gtrsim h_r(k)/(\log k)^{1/\rho}$ and $\ell_{k^\alpha \eta(k)}^q \hookrightarrow \ell_{\{k^\alpha h_r(k)/(\log k)^{1/\rho}\}}^q$.

Proof. The inclusion (3.13) is actually a consequence of (3.9). Assertion (a) is just $2 \Rightarrow 3 \Rightarrow 1$ in the theorem. For assertion (b) notice that in the last step of the proof of $3 \Rightarrow 1$, the right hand inequality of (3.12) can always be replaced by

$$\|1_{\Gamma_n}\|_{\mathbb{B}} \lesssim \left[\sum_{j=0}^n \eta(\kappa^j)^\rho \right]^{1/\rho} \lesssim \eta(\kappa^n) n^{1/\rho}$$

when η is increasing. Thus $h_r(N) \lesssim \eta(N)(\log N)^{1/\rho}$ holds for $N = \kappa^n$, and by the doubling property also for all $N \in \mathbb{N}$. Finally, if $\ell_\omega^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ for some general $\omega \in \mathbb{W}_+$, then given $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$ we trivially have

$$\omega(N) \approx \|1_\Gamma\|_{\ell_\omega^q} \gtrsim \|1_\Gamma\|_{\mathcal{A}_q^\alpha} \geq (N/2)^\alpha \sigma_{N/2}(1_\Gamma) \geq (N/2)^\alpha. \quad \square$$

Remark 3.10. Assertion (b) shows that the inclusion in (3.13) is optimal, except perhaps for a logarithmic loss. The logarithmic loss may actually happen, as there are Banach spaces \mathbb{B} with $h_r(N) \approx (\log N)$ and so that

$$\mathcal{A}_q^\alpha(\mathbb{B}) = \ell_{k^\alpha}^q = \ell_{\{k^\alpha h_r(k)/\log k\}}^q.$$

See Example 5.6 below.

4. LEFT DEMOCRACY AND BERNSTEIN TYPE INEQUALITIES

It is well known that upper inclusions for the approximation spaces \mathcal{A}_q^α , as in (1.5), depend upon Bernstein type inequalities. In this section we show how the left democracy function of \mathcal{B} is linked with these two properties.

We first remark that, for each $\alpha > 0$ and $0 < q \leq \infty$, the approximation classes \mathcal{A}_q^α and \mathcal{G}_q^α satisfy trivial Bernstein inequalities, namely, there exists $C_{\alpha,q} > 0$ such that

$$\|x\|_{\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})} \leq \|x\|_{\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})} \leq C_{\alpha,q} N^\alpha \|x\|_{\mathbb{B}}, \quad \forall x \in \Sigma_N, \quad N = 1, 2, \dots \quad (4.1)$$

This follows easily from the definition of the norms and the trivial estimates $\sigma_N(x) \leq \gamma_N(x) \leq \|x\|_{\mathbb{B}}$.

We start with a preliminary result which is essentially known in the literature (see eg [28]). As usual $\mathcal{B} = \{e_j\}_{j=1}^\infty$ is a fixed (normalized) unconditional basis in \mathbb{B} .

Proposition 4.1. *Let \mathbb{E} be a subspace of \mathbb{B} , endowed with a quasi-norm $\|\cdot\|_{\mathbb{E}}$ satisfying the ρ -triangle inequality for some $\rho = \rho_{\mathbb{E}}$. For each $\alpha > 0$ the following are equivalent:*

1. $\exists C_\alpha > 0$ such that $\|x\|_{\mathbb{E}} \leq C_\alpha N^\alpha \|x\|_{\mathbb{B}}, \forall x \in \Sigma_N, N = 1, 2, \dots$
2. $\mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathbb{E}$.
3. $\mathcal{G}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathbb{E}$.

Proof. “1 \Rightarrow 2” Given $x \in \mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})$, by the representation theorem for approximation spaces [28] one can write $x = \sum_{k=0}^\infty x_k$ with $x_k \in \Sigma_{2^k}, k = 0, 1, 2, \dots$, such that

$$\left(\sum_{k=0}^\infty 2^{k\alpha\rho} \|x_k\|_{\mathbb{B}}^\rho \right)^{1/\rho} \leq C \|x\|_{\mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})}.$$

The hypothesis 1 and the $\rho_{\mathbb{E}}$ -triangular inequality then give

$$\|x\|_{\mathbb{E}}^\rho \leq \sum_{k=0}^\infty \|x_k\|_{\mathbb{E}}^\rho \leq C_\alpha^\rho \sum_{k=0}^\infty 2^{k\alpha\rho} \|x_k\|_{\mathbb{B}}^\rho \leq C' \|x\|_{\mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})}^\rho.$$

“2 \Rightarrow 3”. This follows from the trivial inclusion $\mathcal{G}_\rho^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_\rho^\alpha(\mathcal{B}, \mathbb{B})$.

“3 \Rightarrow 1”. This is immediate using (4.1). \square

Theorem 4.2. *Let $\mathcal{B} = \{e_j\}_{j=1}^\infty$ be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty]$. Then, for any increasing and doubling sequence $\{\eta(k)\}$ the following statements are equivalent:*

1. There exists $C > 0$ such that for all $N = 1, 2, 3, \dots$

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \geq \frac{1}{C} \eta(N), \quad \forall \Gamma \subset \mathbb{N} \text{ with } |\Gamma| = N. \quad (4.2)$$

2. Bernstein type inequality for $\ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B})$: $\exists C_{\alpha,q} > 0$ such that

$$\|x\|_{\ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B})} \leq C_{\alpha,q} N^\alpha \|x\|_{\mathbb{B}}, \quad \forall x \in \Sigma_N, N = 1, 2, 3, \dots \quad (4.3)$$

3. $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B})$.

4. $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B})$.

Proof. “1 \Rightarrow 2”. Let $x = \sum_{k \in \Gamma} c_k e_k \in \Sigma_N$. For any bijection π with $|c_{\pi(k)}|$ decreasing, and any integer $m \in \{1, \dots, N\}$ we have

$$|c_{\pi(m)}| \eta(m) \leq C |c_{\pi(m)}| \left\| \sum_{j=1}^m e_{\pi(j)} \right\|_{\mathbb{B}} \leq C \left\| \sum_{j=1}^m c_{\pi(j)} e_{\pi(j)} \right\|_{\mathbb{B}} \leq C \|x\|_{\mathbb{B}},$$

using (2.3) in the second inequality. This gives

$$\|x\|_{\ell_{k^{\alpha\eta(k)}}^q} = \left[\sum_{m=1}^N (m^\alpha \eta(m) c_m^*)^q \frac{1}{m} \right]^{1/q} \leq C \|x\|_{\mathbb{B}} \left[\sum_{m=1}^N m^{\alpha q} \frac{1}{m} \right]^{1/q} \approx \|x\|_{\mathbb{B}} N^\alpha.$$

“2 \Rightarrow 1”. For any $\Gamma \subset \mathbb{N}$ with $|\Gamma| = N$, applying (4.3) to $1_\Gamma = \sum_{k \in \Gamma} e_k$ we obtain

$$\|1_\Gamma\|_{\mathbb{B}} \geq \frac{1}{C_{\alpha,q}} N^{-\alpha} \|1_\Gamma\|_{\ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B})} \gtrsim \eta(N),$$

where in the last inequality we have used $\|1_\Gamma\|_{\ell_\omega^q} \gtrsim \omega(N)$, when $\omega \in \mathbb{W}$.

“2 \Rightarrow 3”. We have already proved that $1 \Leftrightarrow 2$; since 1 does not depend on α, q , then 2 actually holds for all $\tilde{\alpha} > 0$. In particular, from Proposition 4.1, we have

$$\mathcal{A}_\rho^{\tilde{\alpha}} \hookrightarrow \mathbb{E} := \ell_{k^{\tilde{\alpha}\eta(k)}}^q(\mathcal{B}, \mathbb{B}) \quad (4.4)$$

for $\tilde{\alpha} \in (\frac{\alpha}{2}, \frac{3\alpha}{2})$ and some sufficiently small $\rho > 0$. Now, from the general theory developed in [7], the spaces \mathcal{A}_q^α satisfy a reiteration theorem for the real interpolation method, and in particular

$$\mathcal{A}_q^\alpha = (\mathcal{A}_{q_0}^{\alpha_0}, \mathcal{A}_{q_1}^{\alpha_1})_{1/2, q}, \quad (4.5)$$

when $\alpha = (\alpha_0 + \alpha_1)/2$ with $\alpha_1 > \alpha_0 > 0$, and $q_0, q_1, q \in (0, \infty]$. On the other hand, for the family of weighted Lorentz spaces it is known that

$$(\ell_{\omega_0}^q, \ell_{\omega_1}^q)_{\theta, q} = \ell_\omega^q, \quad 0 < \theta < 1, \quad 0 < q \leq \infty, \quad (4.6)$$

when $\omega_0, \omega_1 \in \mathbb{W}_+$ and $\omega = \omega_0^{1-\theta} \omega_1^\theta$ (see eg [24, Theorem 3]). Thus, for fixed α and q , we can choose the parameters accordingly, and use the inclusion (4.4), to obtain

$$\mathcal{A}_q^\alpha = (\mathcal{A}_\rho^{\alpha_0}, \mathcal{A}_\rho^{\alpha_1})_{1/2, q} \hookrightarrow (\ell_{k^{\alpha_0\eta(k)}}^q, \ell_{k^{\alpha_1\eta(k)}}^q)_{1/2, q} = \ell_{k^{\alpha\eta(k)}}^q(\mathcal{B}, \mathbb{B}).$$

“3 \Rightarrow 4”. This is trivial since $\mathcal{G}_q^\alpha \hookrightarrow \mathcal{A}_q^\alpha$.

“4 \Rightarrow 2”. This is trivial from (4.1). \square

Remark 4.3. Observe that $3 \Rightarrow 4 \Rightarrow 2 \Leftrightarrow 1$ hold with the weaker assumption $\{k^\alpha \eta(k)\} \in \mathbb{W}$.

Corollary 4.4. Optimal inclusions of \mathcal{A}_q^α into ℓ_ω^q .

Let \mathcal{B} be a (normalized) unconditional basis in \mathbb{B} . Fix $\alpha > 0$ and $q \in (0, \infty]$.

(a) If $h_\ell(N)$ is doubling then $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^{\alpha h_\ell(k)}}^q(\mathcal{B}, \mathbb{B})$.

(b) If for some $\omega \in \mathbb{W}$ we have $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_\omega^q(\mathcal{B}, \mathbb{B})$ then necessarily $\omega(k) \lesssim k^\alpha h_\ell(k)$, and hence $\ell_{k^{\alpha h_\ell(k)}}^q \hookrightarrow \ell_\omega^q$.

Proof. Part (a) is an application of $1 \Rightarrow 3$ in the theorem with $\eta = h_\ell$ (which under the doubling assumption satisfies $\{k^\alpha h_\ell(k)\} \in \mathbb{W}_+$ for all $\alpha > 0$). Part (b) is just a restatement of $3 \Rightarrow 1$ in the theorem, setting $\eta(k) = \omega(k)/k^\alpha$ and taking into account Remark 4.3. \square

5. EXAMPLES AND APPLICATIONS

In this section we describe the democracy functions h_ℓ and h_r in various examples which can be found in the literature. Inclusions for $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ and $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ will be obtained immediately from the results of sections 3 and 4. The most interesting case appears when \mathcal{B} is a wavelet basis, and \mathbb{B} a function or distribution space in \mathbb{R}^d which can be characterized by such basis (eg, the general Besov or Triebel-Lizorkin spaces, $B_{p,q}^\alpha$ and $F_{p,q}^s$, and also rearrangement invariant spaces as the Orlicz and Lorentz classes, L^Φ and $L^{p,q}$). Such characterizations provide a description of each \mathbb{B} as a sequence space, so for simplicity we shall work in this simpler setting, reminding in each case the original function space framework.

Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ denote the family of all dyadic cubes Q in \mathbb{R}^d , ie

$$\mathcal{D} = \{ Q_{j,k} = 2^{-j}([0, 1]^d + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d \}.$$

We shall consider sequences indexed by \mathcal{D} , $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{D}}$, endowed with quasi-norms of the following form

$$\left\| \left(\sum_{Q \in \mathcal{D}} (|Q|^{\gamma - \frac{1}{2}} |s_Q| \chi_Q(\cdot))^r \right)^{1/r} \right\|_{\mathbb{X}}, \quad (5.1)$$

where $0 < r \leq \infty$, $\gamma \in \mathbb{R}$ and \mathbb{X} is a suitable quasi-Banach function space in \mathbb{R}^d , such as the ones we consider below. The canonical basis $\mathcal{B}_c = \{\mathbf{e}_Q\}_{Q \in \mathcal{D}}$ is formed by the sequences \mathbf{e}_Q with entry 1 at Q and 0 otherwise. In each of the examples below, the greedy algorithms and democracy functions are considered with respect to the normalized basis $\mathcal{B} = \{\mathbf{e}_Q / \|\mathbf{e}_Q\|_{\mathbb{B}}\}$. Similarly, when stating the corresponding results for the functional setting we shall write \mathcal{W} for the wavelet basis.

Example 5.1. $\mathbb{X} = L^p(\mathbb{R}^d)$, $0 < p < \infty$. In this case, it is customary to consider the sequence spaces $\mathfrak{f}_{p,r}^s$, $s \in \mathbb{R}$, $0 < r \leq \infty$, with quasi-norms given by

$$\|\mathbf{s}\|_{\mathfrak{f}_{p,r}^s} := \left\| \left(\sum_{Q \in \mathcal{D}} (|Q|^{-\frac{s}{d} - \frac{1}{2}} |s_Q| \chi_Q(\cdot))^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d)}.$$

It was proved in [16, 11] that, for all $s \in \mathbb{R}$ and $0 < r \leq \infty$,

$$h_\ell(N; \mathfrak{f}_{p,r}^s) \approx h_r(N; \mathfrak{f}_{p,r}^s) \approx N^{1/p} \quad (5.2)$$

and

$$\mathcal{A}_q^\alpha(\mathfrak{f}_{p,r}^s) = \ell^{\tau,q}(\mathfrak{f}_{p,r}^s) = \left\{ \mathbf{s} : \{s_Q \|e_Q\|_{\mathfrak{f}_{p,r}^s}\}_Q \in \ell^{\tau,q} \right\}, \quad (5.3)$$

if $\frac{1}{\tau} = \alpha + \frac{1}{p}$, as asserted in Theorem 1.2.

It is well-known that $\mathfrak{f}_{p,r}^s$ coincides with the coefficient space under a wavelet basis \mathcal{W} of the (homogeneous) Triebel-Lizorkin space $\dot{F}_{p,r}^s(\mathbb{R}^d)$, defined in terms of Littlewood-Paley theory (see eg [10, 25, 21]). In particular, under suitable decay and smoothness on the wavelet family (so that it is an unconditional basis of the involved spaces) the statement in (5.3) can be translated into

$$\mathcal{A}_q^\alpha(\mathcal{W}, \dot{F}_{p,r}^s(\mathbb{R}^d)) = \mathcal{G}_q^\alpha(\mathcal{W}, \dot{F}_{p,r}^s(\mathbb{R}^d)) = \dot{B}_{q,q}^{s+\alpha d}(\mathbb{R}^d)$$

when $\frac{1}{q} = \frac{\alpha}{d} + \frac{1}{p}$. We refer to [16, 17, 5, 11] for details and further results.

Example 5.2. Weighted Lebesgue spaces $\mathbb{X} = L^p(w)$, $0 < p < \infty$. For weights $w(x)$ in the Muckenhoupt class $A_\infty(\mathbb{R}^d)$, one can define sequence spaces $\mathfrak{f}_{p,r}^s(w)$ with the quasi-norm

$$\|\mathbf{s}\|_{\mathfrak{f}_{p,r}^s(w)} := \left\| \left(\sum_{Q \in \mathcal{D}} (|Q|^{-\frac{s}{d} - \frac{1}{2}} |s_Q| \chi_Q(\cdot))^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^d, w)}.$$

Similar computations as in the previous case in this more general situation will also lead to the identities in (5.2) and (5.3), with $\mathfrak{f}_{p,r}^s$ replaced by $\mathfrak{f}_{p,r}^s(w)$. We refer to [26, 20] for details in some special cases.

When \mathcal{W} is a (sufficiently smooth) orthonormal wavelet basis and w is a weight in the Muckenhoupt class $A_p(\mathbb{R}^d)$, $1 < p < \infty$, then $\mathfrak{f}_{p,2}^0(w)$ becomes the coefficient space of the weighted Lebesgue space $L^p(w)$ (see eg [1]). One then obtains as special case

$$h_\ell(N; \mathcal{W}, L^p(w)) \approx h_r(N; \mathcal{W}, L^p(w)) \approx N^{\frac{1}{p}}.$$

Moreover, if $\omega \in A_\tau(\mathbb{R}^d)$,

$$\mathcal{A}_\tau^\alpha(\mathcal{W}, L^p(w)) \approx \mathcal{G}_\tau^\alpha(\mathcal{W}, L^p(w)) \approx \dot{B}_{\tau,\tau}^{\alpha d}(w^{\tau/p}), \quad \text{if } \frac{1}{\tau} = \alpha + \frac{1}{p},$$

where $\dot{B}_{\tau,q}^\alpha(w)$ denotes a weighted Besov space (see [26] for details).

Example 5.3. Orlicz spaces $\mathbb{X} = L^\Phi(\mathbb{R}^d)$. Following [12], we denote by \mathfrak{f}^Φ the sequence space with quasi-norm

$$\|\mathbf{s}\|_{\mathfrak{f}^\Phi} := \left\| \left(\sum_{Q \in \mathcal{D}} (|s_Q| \frac{\chi_Q(\cdot)}{|Q|^{1/2}})^2 \right)^{1/2} \right\|_{L^\Phi(\mathbb{R}^d)},$$

where L^Φ is an Orlicz space with non-trivial Boyd indices. If we denote by $\varphi(t) = 1/\Phi^{-1}(1/t)$, the fundamental function of L^Φ , then it is shown in [12] that

$$h_\ell(N; \mathfrak{f}^\Phi) \approx \inf_{s>0} \frac{\varphi(Ns)}{\varphi(s)} \quad \text{and} \quad h_r(N; \mathfrak{f}^\Phi) \approx \sup_{s>0} \frac{\varphi(Ns)}{\varphi(s)},$$

with the two expressions being equivalent iff $\varphi(t) = t^{1/p}$ (ie, iff $L^\Phi = L^p$). Thus, these are first examples of non-democratic spaces, with a wide range of possibilities for the democracy functions. The theorems in sections 3 and 4 recover the embeddings obtained in [12] for the approximation classes $\mathcal{A}_q^\alpha(\mathfrak{f}^\Phi)$ and $\mathcal{G}_q^\alpha(\mathfrak{f}^\Phi)$ in terms of weighted discrete Lorentz spaces. When using suitable wavelet bases, these lead to corresponding inclusions for $\mathcal{A}_q^\alpha(\mathcal{W}, L^\Phi)$ and $\mathcal{G}_q^\alpha(\mathcal{W}, L^\Phi)$, some of which can be expressed in terms of Besov spaces of generalized smoothness (see [12] for details).

Example 5.4. Lorentz spaces $\mathbb{X} = L^{p,q}(\mathbb{R}^d)$, $0 < p, q < \infty$. Consider sequence spaces $\mathfrak{l}^{p,q}$ defined by the following quasi-norms

$$\|\mathbf{s}\|_{\mathfrak{l}^{p,q}} := \left\| \left(\sum_{Q \in \mathcal{D}} (|s_Q| \frac{\chi_Q(\cdot)}{|Q|^{1/2}})^2 \right)^{1/2} \right\|_{L^{p,q}(\mathbb{R}^d)}.$$

Their democracy functions have been computed in [14], obtaining

$$h_\ell(N; \mathfrak{l}^{p,q}) \approx N^{\frac{1}{\max(p,q)}} \quad \text{and} \quad h_r(N; \mathfrak{l}^{p,q}) \approx N^{\frac{1}{\min(p,q)}}.$$

These imply corresponding inclusions for the classes $\mathcal{A}_s^\alpha(\mathfrak{l}^{p,q})$ and $\mathcal{G}_s^\alpha(\mathfrak{l}^{p,q})$ in terms of discrete Lorentz spaces $\ell^{\tau,s}$ (as described in the theorems of sections 3 and 4). The spaces $\mathfrak{l}^{p,q}$ characterize, via wavelets, the usual Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ when $1 < p < \infty$ and $1 \leq q < \infty$ ([31]). Hence inclusions for $\mathcal{A}_s^\alpha(\mathcal{W}, L^{p,q})$ and $\mathcal{G}_s^\alpha(\mathcal{W}, L^{p,q})$ can be obtained using standard Besov spaces.

Example 5.5. Hyperbolic wavelets. For $0 < p < \infty$, consider now the sequence space

$$\|\mathbf{s}\|_{\mathfrak{f}_{\text{hyp}}^p} := \left\| \left(\sum_R (|s_R| \frac{\chi_R(\cdot)}{|R|^{1/2}})^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}.$$

where R runs over the family of all dyadic rectangles of \mathbb{R}^d , that is $R = I_1 \times \dots \times I_d$, with $I_i \in \mathcal{D}(\mathbb{R})$, $i = 1, \dots, d$. This gives another example of non-democratic basis. In fact, the following result is proved in [36, Proposition 11] (see also [33]):

(a) If $0 < p \leq 2$,

$$h_\ell(N; \mathfrak{f}_{\text{hyp}}^p) \approx N^{1/p} (\log N)^{\left(\frac{1}{2} - \frac{1}{p}\right)(d-1)} \quad \text{and} \quad h_r(N; \mathfrak{f}_{\text{hyp}}^p) \approx N^{1/p}.$$

(b) If $2 \leq p < \infty$,

$$h_\ell(N; \mathfrak{f}_{\text{hyp}}^p) \approx N^{1/p} \quad \text{and} \quad h_r(N; \mathfrak{f}_{\text{hyp}}^p) \approx N^{1/p} (\log N)^{\left(\frac{1}{2} - \frac{1}{p}\right)(d-1)}.$$

If \mathcal{H}_d denotes the multidimensional (hyperbolic) Haar basis, then $\mathfrak{f}_{\text{hyp}}^p$ becomes the coefficient space of the usual $L^p(\mathbb{R}^d)$ if $1 < p < \infty$ (and the dyadic Hardy space $H^p(\mathbb{R}^d)$ if $0 < p \leq 1$). In this case, one obtains corresponding inclusions for the classes $\mathcal{A}_q^\alpha(\mathcal{H}_d, L^p)$ and $\mathcal{G}_q^\alpha(\mathcal{H}_d, L^p)$ (see also [18, Thm 5.2]), some of which could possibly be expressed in terms of Besov spaces of bounded mixed smoothness [18, 6].

Example 5.6. Bounded mean oscillation. Let bmo denote the space of sequences $\mathbf{s} = \{s_I\}_{I \in \mathcal{D}}$ with

$$\|\mathbf{s}\|_{bmo} = \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \subset I, J \in \mathcal{D}} |s_J|^2 |J| \right)^{1/2} < \infty. \quad (5.4)$$

This sequence space gives the correct characterization of $BMO(\mathbb{R})$ for sufficiently smooth wavelet bases appropriately normalized (see [35, 10, 16]). Their democracy functions are determined by

$$h_\ell(N; bmo) \approx 1, \quad h_r(N; bmo) \approx (\log N)^{1/2}. \quad (5.5)$$

The first part of (5.5) is easy to prove, and the second follows, for instance, by an argument similar to the one presented in the proof of [27, Lemma 3]. Our results of sections 3 and 4 give in this case the inclusions:

$$\ell_{k^\alpha \sqrt{\log k}}^q \hookrightarrow \mathcal{G}_q^\alpha(bmo) \hookrightarrow \mathcal{A}_q^\alpha(bmo) \hookrightarrow \ell_{k^\alpha}^q = \ell^{1/\alpha, q}. \quad (5.6)$$

However, this is not the best one can say for the approximation classes \mathcal{A}_q^α . A result proved in [29] (see also Proposition 11.6 in [16]) shows that one actually has

$$\mathcal{A}_q^\alpha(bmo) = \mathcal{A}_q^\alpha(\ell^\infty) = \ell^{1/\alpha, q},$$

for all $\alpha > 0$ and $q \in (0, \infty]$. For $0 < r < \infty$ one can define the space $bmo(r)$ replacing the 2 by r in (5.4); it can then be shown that $h_r(N; bmo(r)) \approx (\log N)^{1/r}$.

6. DEMOCRACY FUNCTIONS FOR $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ AND $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$

As usual, we fix a (normalized) unconditional basis $\mathcal{B} = \{e_j\}_{j=1}^\infty$ in \mathbb{B} . In this section we compute the democracy functions for the spaces $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ and $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$, in terms of the democracy functions in the ambient space \mathbb{B} . To distinguish among these notions we shall use, respectively, the notations

$$h_\ell(N; \mathcal{A}_q^\alpha), \quad h_\ell(N; \mathcal{G}_q^\alpha) \quad \text{and} \quad h_\ell(N; \mathbb{B}),$$

and similarly for h_r (recall the definitions in section 2.5). Since we shall use the embeddings in sections 3 and 4, observe first that

$$h_\ell(N; \ell_\omega^q(\mathcal{B}, \mathbb{B})) \approx h_r(N; \ell_\omega^q(\mathcal{B}, \mathbb{B})) \approx \omega(N), \quad (6.1)$$

for all $\omega \in \mathbb{W}_+$ and $0 < q \leq \infty$. This is immediate from the definition of the spaces $\ell_\omega^q(\mathcal{B}, \mathbb{B})$ and Lemma 2.3.

Proposition 6.1. *Fix $\alpha > 0$ and $0 < q \leq \infty$. If $h_\ell(\cdot; \mathbb{B})$ is doubling then*

$$(a) \quad h_\ell(N; \mathcal{G}_q^\alpha) \approx N^\alpha h_\ell(N; \mathbb{B}).$$

$$(b) \ h_r(N; \mathcal{G}_q^\alpha) \approx N^\alpha h_r(N; \mathbb{B}).$$

In particular, \mathcal{B} is democratic in $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ if and only if \mathcal{B} is democratic in \mathbb{B} .

Proof. The inequalities “ \gtrsim ” in (a), and “ \lesssim ” in (b) follow immediately from the embeddings

$$\ell_{k^\alpha h_r(k)}^q(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^\alpha h_\ell(k)}^q(\mathcal{B}; \mathbb{B})$$

and the remark in (6.1). Thus we must show the converse inequalities. To establish (a), given $N = 1, 2, 3, \dots$ choose Γ with $|\Gamma| = N$ and so that $\|1_\Gamma\|_{\mathbb{B}} \leq 2h_\ell(N; \mathbb{B})$. Then, using the trivial bound in (4.1) we obtain

$$h_\ell(N; \mathcal{G}_q^\alpha) \leq \|1_\Gamma\|_{\mathcal{G}_q^\alpha} \lesssim N^\alpha \|1_\Gamma\|_{\mathbb{B}} \approx N^\alpha h_\ell(N; \mathbb{B}).$$

We now prove “ \gtrsim ” in (b). Given $N = 1, 2, \dots$, choose first Γ with $|\Gamma| = N$ and $\|1_\Gamma\|_{\mathbb{B}} \geq \frac{1}{2}h_r(N; \mathbb{B})$, and then any Γ' disjoint with Γ with $|\Gamma'| = N$. Then

$$h_r(2N; \mathcal{G}_q^\alpha) \geq \|1_{\Gamma \cup \Gamma'}\|_{\mathcal{G}_q^\alpha} \gtrsim N^\alpha \gamma_N(1_{\Gamma \cup \Gamma'}; \mathbb{B}) \gtrsim N^\alpha \|1_\Gamma\|_{\mathbb{B}} \approx N^\alpha h_r(N; \mathbb{B}).$$

The required bound then follows from the doubling property of h_r . \square

Proposition 6.2. Fix $\alpha > 0$ and $0 < q \leq \infty$, and assume that $h_\ell(\cdot; \mathbb{B})$ is doubling. Then

$$(a) \ h_\ell(N; \mathcal{A}_q^\alpha) \approx N^\alpha h_\ell(N; \mathbb{B}).$$

$$(b) \ h_r(N; \mathcal{A}_q^\alpha) \lesssim N^\alpha h_r(N; \mathbb{B}).$$

In particular, if \mathcal{B} is democratic in \mathbb{B} then \mathcal{B} is democratic in $\mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$.

Proof. As before, “ \gtrsim ” in (a), and “ \lesssim ” in (b) follow immediately from the embeddings

$$\ell_{k^\alpha h_r(k)}^q(\mathcal{B}; \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \ell_{k^\alpha h_\ell(k)}^q(\mathcal{B}; \mathbb{B}).$$

The converse inequality in (a) follows from the previous proposition and the trivial inclusion $\mathcal{G}_q^\alpha \hookrightarrow \mathcal{A}_q^\alpha$. \square

As shown in Example 5.6, the converse to the last statement in Proposition 6.2 is not necessarily true. The space $\mathbb{B} = bmo$ is not democratic, but their approximation classes $\mathcal{A}_q^\alpha(bmo) = \ell^{1/\alpha, q}$ are democratic. Moreover, this example shows that the converse to the inequality in (b) does not necessarily hold, since

$$h_r(N; \mathcal{A}_\alpha^q(bmo)) = N^\alpha \quad \text{but} \quad N^\alpha h_r(N; bmo) \approx N^\alpha (\log N)^{1/2}.$$

Nevertheless, we can give a sufficient condition for $h_r(N; \mathcal{A}_q^\alpha) \approx N^\alpha h_r(N; \mathbb{B})$, which turns out to be easily verifiable in all the other examples presented in §5.

PROPERTY (H). We say that \mathcal{B} satisfies the **Property (H)** if for each $n = 1, 2, 3, \dots$ there exist $\Gamma_n \subset \mathbb{N}$, with $|\Gamma_n| = 2^n$, satisfying the property

$$\|1_{\Gamma'}\|_{\mathbb{B}} \approx h_r(2^{n-1}; \mathbb{B}), \quad \forall \Gamma' \subset \Gamma_n \quad \text{with} \quad |\Gamma'| = 2^{n-1}.$$

Proposition 6.3. Assume that \mathcal{B} satisfies the Property (H). Then, for all $\alpha > 0$ and $0 < q \leq \infty$

$$h_r(N; \mathcal{A}_q^\alpha) \approx N^\alpha h_r(N; \mathbb{B})$$

Proof. We must show “ \gtrsim ”, for which we argue as in the proof of Proposition 6.1. Given $N = 2^n$, select Γ_n as in the definition of Property (H). Then,

$$h_r(N; \mathcal{A}_q^\alpha) \geq \|1_{\Gamma_n}\|_{\mathcal{A}_q^\alpha} \gtrsim N^\alpha \sigma_{N/2}(1_{\Gamma_n}).$$

Now, the property (H) (and the remark in (2.4)) give

$$\sigma_{N/2}(1_{\Gamma_n}) = \inf \{ \|1_{\Gamma'}\|_{\mathbb{B}} : \Gamma' \subset \Gamma, |\Gamma'| = N/2 \} \approx h_r(N/2; \mathbb{B}) \approx h_r(N; \mathbb{B}).$$

Combining these two facts the proposition follows for $N = 2^n$. For general N use the result just proved and the doubling property of h_r . \square

As an immediate consequence, the property (H) allows to remove the possible logarithmic loss for the embedding $\ell_{k^\alpha h_r(k)}^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$ discussed in Corollary 3.9.

Corollary 6.4. More about optimality for inclusions into \mathcal{A}_q^α .

Assume that $(\mathbb{B}, \mathcal{B})$ satisfies property (H). If for some $\alpha > 0$, $q \in (0, \infty]$ and $\omega \in \mathbb{W}_+$ we have $\ell_\omega^q(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, then necessarily $\omega(k) \gtrsim k^\alpha h_r(k)$, and therefore $\ell_\omega^q \hookrightarrow \ell_{k^\alpha h_r(k)}^q$.

The following examples show that Property (H) is often satisfied.

Example 6.1. Wavelet bases in Orlicz spaces $L^\Phi(\mathbb{R}^d)$ satisfy the property (H). Indeed, recall from [12, Thm 1.2] (see also Example 5.3) that

$$h_r(N; L^\Phi) \approx \sup_{s>0} \varphi(Ns)/\varphi(s). \quad (6.2)$$

Moreover, any collection Γ of N pairwise disjoint dyadic cubes *with the same fixed size* $a > 0$ satisfies

$$\|1_\Gamma\|_{L^\Phi} \approx \varphi(Na)/\varphi(a), \quad (6.3)$$

(see eg [12, Lemma 3.1]). Thus, for each $N = 2^n$, we first select $a_n = 2^{jn^d}$ so that $h_r(2^n; L^\Phi) \approx \varphi(2^n a_n)/\varphi(a_n)$, and then we choose as Γ_n any collection of 2^n pairwise disjoint cubes with constant size a_n . Then, any subfamily $\Gamma' \subset \Gamma_n$ with $|\Gamma'| = N/2$, satisfies

$$\|1_{\Gamma'}\|_{L^\Phi} \approx \varphi((N/2)a_n)/\varphi(a_n) \approx \varphi(Na_n)/\varphi(a_n) \approx h_r(N) \approx h_r(N/2),$$

by (6.3) and the doubling property of φ and h_r .

Example 6.2. Wavelet bases in Lorentz spaces $L^{p,q}(\mathbb{R}^d)$, $1 < p, q < \infty$. These also satisfy the property (H). Indeed, it can be shown that any set Γ consisting of N *disjoint cubes of the same size* has

$$\|1_\Gamma\|_{L^{p,q}} \approx N^{\frac{1}{p}},$$

while sets Δ consisting of N disjoint cubes *all having different sizes* satisfy

$$\|1_\Delta\|_{L^{p,q}} \approx N^{\frac{1}{q}}.$$

(see [14, (3.6) and (3.8)]). Since $h_r(N) \approx N^{1/(p \wedge q)}$, we can define the Γ_n 's with sets of the first type when $p \leq q$, and with sets of the second type when $q < p$, to obtain in both cases a collection satisfying the hypotheses of property (H).

Example 6.3. The hyperbolic Haar system in $L^p(\mathbb{R}^d)$ from Example 5.5 also satisfies property (H). In this case, again, any set Γ consisting of N disjoint rectangles has

$$\|1_\Gamma\|_{L^p(\mathbb{R}^d)} = N^{\frac{1}{p}}.$$

On the other hand, if Δ_n denotes the set of all the dyadic rectangles in the unit cube with fixed size 2^{-n} , then

$$\|1_{\Delta_n}\|_{L^p(\mathbb{R}^d)} \approx 2^{n/p} n^{(d-1)/2} \approx |\Delta_n|^{1/p} (\log |\Delta_n|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}. \quad (6.4)$$

Moreover, it is not difficult to show that any $\Delta' \subset \Delta_n$ with $|\Delta'| = |\Delta_n|/2$ also satisfies (6.4) (with Δ_n replaced by Δ'). Hence, combining these two cases and using the description of $h_r(N)$ in Example 5.5, one easily establishes the property (H).

7. COUNTEREXAMPLES FOR THE CLASSES $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$

7.1. Conditions for $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$. Recall from section 2.3 that $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{A}_q^\alpha(\mathcal{B}, \mathbb{B})$, with equality of the spaces when \mathcal{B} is a democratic basis. It is known that there are some *conditional* non-democratic bases for which $\mathcal{G}_q^\alpha = \mathcal{A}_q^\alpha$ (see [13, Remark 6.2]). For unconditional bases, however, one could ask whether non-democracy necessarily implies that $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$. We do not know how to prove such a general result, but we can show that the inclusion $\mathcal{A}_q^\alpha \hookrightarrow \mathcal{G}_q^\alpha$ must fail whenever the gap between $h_\ell(N)$ and $h_r(N)$ is at least logarithmic (and even less than that). More precisely, we have the following.

Proposition 7.1. *Let \mathcal{B} be an unconditional basis in \mathbb{B} and $\alpha > 0$. Suppose that there exist integers $p_N \geq q_N \geq 1$, $N = 1, 2, \dots$ such that*

$$\lim_{N \rightarrow \infty} \frac{p_N}{q_N} = \infty \quad \text{and} \quad \frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \left(\frac{p_N}{q_N}\right)^\alpha. \quad (7.1)$$

Then the inclusion $\mathcal{A}_\tau^\alpha(\mathcal{B}, \mathbb{B}) \hookrightarrow \mathcal{G}_\tau^\alpha(\mathcal{B}, \mathbb{B})$ does not hold for any $\tau \in (0, \infty]$.

Proof. For each N , choose $\Gamma_l, \Gamma_r \subset \mathbb{N}$ with $|\Gamma_l| = p_N$, $|\Gamma_r| = q_N$, and such that

$$\|1_{\Gamma_l}\|_{\mathbb{B}} \leq 2h_\ell(p_N), \quad \|1_{\Gamma_r}\|_{\mathbb{B}} \geq \frac{1}{2}h_r(q_N). \quad (7.2)$$

Set $x_N = 1_{\Gamma_r} + 2 \cdot 1_{\Gamma_l - \Gamma_l \cap \Gamma_r}$. Since $\#(\Gamma_l - \Gamma_l \cap \Gamma_r) \geq p_N - q_N$, when $k \in [1, p_N - q_N]$ we have

$$\|x_N - G_k(x_N)\|_{\mathbb{B}} \geq \|1_{\Gamma_r}\|_{\mathbb{B}} \geq \frac{1}{2}h_r(q_N).$$

Therefore, using $p_N - q_N > p_N/2$ (since $p_N/q_N > 2$ for N large), we obtain that

$$\|x_N\|_{\mathcal{G}_\tau^\alpha(\mathcal{B}, \mathbb{B})} \geq \frac{1}{2} \left[\sum_{k=1}^{p_N/2} (k^\alpha h_r(q_N))^\tau \frac{1}{k} \right]^{\frac{1}{\tau}} \gtrsim h_r(q_N) p_N^\alpha. \quad (7.3)$$

On the other hand, we can estimate the norm of x_N as follows:

$$\|x_N\|_{\mathbb{B}} \lesssim \|1_{\Gamma_r}\|_{\mathbb{B}} + \|1_{\Gamma_l - \Gamma_l \cap \Gamma_r}\|_{\mathbb{B}} \leq h_r(q_N) + 2h_\ell(p_N) \lesssim h_r(q_N) \quad (7.4)$$

where the last inequality is true for N large due to (7.1). Thus

$$\sigma_k(x_N) \leq \|x_N\|_{\mathbb{B}} \lesssim h_r(q_N). \quad (7.5)$$

Next, if $k \geq q_N$, by (7.2)

$$\sigma_k(x_N) \leq 2\|1_{\Gamma_l - \Gamma_l \cap \Gamma_r}\|_{\mathbb{B}} \leq 2\|1_{\Gamma_l}\|_{\mathbb{B}} \lesssim h_\ell(p_N). \quad (7.6)$$

Combining (7.4), (7.5), and (7.6) we see that

$$\begin{aligned} \|x_N\|_{\mathcal{A}_\tau^\alpha(\mathcal{B}, \mathbb{B})} &\lesssim h_r(q_N) + \left[\sum_{k=1}^{q_N-1} (k^\alpha h_r(q_N))^\tau \frac{1}{k} + \sum_{k=q_N}^{p_N+q_N} (k^\alpha h_\ell(p_N))^\tau \frac{1}{k} \right]^{\frac{1}{\tau}} \\ &\lesssim h_r(q_N) + [h_r(q_N)^\tau (q_N)^{\alpha\tau} + h_\ell(p_N)^\tau (p_N)^{\alpha\tau}]^{\frac{1}{\tau}} \\ &\lesssim h_r(q_N) + h_r(q_N)(q_N)^\alpha \lesssim h_r(q_N)(q_N)^\alpha \end{aligned} \quad (7.7)$$

where in the second inequality we have used the elementary fact $\sum_{k=a}^{a+b} k^{\gamma-1} \lesssim b^\gamma$ if $b \geq a$, and the third inequality is due to (7.1). Therefore, from (7.3) and (7.7) we deduce

$$\frac{\|x_N\|_{\mathcal{G}_\tau^\alpha}}{\|x_N\|_{\mathcal{A}_\tau^\alpha}} \gtrsim \frac{h_r(q_N)(p_N)^\alpha}{h_r(q_N)(q_N)^\alpha} = \left(\frac{p_N}{q_N}\right)^\alpha \longrightarrow \infty$$

as $N \rightarrow \infty$. This shows the desired result. \square

Corollary 7.2. *Let \mathcal{B} be an unconditional basis such that $h_\ell(N) \lesssim N^{\beta_0}$ and $h_r(N) \gtrsim N^{\beta_1}$, for some $\beta_1 > \beta_0 \geq 0$. Then, $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$, for all $\alpha > 0$ and all $\tau \in (0, \infty]$.*

Proof. Choose $r, s \in \mathbb{N}$, such that $\frac{\alpha+\beta_0}{\alpha+\beta_1} < \frac{r}{s} < 1$. Take $p_N = N^s$ and $q_N = N^r$. Then, $\lim_{N \rightarrow \infty} \frac{p_N}{q_N} = \lim_{N \rightarrow \infty} N^{s-r} = \infty$ and

$$\frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \frac{N^{r\beta_1}}{N^{s\beta_0}} > N^{\alpha(s-r)} = \left(\frac{N^s}{N^r}\right)^\alpha = \left(\frac{p_N}{q_N}\right)^\alpha,$$

which proves (7.1) in this case, so that we can apply Proposition 7.1. \square

Corollary 7.3. *Let \mathcal{B} be an unconditional basis such that for some $\beta \geq 0$ and $\gamma > 0$ we have either*

- (i) $h_r(N) \gtrsim N^\beta (\log N)^\gamma$ and $h_\ell(N) \lesssim N^\beta$, or
- (ii) $h_r(N) \gtrsim N^\beta$ and $h_\ell(N) \lesssim N^\beta (\log N)^{-\gamma}$.

Then, $\mathcal{G}_q^\alpha \neq \mathcal{A}_q^\alpha$ for all $\alpha > 0$ and all $\tau \in (0, \infty]$.

Proof. *i)* Choose $a, b \in \mathbb{N}$ such that $0 < \frac{a}{b} < \frac{\gamma}{\alpha+\beta}$. Let $p_N = N^a 2^{N^b}$ and $q_N = 2^{N^b}$. Then, $\lim_{N \rightarrow \infty} \frac{p_N}{q_N} = \lim_{N \rightarrow \infty} N^a = \infty$ and

$$\frac{h_r(q_N)}{h_\ell(p_N)} \gtrsim \frac{(2^{N^b})^\beta (\log 2^{N^b})^\gamma}{N^{a\beta} (2^{N^b})^\beta} \approx \frac{N^{b\gamma}}{N^{a\beta}} = N^{b\gamma-a\beta} > N^{a\alpha} = \left(\frac{p_N}{q_N}\right)^\alpha$$

which proves (7.1) in this case, so that we can apply Proposition 7.1 to conclude the result. The proof of *ii)* is similar with the same choice of p_N and q_N . \square

7.2. Non linearity of $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$. We conclude by showing with simple examples that $\mathcal{G}_q^\alpha(\mathcal{B}, \mathbb{B})$ may not even be a linear space when the basis \mathcal{B} is not democratic.

Let $\mathbb{B} = \ell^p \oplus_{\ell^1} \ell^q$, $0 < q < p < \infty$; that is, \mathbb{B} consists of pairs $(a, b) \in \ell^p \times \ell^q$, endowed with the quasi-norm $\|a\|_{\ell^p} + \|b\|_{\ell^q}$. We consider the canonical basis in \mathbb{B} .

Now, set $\beta = \alpha + \frac{1}{p}$ and $x = \{(k^{-\beta}, 0)\}_{k \in \mathbb{N}} \in \mathbb{B}$. For $N = 1, 2, 3, \dots$ we have

$$\gamma_N(x) = \left(\sum_{k>N} \frac{1}{k^{\beta p}} \right)^{1/p} \approx \left(\frac{1}{N^{\beta p-1}} \right)^{1/p} = N^{-\alpha}.$$

This shows that $x \in \mathcal{G}_\infty^\alpha(\mathcal{B}, \mathbb{B})$. Similarly, if we let $\gamma = \alpha + \frac{1}{q}$, then $y = \{(0, j^{-\gamma})\}_{j \in \mathbb{N}}$ belongs to $\mathcal{G}_\infty^\alpha$. We will show, however, that $x + y \notin \mathcal{G}_\infty^\alpha$. In fact, we will find a subsequence N_J of natural numbers so that

$$\gamma_{N_J}(x + y) \approx \frac{1}{N_J^{\alpha\beta/\gamma}} \quad (7.8)$$

(notice that $\beta < \gamma$ since we chose $q < p$). To prove (7.8) let $A_1 = \{1\}$ and

$$A_j = \left\{ k \in \mathbb{N} : \frac{1}{j^\gamma} \leq \frac{1}{k^\beta} < \frac{1}{(j-1)^\gamma} \right\}, \quad j = 2, 3, \dots$$

The number of elements in A_j is

$$|A_j| \approx j^{\gamma/\beta} - (j-1)^{\gamma/\beta} \approx j^{\frac{\gamma}{\beta}-1}, \quad j = 1, 2, 3, \dots \quad (7.9)$$

For $J = 2, 3, 4, \dots$ let $N_J = \sum_{j=1}^J |A_j| + J$. From (7.9) we obtain

$$N_J \approx \sum_{j=1}^J j^{\frac{\gamma}{\beta}-1} + J \approx J^{\frac{\gamma}{\beta}} + J \approx J^{\frac{\gamma}{\beta}},$$

since $\gamma > \beta$. Thus,

$$\begin{aligned} \gamma_{N_J}(x + y) &\approx \left(\sum_{k > J^{\frac{\gamma}{\beta}}} k^{-\beta p} \right)^{1/p} + \left(\sum_{j > J} j^{-\gamma q} \right)^{1/q} \approx [(J^{\gamma/\beta})^{-\beta p + 1}]^{1/p} + [J^{-\gamma q + 1}]^{1/q} \\ &= J^{-\alpha\gamma/\beta} + J^{-\alpha} \approx J^{-\alpha} \approx (N_J)^{-\alpha\beta/\gamma}, \end{aligned}$$

proving (7.8).

A simple modification of the above construction can be used to show that the set $\mathcal{G}_s^\alpha(\mathcal{B}, \mathbb{B})$ is not linear, for any $\alpha > 0$ and any $s \in (0, \infty)$.

REFERENCES

- [1] H.A. AIMAR, A.L. BERNARDIS, AND F.J. MARTÍN-REYES, *Multiresolution approximation and wavelet bases of weighted Lebesgue spaces*, J. Fourier Anal. and Appl., 9, No.5, (2003), 497-510.
- [2] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*. Academic Press Inc, 1988.
- [3] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces. An introduction*, No. 223, Springer-Verlag, Berlin-New York, 1976.
- [4] M. J. CARRO, J. RAPOSO AND J. SORIA, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Mem. Amer. Math. Soc., no. 877, 187(2007).
- [5] R.A. DEVORE, *Nonlinear Approximation*, Acta Numer., 7, (1998), 51-150.
- [6] R. DEVORE, S. KONYAGIN, V. TEMLYAKOV, *Hyperbolic wavelet approximation*. Constr. Approx., 14, (1998), 1-26.
- [7] R. DEVORE AND V.A. POPOV, *Interpolation spaces and nonlinear approximation*, Function spaces and applications (Lund, 1986), Lecture Notes in Math., 1302, Springer, Berlin, (1988), 191-205.
- [8] R. A. DEVORE AND V. N. TEMLYAKOV, *Some remarks on greedy algorithms*, Adv. Comp. Math. 5, (2-3), (1996), 113-187.
- [9] S.J. DILWORTH, N.J. KALTON, D. KUTZAROVA, AND V.N. TEMLYAKOV, *The Thresholding Greedy Algorithm, Greedy Bases, and Duality*, Constr. Approx., 19, (2003), 575-597.
- [10] M. FRAZIER AND B. JAWERTH, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93, (1990), 34-170.
- [11] G. GARRIGÓS, E. HERNÁNDEZ, *Sharp Jackson and Bernstein Inequalities for n -term Approximation in Sequence spaces with Applications*, Indiana Univ. Math. J. 53, (2004), 1739-1762.

- [12] G. GARRIGÓS, E. HERNÁNDEZ, AND J.M. MARTELL, *Wavelets, Orlicz spaces and greedy bases*, Appl. Compt. Harmon. Anal., 24, (2008), 70–93.
- [13] R. GRIBONVAL, M. NIELSEN, *Some remarks on non-linear approximation with Schauder bases*, East. J. of Approximation, 7(2), (2001), 1–19.
- [14] E. HERNÁNDEZ, J.M. MARTELL AND M. DE NATIVIDADE, *Quantifying Democracy of Wavelet Bases in Lorentz Spaces*, Preprint (2009).
- [15] E. HERNÁNDEZ AND G. WEISS, *A first course on wavelets*, CRC Press, Boca Raton FL, 1996.
- [16] C. HSIAO, B. JAWERTH, B.J. LUCIER, AND X.M. YU, *Near optimal compression of almost optimal wavelet expansions*, Wavelet: mathematics and applications, Stud. Adv. Math., CRC, Boca Raton, FL, 133, (1994), 425–446.
- [17] B. JAWERTH AND M. MILMAN, *Wavelets and best approximation in Besov spaces*, in *Interpolation spaces and related topics (Haifa, 1990)*, 107–112, Israel Math. Conf. Proc., 5, Bar-Ilan Univ., Ramat Gan, 1992.
- [18] A. KAMONT AND V.N. TEMLYAKOV, *Greedy approximation and the multivariate Haar system*, Studia Math, 161 (3), (2004), 199–223.
- [19] G. KERKYACHARIAN AND D. PICARD, *Entropy, Universal Coding, Approximation, and Bases Properties*, Const. Approx. 20, (2004), DOI: 10.1007/s00365-003-0556-z 1–37.
- [20] G. KERKYACHARIAN AND D. PICARD, *Nonlinear Approximation and Muckenhoupt Weights*, Constr. Approx. 24, (2006), 123–156 DOI: 10.1007/s00365-005-0618-5.
- [21] G. KYRIAZIS, *Multilevel characterization of anisotropic function spaces*, SIAM J. Math. Anal. 36, (2004), 441–462.
- [22] S.V. KONYAGIN AND V.N. TEMLYAKOV, *A remark on greedy approximation in Banach spaces*, East. J. Approx. 5, (1999), 365–379.
- [23] S. KREIN, J. PETUNIN AND E. SEMENOV, *Interpolation of Linear Operators*, Translations Math. Monographs, vol. 55, Amer. Math. Soc., Providence, RI, (1992).
- [24] C. MERUCCI, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. Interpolation spaces and allied topics in analysis*, Lecture Notes in Math., 1070, Springer, Berlin, (1984), 183–201.
- [25] Y. MEYER, *Ondelettes et operateurs. I: Ondelettes*, Hermann, Paris, (1990). [English translation: *Wavelets and operators*, Cambridge University Press, (1992).]
- [26] M. DE NATIVIDADE, *Best approximation with wavelets in weighted Orlicz spaces*, Preprint (2008).
- [27] P. OSWALD, *Greedy Algorithms and Best m -Term Approximation with Respect to Biorthogonal Systems*, J. Fourier Anal. and Appl., 7, No.4, (2001), 325–341.
- [28] A. PIETSCH, *Approximation spaces*, J. Approximation Theory, 32, (1981), 113–134.
- [29] R. ROCHBERG AND M. TAIBLESON, *An averaging operator on a tree*, in *Harmonic analysis and partial differential equations (El Escorial, 1987)*, 207–213, Lecture Notes in Math., 1384, Springer, Berlin, 1989.
- [30] S. B. STECHKIN, *On absolute convergence of orthogonal series*, Dokl. Akad. Nauk SSSR, 102, (1955), 37–40.
- [31] P. SOARDI, *Wavelet bases in rearrangement invariant function spaces*, Proc. Amer. Math. Soc. 125 no. 12, (1997), 3669–3973.
- [32] V. N. TEMLYAKOV, *The best m -term approximation and greedy algorithms*, Adv. in Comp. Math., 8, (1998), 249–265.
- [33] V. N. TEMLYAKOV, *Nonlinear m -term approximation with regard to the multivariate Haar system*, East J. Approx., 4, (1998), 87–106.
- [34] V. N. TEMLYAKOV, *Nonlinear methods of approximation*, Found. Comp. Math., 3 (1), (2003), 33–107.
- [35] P. WOJSTASZCZYK, *The Franklin system is an unconditional basis in H^1* , Arkiv Mat., 20, (1982), 293–300.
- [36] P. WOJSTASZCZYK, *Greedy Algorithm for General Biorthogonal Systems*, Journal of Approximation Theory, 107, (2000), 293–314.
- [37] P. WOJSTASZCZYK, *Greediness of the Haar system in rearrangement invariant spaces*, Banach Center Publications, Warszawa, 72, (2006), 385–395.

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