Band-Limited Wavelets with Subexponential Decay

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Abstract

It is well known that the compactly supported wavelets cannot belong to the class $C^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$. This is also true for wavelets with exponential decay. We show that one can construct wavelets in the class $C^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$ that are "almost" of exponential decay and, moreover, they are band-limited. We do this by showing that we can adapt the construction of the Lemarié-Meyer wavelets [LM] that is found in [BSW] so that we obtain band-limited, C^∞-wavelets on $\mathbf{R}$ that have subexponential decay, that is, for every $0 < \varepsilon < 1$, there exits $C_\varepsilon > 0$ such that $|\psi(x)| \le C_\varepsilon \, e^{-|x|^{1-\varepsilon}}, x \in \mathbf{R}$. Moreover, all of its derivatives have also subexponential decay. The proof is constructive and uses the Gevrey classes of functions.

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1 Introduction.

An orthonormal wavelet ψ is said to have **exponential decay** if there exist $c > 0$ and $\alpha > 0$ such that $|\psi(x)| \leq c\, e^{-\alpha|x|}$ for all $x \in \mathbf{R}$. The spline wavelets have exponential decay ([Le]) as well as the compactly supported wavelets ([Da]). But, **there is no orthonormal wavelet with exponential decay belonging** to $C^\infty(\mathbf{R})$ such that all its derivatives are bounded. To see this, suppose that such a wavelet ψ exists. The exponential decay of ψ would imply that

$$\hat{\psi}(z) = \int_{\mathbf{R}} e^{-izx}\ \psi(x)\ dx$$

is a holomorphic function on $|\text{Im } z| < \alpha$. Moreover, the smoothness and decay of ψ would imply that all the moments of ψ are zero. (See Theorem 3.4, Chapter 2, in [HW]). Hence, $\frac{d^n \hat{\psi}}{d\xi^n}(0) = 0$ for all $n = 0, 1, 2, \cdots$. The expansion of $\hat{\psi}(z)$ in powers of z around the origin shows that $\hat{\psi} \equiv 0$ in a neighborhood of $z = 0$. Since $\{z \in \mathbf{C} : |\text{Im } z| < \alpha\}$ contains the real line in its interior, ψ must be the zero function on $\mathbf{R}$.

Orthonormal wavelets ψ that belong to $C^\infty(\mathbf{R})$ have been exhibited in [LM]. They are band-limited (i.e. the supports of their Fourier transforms are bounded) and belong to the Schwartz class $\mathcal{S}$. They can be constructed using smooth "bell" functions as explained in [AWW] or [HW]. It is impossible, however, for any one of these wavelets to have exponential decay (since $\hat{\psi} \equiv 0$ in a neighborhood of the origin).

DEFINITION 1.1 *A real-valued function f defined on* $\mathbf{R}$ *is said to have* **subexponential decay** *if whenever $0 < \varepsilon < 1$, there exists $C_\varepsilon > 0$ such that*

$$|\psi(x)| \leq C_\varepsilon\ e^{-|x|^{1-\varepsilon}}$$

for all $x \in$ $\mathbf{R}$.

We shall show how to construct band-limited, orthonormal wavelets with subexponential decay belonging to $C^\infty(\mathbf{R})$. The construction is obtained by finding an appropriate "bell" function b whose Fourier transform has subexponential decay. This is accomplished by means of the Gevrey classes of functions, whose definition and properties are presented in the next section.

2 The Gevrey Classes.

DEFINITION 2.1 *For $\delta > 0$, the* **Gevrey** *class Γ^δ is the set of all C^∞ real-valued functions defined on* $\mathbf{R}$ *such that for every compact set $K \subset \mathbf{R}$ there is a constant C_K satisfying*

$$|D^n f(x)| \leq C_K \ C_K^n \ n^{n\delta} \ ,$$

for all $x \in K$ and for all $n = 1, 2, 3, \cdots$.

DEFINITION 2.2 *For $\delta > 0$, the (***small***)* **Gevrey** *class γ^δ is the set of all C^∞ real-valued functions defined on* $\mathbf{R}$ *such that for every compact set $K \subset \mathbf{R}$ and every $\varepsilon > 0$, there is a constant $C_{K,\varepsilon}$ satisfying*

$$|D^n f(x)| \leq C_{K,\varepsilon} \ \varepsilon^n (n!)^\delta \ ,$$

for all $x \in K$ and for all $n = 1, 2, 3, \cdots$.

We have taken the above definitions from [Ho1] (pp. 280–281) and [Ho2] (p. 137). Since $n! \leq n^n$ it is clear that for every $\delta > 0$

$$\gamma^\delta \subset \Gamma^\delta \ . \tag{2.3}$$

LEMMA 2.4 *If $0 < \delta' < \delta$ then $\Gamma^{\delta'} \subset \gamma^\delta$.*

Proof: Let $K \subset \mathbf{R}$ be compact and $\varepsilon > 0$. For $f \in \Gamma^{\delta'}$ we can find $C_K > 0$ such that

$$|D^n f(x)| \leq C_K \ C_K^n \ n^{\delta' n} \ ,$$

for all $x \in K$ and all $n = 1, 2, 3, \cdots$. By Stirling's formula ($n! \sim \sqrt{2\pi n} \, n^n \, e^{-n}$) we can write

$$|D^n f(x)| \leq C'_K (n!)^\delta (C'_K)^n \ \frac{(n!)^{\delta' - \delta} \ e^{n\delta'}}{(\sqrt{2\pi n})^{\delta'}} \ , \qquad x \in K \ .$$

The sequence $A_n = C'_K (n!)^{\frac{\delta' - \delta}{n}} \ e^{\delta'} \Big/ (\sqrt{2\pi n})^{\delta'/n}$ tends to zero as $n \to \infty$ since $\delta' < \delta$. Thus, there exists $N(\varepsilon) \in \mathbf{N}$ such that for all $n \geq N(\varepsilon)$, $A_n \leq \varepsilon$. Hence, for all $x \in K$,

$$|D^n f(x)| \leq C'_K \ \varepsilon^n (n!)^\delta \ ,$$

for all $n \geq N(\varepsilon)$. This inequality is also true for $n = 1, 2, \dots N(\varepsilon) - 1$ by enlarging the constant if necessary. $\square$

The Gevrey classes satisfy $\gamma^{\delta_1} \subset \gamma^{\delta_2}$ and $\Gamma^{\delta_1} \subset \Gamma^{\delta_2}$ when $0 < \delta_1 < \delta_2$. When $\delta > 1$ the classes γ^{δ} and Γ^{δ} contain "cutoff" functions. This follows from Theorem 1.3.5 in [Ho1]. We feel that it is worthwhile for the reader to present the essential ingredients of this result. With $\chi = \chi_{[0,1]}$ write $\chi_a = \frac{1}{a}\chi\left(\frac{x}{a}\right)$. For any sequence $a_1 \geq a_2 \geq \cdots > 0$ such that $a = \sum_{j=1}^{\infty} a_j < \infty$, the function

$$\varphi_k = \chi_{a_1} * \dots * \chi_{a_k}$$

belongs to $C^{k-1}(\mathbf{R})$, has support in $[0, a]$ and converges as $k \to \infty$ to a function $\varphi \in C^{\infty}(\mathbf{R})$, with support in $[0, a]$, such that $\int_{\mathbf{R}} \varphi(x)\, dx = 1$ and

$$|D^n \varphi(x)| \leq \frac{2^n}{a_1 \dots a_n} . \tag{2.5}$$

By taking $a_n = n^{-\delta}$ in the above construction it follows that $\varphi \in \Gamma^{\delta}$ when $\delta > 1$. (Observe that in this case $\sum_{n=1}^{\infty} n^{-\delta}$ is a convergent series.)

This result shows that there are "cutoff" functions in every class Γ^{δ} and γ^{δ} when $\delta > 1$. A modification of the above regularization procedure shows that there exists a "cutoff" function which belongs to every Γ^{δ} and γ^{δ} for all $\delta > 1$.

PROPOSITION 2.6 *For every $a > 0$ there exists $\varphi_a \in \Gamma^{\delta}$ for every $\delta > 1$. Moreover, $\varphi_a \geq 0$, $\sup \varphi_a \subset [-a, a]$ and $\int_{\mathbf{R}} \varphi_a(x)\, dx = \pi/2$.*

Proof: Since Γ^{δ} is invariant under dilations and multiplication by constants, it is enough to show the result for $a = 1$ and show that $\int_{\mathbf{R}} \varphi_a(x)\, dx < \infty$. Let h be an even function such that $h \in C^{\infty}([-1, 1]), h \geq 0$, and $\int_{-1}^{1} h(x)\, dx = 1$. Choose $\delta_m = 1 + \frac{1}{m}$ and let N_m be an increasing sequence of positive integers such that

$$\sum_{n \geq N_m} \frac{1}{n^{\delta_m}} < \frac{1}{2^m} .$$

Choose $a_n = n^{-\delta_m}$ when $N_m \leq n < N_{m+1}$. Observe that

$$\sum_{n \geq N_1} a_n \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1 .$$

Define

$$\varphi_{(n)} = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_n}$$

where $h_a(x) = \frac{1}{a} h\left(\frac{x}{a}\right)$, so that $\int_{\mathbf{R}} h_a(x)\ dx = 1$. Obviously $\sup \varphi_{(n)} \subset [-1,1]$. We shall show that for every $\delta > 1$, there exists $C = C_\delta$ such that for all $x \in \mathbf{R}$ and all $N = 1, 2, 3, \cdots$,

$$|D^N \varphi_{(n)}(x)| \leq C_\delta (C_\delta)^N N^{\delta N}\ , \tag{2.7}$$

for all $n \geq n(C_\delta\ , N)$. Take m and n so large that $\delta_m < \delta$, and $N_m + N < n$. Then,

$$D^N \varphi_{(n)} = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_{N_m}} * Dh_{a_{N_m+1}} * \cdots * Dh_{N_m+N} * \cdots * h_{a_n}\ .$$

We have

$$\| Dh_{a_n} \|_1 = \frac{1}{a_n} \int_{\mathbf{R}} \frac{1}{a_n} \left| Dh\left(\frac{x}{a_n}\right)\right|\ dx \leq \frac{C}{a_n} \leq C n^{\delta_m}$$

if $n \geq N_m$. Thus, using $\int u * v = (\int u)(\int v)$, and $\int_{\mathbf{R}} h_a = 1$, we deduce,

$$\begin{aligned} |D^N \varphi_{(n)}(x)| &\leq C^N\ (N_m+1)^{\delta_m}\ \cdots\ (N_m+N)^{\delta_m} \\ &\leq C^N (N_m+N)^{\delta_m N} \leq C^N\ N_m^{\delta_m N} N^{\delta N} \\ &\leq C^N\ N_m^{2N} N^{\delta N} \leq C_\delta (C_\delta)^N N^{\delta N}\ , \end{aligned}$$

where $C_\delta = C N_m^2$ (observe that N_m depends on δ). One can show that $\{D^N \varphi_{(n)} : n = N_{1,\ldots}\}$ is a Cauchy sequence for every $N = 0, 1, 2, \ldots$. Thus, $\varphi_{(n)}$ converges to a function φ which satisfies 2.7 with $\varphi_{(n)}$ replaced by φ . Hence, $\varphi \in \Gamma^\delta$ for all $\delta > 1$ and $\sup \varphi_{(n)} \subset [-1,1]$. □

The behaviour of the Fourier transforms of functions with compact support that are contained in γ^δ is given in the following result.

PROPOSITION 2.8 *Let $\delta > 0$. Suppose f is a function such that* $\sup f \subset [-A, A]$ *and $f \in \gamma^\delta$. Then, for every $B > 0$ there exists a constant C_B such that*

$$|\hat{f}(z)| \leq C_B\ e^{A|\mathrm{Im}(z)|}\ e^{-B|\mathrm{Re}\ z|^{1/\delta}}\ , \qquad z \in \mathbf{C}\ .$$

Proposition 2.8 is a generalization of one of the implications in the Paley-Wiener theorem and its proof can be found in Lemma 12.7.4. of [Ho2].

3 The Construction.

For fixed $a > 0$ choose a "cutoff" function φ_a as in Proposition 2.6. In particular, $\varphi_a \in \Gamma^\delta$ for every $\delta > 1$. Set

$$\theta_a(x) = \int_{-\infty}^{x} \varphi_a(t)\ dt\ .$$

Observe that $\theta_a \in \Gamma^\delta$ for every $\delta > 1$. As in [AWW] we consider $S_a(x) = \sin(\theta_a(x))$ and $C_a(x) = \cos(\theta_a(x))$, so that

$$b_a(x) = S_a(x - \pi)\ C_{2a}(x - 2\pi), \qquad a \leq \frac{\pi}{3}\ , \tag{3.1}$$

is a bell function associated with the interval $[\pi, 2\pi]$ as considered in [AWW] or [BSW]. Let us assume for the moment (see Theorem 3.3 below) that S_a and C_a belong to Γ^δ for every $\delta > 1$. Since Γ^δ is an algebra (Proposition 8.4.1 in [Ho1]) and it is invariant under traslations, it follows that $b_a \in \Gamma^\delta$ for every $\delta > 1$. Extending b_a evenly to $[-\infty, 0]$ it is proved in [AWW] (see also Corollary 4.7 of Chapter 1 in [HW]) that the function ψ defined by

$$\hat{\psi}_a(\xi) = e^{i\xi/2}\ b_a(\xi) \tag{3.2}$$

is an orthonormal wavelet in $L^2(\mathbf{R})$.

The following result shows, as a particular case, that the functions S_a and C_a, constructed as the composition of the sine and cosine functions with θ_a, belong to Γ^δ for every $\delta > 1$.

THEOREM 3.3 *Let $\delta \geq 1$. Suppose that F is an entire function and $f \in \Gamma^\delta$. Then, $g(x) = F(f(x)) \in \Gamma^\delta$.*

Proof: We have to show that for every compact set $K \subset \mathbf{R}$ there is a constant C_0 such that

$$|D^N g(x_0)| \leq C_0\ C_0^N N^{\delta N}$$

for all $x_0 \in K$ and all $N = 1, 2, \cdots$. Using the Taylor expansion we can write

$$f(x) = \sum_{n=0}^{N} \frac{1}{n!}\ D^n f(x_0)(x - x_0)^n + R_N(x; x_0) \equiv f_N(x) + R_N(x).$$

Obviously, $D^N g(x_0) = D^N[F(f_N)](x_0)$. By the assumption, $F(f_N(z)), z \in$ $\mathbf{C}$, is analytic, and by the Cauchy formula we can write

$$D^N g(x_0) = \frac{N!}{2\pi i} \int_{\omega_N} \frac{F(f_N(z))}{(z - x_0)^{N+1}} \, dz \ ,$$

where $\omega_N = \{z \in \mathbf{C} : |z - x_0| = \frac{1}{2eC} \ N^{1-\delta}\}$ and C is the constant such that $|D^n f(x)| \leq C \ C^n \ n^{\delta n}$ for all $x \in K$ and all $n = 1, 2, \cdots$. If $z \in \omega_N$, we use Stirling's formula to obtain

$$\begin{aligned} |f_N(z)| &\leq \sum_{n=0}^{N} \frac{1}{n!} \ C \ C^n \ n^{\delta n} \left(\frac{1}{2eC} \ N^{1-\delta} \right)^n \\ &\leq \sum_{n=0}^{N} \frac{1}{n!} \ C \ n^{\delta n} \frac{1}{(2e)^n} \ N^{n-n\delta} \\ &\leq C' \sum_{n=0}^{N} n^{\delta n - n} \ e^n \ \frac{1}{(2e)^n} \ N^{n-n\delta} \\ &= C' \sum_{n=0}^{N} \frac{1}{2^n} \left(\frac{n}{N} \right)^{\delta n - n} . \end{aligned}$$

Since $\delta \geq 1$, we have $|f_N(z)| \leq C' \sum_{n=0}^{\infty} \frac{1}{2^n} = 2C'$. Since F is analytic we obtain $|F(f_N(z))| \leq C''$ on ω_N . Thus,

$$\begin{aligned} |D^N g(x_0)| &\leq \frac{N!}{2\pi} \ C'' \frac{2\pi}{2eC} \ N^{1-\delta} \left(\frac{1}{2eC} \ N^{1-\delta} \right)^{-(N+1)} \\ &\leq C_1 \ N!(2eC)^N N^{(\delta-1)N} . \end{aligned}$$

Using $N! \leq N^n$ we obtain

$$|D^N g(x_0)| \leq C_1 (2eC)^N N^{\delta N} \leq C_0 \ C_0^N N^{\delta N}$$

where $C_0 = \max\{C_1, 2eC\}$. □

REMARK. One can find in the literature that if F is an entire function and $f \in \Gamma^\delta$, then $h(x) = f(F(x)) \in \Gamma^\delta$ (see Proposition 8.4.1 in [Ho1]). The result contained in Theorem 3.3 seems to be knew.

COROLLARY 3.4 *There exist band-limited, C^∞, orthonormal wavelets in $L^2(\mathbf{R})$ with subexponential decay. Moreover all of its derivatives have also exponential decay.*

Proof: Let $0 < \varepsilon < 1$ and choose $\delta = \frac{1}{1-\varepsilon}(\delta > 1)$. The function b_a defined by 3.1, as well as its even extension to $(-\infty\ ,0]$ belong to Γ^δ for every $\delta > 1$ by Theorem 3.3. By Lemma 2.4, $b_a \in \gamma^\delta$ for every $\delta > 1$. By Proposition 2.8 (with $B = 1$) the orthonormal wavelet ψ_a given by 3.2 satisfies

$$|\psi_a(x)| = C \left| \hat{b}_a \left(x + \frac{1}{2} \right) \right| \le C_\varepsilon\ e^{-|x+\frac{1}{2}|^{1/\delta}} \le C_\varepsilon\ e^{-|x|^{1-\varepsilon}} , \qquad x \in \mathbf{R} .$$

That ψ_a is band-limited is obvious from the definition of b_a . The fact that all of its derivatives have also exponential decay follows from

$$|D^n \psi_a(x)| = C \left| \left(\xi^n e^{i\xi/2} b_a(\xi) \right)\hat{}(x) \right|$$

and

$$\xi^n e^{i\xi/2} b_a(\xi) \in \gamma^\delta$$

for every $\delta > 1$. □

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