Smoothing Minimally Supported Frequency (MSF) Wavelets : Part II

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Abstract

The main purpose of this paper is to give a procedure to "mollify" the low-pass filters of a large number of Minimally Supported Frequency (MSF) wavelets, so that the smoother functions obtained in this way are also low-pass filters for an MRA. Hence, we are able to approximate (in the L^2-norm) MSF wavelets by wavelets with any desired degree of smoothness on the Fourier transform side. Although the MSF wavelets we consider are band-limited, this may not be true for their smooth approximations. This phenomena is related to the invariant cycles under the transformation $x \mapsto 2x \,(\mathrm{mod}\, 2\pi)$. We also give a characterization of all low-pass filters for MSF wavelets. Throughout the paper new and interesting examples of wavelets are described.

1 Introduction

An orthonormal wavelet, or simply a wavelet, is defined to be a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{j,k} = 2^{\frac{j}{2}} \psi(2^j x - k), \qquad j, k \in \mathbb{Z}.$$

In the first paper of this series ([HWW]) we characterized all orthonormal wavelets having Fourier transforms supported on the set $S_\alpha = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, $0 < \alpha \leq \pi$ (or,

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symmetrically, $\tilde{S}_\beta = [-4\pi + \frac{4}{3}\beta, \frac{8}{3}\beta],\ 0 < \beta \le \pi$). This set is natural since it includes the support of the Fourier transform of the wavelets known as the Lemarié-Meyer wavelets ([LM]) and, as explained in [HWW], it is "optimal" for this purpose. In fact, we showed that the support excludes the interval $H_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha]$ and each such wavelet can be obtained by a simple procedure from a particular wavelet ψ^a whose Fourier transform is nonzero for almost every $\xi \in I_a = [-2a, -a] \cup [2\pi - a, 4\pi - 2a]$, $0 < a < 2\pi$. Observe that this set has measure 2π and it is easy to see that $|\widehat{\psi^a}(\xi)| = \chi_{I_a}(\xi)$ almost everywhere. More generally, for any wavelet ψ, the support of $\hat{\psi}$ must have measure that is at least 2π, and, when it is 2π, $|\hat{\psi}|$ must be the characteristic function of this supporting set ([BSW]). We have called wavelets whose Fourier transform have support that is a set of measure 2π Minimally Supported Frequency (MSF) wavelets. Observe that $I = I_\pi = [-2\pi, -\pi] \cup [\pi, 2\pi]$ is the support of the Fourier transform of the Shannon wavelet, $\psi^{(S)}$, and the Lemarié-Meyer wavelets, ψ, are obtained by constructing a smooth non-negative approximation b of χ_I, so that $\hat{\psi} = e^{i\frac{\xi}{2}} b(\xi)$ (see [BSW]). This process leads us to consider all wavelets that "arise" from the MSF wavelets ψ^a. We described in [HWW] how to construct such wavelets based on the two equations

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}; \tag{1.1}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi + 2k\pi))} = 0, \qquad a.e.\ \xi \in \mathbb{R}, \quad \text{for each } \ k \in 2\mathbb{Z}+1, \tag{1.2}$$

that characterize all wavelets ([Wan]). Being a characterizarion of all orthonormal wavelets, (1.1) and (1.2) imply the orthonormality of the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$, which is equivalent (see [HW]) to another pair of equations, namely,

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1, \qquad a.e.\ \xi \in \mathbb{R}; \tag{1.3}$$

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(2^j(\xi + 2k\pi))\,\overline{\hat{\psi}(\xi + 2k\pi)} = 0, \qquad a.e.\ \xi \in \mathbb{R}, \quad \text{for each } \ j \ge 1. \tag{1.4}$$

In these equations, and throughout this paper, the Fourier transform of a function f is defined by $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx$, so that the Plancherel theorem takes the form $2\pi \|f\|_2^2 = \|\hat{f}\|_2^2$.

Many natural questions arise from these considerations. Let us list some of them:

(1) What measurable sets $K \subset \mathbb{R}$ are the supports of the Fourier transforms of MSF wavelets?

(2) When can such an MSF wavelet associated with the set K be approximated (in the L^2-norm) by one that has a smooth Fourier transform supported in a set that is "slightly" larger than K?

(3) Does there exist an MSF wavelet that is not band-limited?

(4) Can K be a nowhere dense perfect set?

(5) Is the collection of all wavelets a closed subset (in $L^2(\mathbb{R})$) of the unit ball? Is it connected?

Answers to (1) and (3) can be found in [FW]. Negative answers to (2), positive answer to (4) and negative one to (5) are provided in [HWW].

These answers were derived rather directly by making use of (1.1) and (1.2). We shall consider other questions as well; however, one of our purposes in this paper is to apply another construction of wavelets to study these questions as well as related ones. As we shall see, this will give us a deeper understanding of the properties of wavelets connected with the smoothness of their Fourier transforms. The basic idea of this method is to use the notion of an MRA and the associated quadrature mirror filter construction. We assume that the reader is familiar with these notions, and refer to their excellent treatment in [Dau] and [Mey]. In particular, the fact that a "low-pass filter" m is a 2π-periodic function satisfying

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \qquad \text{for } \ a.e. \ \xi \in \mathbb{R}, \tag{1.5}$$

motivates one to ask what other properties guarantee that such a periodic function is a low-pass filter associated with an MRA.

In much the same spirit as in [HWW], we ask what "deformations" of such a filter give us "near by" filters. Of course, many authors have consider the question of what 2π-periodic functions satisfying (1.5) are MRA low-pass filters. We shall be particularly interested in A. Cohen's characterization of C^∞ filters ([Co1], [Co2]) and will develop some analogous results suited to our purpose of smoothing the filters associated with MSF wavelets. These results, which will be explained in section 2, allow us to construct a new family of MSF wavelets that answer a question connected with the behaviour around the origin of the support of the Fourier transform of a band-limited wavelet. The smoothing procedure, that will be discussed in section 4, will lead us to some surprising results; perhaps the most striking is that, even when the MSF wavelet have Fourier transform supported in a very simple bounded set (say, four disjoint intervals), the "neighboring" wavelets obtained by our method with smoother Fourier transform cannot be band-limited. This behaviour is connected with the invariant cycles of the transformation $x \mapsto 2x \pmod{2\pi}$, a subject that will be the main theme of the last section of this paper. It is difficult to describe fully this phenomenon without using the details of our construction. Thus we postpone further discussion of this feature.

2 Minimally supported low-pass filters

We have mentioned in the introduction that A. Cohen has studied the C^∞ filters that give rise to MRA's. Let us begin this section by presenting a precise statement of his result, which can be found in [Co1] or [Co2].

Theorem 2.1 *Let m be a 2π-periodic C^{r+1} function, $r = 0, 1, 2, ...\infty$, defined on $\mathbb{R}$ such that $m(0) = 1$. Then m is a low-pass filter for a wavelet if and only if m satisfies* (1.5) *and there exits a compact set K, which contains* 0 *in its interior, such that*

$$\sum_{\ell\in\mathbb{Z}} \chi_K(\xi + 2\ell\pi) = 1 \qquad \textit{for all } \ \xi \in \mathbb{R}, \tag{2.2}$$

and

$$m(2^{-j}\xi) \neq 0 \qquad \textit{for all } \ \xi \in K \quad \textit{ and all } \ j = 1, 2, \cdots. \tag{2.3}$$

Moreover, the scaling function φ can be chosen so that $\hat{\varphi} \in C^r$.

A. Cohen's result assumes $m \in C^\infty(\mathbb{R})$. The fact that we can replace C^∞ by C^{r+1}, $r = 0, 1, 2, ...$, is easily obtained from his arguments (see [HW]).

We shall obtain an analogous result for the case of filters that produce MSF wavelets associated with MRA's. In order to do this it is necessary to say a few words about MRA (as we said above, details can be found in [Dau] and [Mey]).

A multiresolution analysis (MRA) consists of a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{2.4}$$

$$f \in V_j \ \text{ if and only if } \ f(2(\cdot)) \in V_{j+1}, \quad j \in \mathbb{Z}, \tag{2.5}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{2.6}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \tag{2.7}$$

$$\left.\begin{array}{l}\text{There exists a } \varphi \in V_o, \text{ such that } \{\varphi(\cdot - k) : k \in \mathbb{Z}\} \\ \text{is an orthonormal basis for } V_o.\end{array}\right\} \tag{2.8}$$

The function φ is called a scaling function for the MRA. These five conditions are not independent. For example, (2.6) is a consequence of (2.4), (2.5) and (2.8). In case $|\hat{\varphi}|$ is continuous at zero (and this is generally true and assumed in this paper) it can be shown that (2.7) is equivalent to $|\hat{\varphi}(0)| = 1$ as long as the other conditions of an MRA are true (see [HWW] or [Wan] for details). Thus, multiplying $\hat{\varphi}$ by a unimodular constant we might as well assume (and do so in this paper) that $\hat{\varphi}(0) = 1$.

It follows from (2.5) that (2.4) is equivalent to $V_{-1} \subset V_0$. When this is the case, $\varphi(\frac{\bullet}{2}) \in V_0$ and if we develop this function in terms of the basis $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ and express this in terms of the Fourier transform we obtain

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) \qquad \text{for } \textit{a.e. } \xi \in \mathbb{R}, \tag{2.9}$$

where m is a 2π-periodic function satisfying (1.5) which is called the low-pass filter associated with this MRA. We recall that the wavelet corresponding to this scaling function is obtained from the equation

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\, \overline{m(\tfrac{\xi}{2} + \pi)}\, \hat{\varphi}(\tfrac{\xi}{2}). \tag{2.10}$$

We shall make use of the fact that the scaling function φ can be expressed in terms of the low-pass filter m

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi). \tag{2.11}$$

Moreover $|\hat{\varphi}|$ can also be recovered from the wavelet ψ via

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2. \tag{2.12}$$

We shall also use the following result:

Lemma 2.13 *Let $\varphi \in L^2(\mathbb{R})$ be such that $|\hat{\varphi}(0)| = 1$ and $|\hat{\varphi}|$ is continuous at zero. For $j \in \mathbb{Z}$, define V_j as the closed span in $L^2(\mathbb{R})$ of $\{\varphi_{j,k} : k \in \mathbb{Z}\}$. Then, the family of subspaces $\{V_j : j \in \mathbb{Z}\}$ is an* MRA *with scaling function φ if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1 \qquad \textit{for a.e. } \xi \in \mathbb{R}, \tag{2.14}$$

and (2.9) holds for a 2π-periodic function m.

For MSF wavelets we cannot apply Theorem 2.1, since $m \notin C^1$. In fact, it follows from (2.12) and (2.9) that the filters m associated with these wavelets must be of the form $|m| = \chi_E$, for some measurable set $E \subset \mathbb{R}$. In this situation, the 2π-periodicity of m implies $E = E + 2\pi$, and (1.5) can be rewritten as

$$\chi_E(\xi) + \chi_E(\xi + \pi) = 1 \qquad \text{for } \; a.e. \; \xi \in \mathbb{R}. \tag{2.15}$$

It follows from equation (2.11) that $|\hat{\varphi}| = \chi_S$, where $S = \bigcap_{j=1}^{\infty} 2^j E$. Since $\|\hat{\varphi}\|_2^2 = 2\pi\|\varphi\|_2^2 = 2\pi$, we immediately obtain $|S| = 2\pi$. The orthonormality of the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ implies

$$\sum_{k\in\mathbb{Z}} \chi_S(\xi + 2k\pi) = 1 \qquad \text{for } \; a.e. \; \xi \in \mathbb{R}, \tag{2.16}$$

since this is equivalent to (2.14) when $|\hat{\varphi}| = \chi_S$. This tells us that $\{S + 2k\pi : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ almost everywhere. Moreover, (2.11) implies

$$|m(2^{-j}\xi)| = 1 \qquad \text{for } \; a.e. \; \xi \in S \quad \text{and all } \; j \geq 1. \tag{2.17}$$

Finally observe that, if for an MSF wavelet, we have $|\hat{\varphi}(0)| = 1$ and $|\hat{\varphi}|$ continuous at 0, we must have that 0 is an interior point of S. We have proved several results concerning the low-pass filter of an MRA, which are collected in the next result:

Proposition 2.18 *Suppose that ψ is an* MSF *wavelet associated with an* MRA *for which $|\hat{\varphi}|$ is continuous at* 0 *and $|\hat{\varphi}(0)| = 1$. Then, the low-pass filter m associated with this wavelet must be of the form $|m| = \chi_E$, where $E \subset \mathbb{R}$ is a measurable set that satisfies $E = E + 2\pi$ (2π-periodicity) and* (2.15)*. Also, the set $S = \bigcap_{j=1}^{\infty} 2^j E$ is the support of $\hat{\varphi}$, $|\hat{\varphi}| = \chi_S$, $|S| = 2\pi$, S contains* 0 *in its interior, and satisfies conditions* (2.16) *and* (2.17)*.*

Remark 1 The set E that appears in proposition (2.18) does not have finite measure. But, when restricted to a period interval, the new set $F = E \cap [-\pi, \pi]$ has measure π. This follows from (2.15):

$$|F| = \int_{-\pi}^{\pi} \chi_E(\xi)\, d\xi = \int_{-\pi}^{0} \chi_E(\xi)\, d\xi + \int_{0}^{\pi} \chi_E(\xi)\, d\xi = \int_{0}^{\pi} [\chi_E(\xi + \pi) + \chi_E(\xi)]\, d\xi = \pi.$$

Remark 2 If ψ is an MSF wavelet and we define $\tilde{\psi}$ by $\hat{\tilde{\psi}}(\xi) = e^{i\alpha(\xi)}\hat{\psi}(\xi)$ for any real-valued measurable function α, then $\tilde{\psi}$ is also an MSF wavelet. This follows from the fact that (1.1) and (1.3) characterize MSF wavelets (see [HKLS] or [HW]). Thus, the phase of an MSF wavelet is arbitrary.

The next result characterizes the 2π-periodic measurable fucntions that are low-pass filters for MSF wavelets.

Theorem 2.19 *Let m be a 2π-periodic measurable function defined on $\mathbb{R}$ such that m is continuous at 0 and $|m(0)| = 1$. Then m is a low-pass filter for an* MSF *wavelet if and only if $|m| = \chi_E$, where $E \subset \mathbb{R}$ is a measurable set that satisfies* (2.15) *and*

$$\Big|\bigcap_{j=1}^{\infty} 2^j E\Big| = 2\pi. \tag{2.20}$$

In view of Proposition (2.18) we only need to show that if $|m| = \chi_E$, with E satisfying (2.15) and (2.20), then m is a low-pass filter for an MSF wavelet. In fact, we only need to show that m is a low-pass filter for a wavelet, since this will be an MSF wavelet by (2.11) and (2.10). We start with a lemma.

Lemma 2.21 *Suppose that E is a measurable set contained in $\mathbb{R}$ such that $E = E + 2\pi$ and E satisfies* (2.15)*. If $S = \bigcap_{j=1}^{\infty} 2^j E$, then we have*

(i) $|S \cap (S + 2k\pi)| = 0$ *for all* $k \neq 0,\ k \in \mathbb{Z}$;

(ii) $|S| \leq 2\pi$.

Proof : Given $k \in \mathbb{Z}$, $k \neq 0$, let $B_k = S \cap (S + 2k\pi)$, and write $k = 2^p q$ with $p \geq 0$ and q odd, $p, q \in \mathbb{Z}$. For all $\xi \in B_k$, ξ belongs to both S and $S + 2k\pi$. From the definition of S we then deduce that $\xi \in 2^{p+1}E$ and $\xi \in 2^{p+1}E + 2^{p+1}q\pi$. Thus, $2^{-p-1}\xi \in E \cap (E + q\pi)$, which shows that $2^{-p-1}B_k \subset E \cap (E + \pi)$, since E is 2π-periodic. But (2.15) implies $|E \cap (E + \pi)| = 0$; hence, $|2^{-p-1}B_k| = 0$ and, consequently, $|S \cap (S + 2k\pi)| = |B_k| = 0$. This proves (i).

Inequality (ii) follows from (i) since this implies that the sets $S + 2k\pi$, $k \in \mathbb{Z}$, are mutually disjoint almost everywhere, and, hence,

$$\sum_{k\in\mathbb{Z}} \chi_S(\xi + 2k\pi) \leq 1 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}.$$

The argument is as follows:

$$|S| = \int_{\mathbb{R}} \chi_S(\xi)\, d\xi = \sum_{k\in\mathbb{Z}} \int_0^{2\pi} \chi_S(\xi + 2k\pi)\, d\xi = \int_0^{2\pi} \sum_{k\in\mathbb{Z}} \chi_S(\xi + 2k\pi)\, d\xi \leq 2\pi.$$

□

We now continue the proof of Theorem 2.19. Let us define

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi). \tag{2.22}$$

Then $|\hat{\varphi}| = \chi_S$, where $S = \bigcap_{j=1}^{\infty} 2^j E$. We claim that

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}. \tag{2.23}$$

If we assume (2.23), the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is orthonormal. Since m is continuous at 0 and $|m(0)| = 1$, we have $|\hat{\varphi}(0)| = 1$ and $|\hat{\varphi}|$ is continuous at 0. By Lemma 2.13 we only need to show that $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$; but this follows immediately from the definition of $\hat{\varphi}$ given in (2.22).

It remains to show (2.23). By part (i) of Lemma (2.21), the sets $S + 2k\pi$, $k \in \mathbb{Z}$, are disjoint almost everywhere and, hence,

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \chi_{S+2k\pi}(\xi) \le 1 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}.$$

On the other hand, (2.20) implies

$$\int_0^{2\pi} \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \, d\xi = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \, d\xi = |S| = \Big|\bigcap_{j=1}^{\infty} 2^j E\Big| = 2\pi,$$

which shows (2.23) for $a.e.\ \xi \in [0, 2\pi]$. Since the function on the left of (2.23) is 2π-periodic, the desired results follows. This finishes the proof of Theorem 2.19.

There is an immediate rephrasal of this theorem, which is most useful for applications.

Corollary 2.24 *Let m be a 2π-periodic measurable function defined on $\mathbb{R}$ such that m is continuous at 0 and $|m(0)| = 1$. Then m is a low-pass filter for an* MSF *wavelet if and only if $|m| = \chi_E$, where $E \subset \mathbb{R}$ is a measurable set that satisfies* (2.15) *and there exits a measurable set $C \subset \mathbb{R}$ such that $|C| = 2\pi$ and*

$$|m(2^{-j}\xi)| = 1 \qquad \textit{for } \ a.e.\ \xi \in C \quad \textit{and all } \ j \ge 1.$$

Proof : The necessity follows from Proposition 2.18. For the sufficiency we only need to show (2.20). Since $|m(2^{-j}\xi)| = 1$ for $a.e.\ \xi \in C$ and all $j \ge 1$, we obtain that $C \subset \bigcap_{j=1}^{\infty} 2^j E$ almost everywhere. Since $|C| = 2\pi$, Lemma (2.21) gives us

$$2\pi = |C| \le \Big|\bigcap_{j=1}^{\infty} 2^j E\Big| \le 2\pi.$$

□

Remark 3 Corollary (2.24) is stated in a form that is closer to the conditions in Theorem 2.1. Since $C = S = \bigcap_{j=1}^{\infty} 2^j E$ almost everywhere, (2.16) implies that $\{C + 2k\pi : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ almost everywhere.

Corollary 2.24 allows us to construct MSF wavelets. The examples we give in this section are not new, but they are included since we shall use them in the sequel. In the next section we shall use Corollary 2.24 to obtain a new family of MSF wavelets.

Example 1 Let $0 < a < 2\pi$ and $F = [-\frac{a}{2}, \pi - \frac{a}{2}]$. Define

$$m(\xi) = \sum_{k\in\mathbb{Z}} \chi_F(\xi + 2k\pi),$$

so that m is 2π-periodic. The set C described in Corollary 2.24 is $C = [-a, 2\pi - a]$, and we obtain the wavelet ψ^a for which $\widehat{\psi^a}(\xi) = e^{i\frac{\xi}{2}}\chi_{I_a}$, where $I_a = [-2a, -a] \cup [2\pi - a, 4\pi - 2a]$.

Example 2 Let $F = [-\frac{2}{3}\pi, -\frac{1}{2}\pi] \cup [-\frac{1}{3}\pi, \frac{1}{3}\pi] \cup [\frac{1}{2}\pi, \frac{2}{3}\pi]$, and define

$$m(\xi) = \sum_{k\in\mathbb{Z}} \chi_F(\xi + 2k\pi),$$

Then the set C described in Corollary 2.24 is $C = [-\frac{4}{3}\pi, -\pi] \cup [-\frac{2}{3}\pi, \frac{2}{3}\pi] \cup [\pi, \frac{4}{3}\pi]$, and we obtain the wavelet ψ for which $\hat{\psi} = e^{i\frac{\xi}{2}}\chi_K$, where $K = [-\frac{8}{3}\pi, -2\pi] \cup [-\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \pi] \cup [2\pi, \frac{8}{3}\pi]$. The reader can check that all the wavelets described in example 2 of [HWW] can also be obtained from Corollary 2.24.

3 A new family of MSF wavelets

In this section we use Corollary 2.24 to construct a new family of band-limited MSF wavelets. For a band-limited wavelet, ψ, such that $|\hat{\psi}|$ is continuous at zero, it is known (see [BSW]) that there is a neighborhood of the origin such that the Fourier transfrom of ψ must be zero in that neighborhood. The family of wavelets that we shall construct in this section shows that this neighborhood can be arbitrarily small (this was observed in [BSW]; perhaphs the approach here is more direct).

The inequalities

$$\frac{(j-1)\pi}{2^n} < \frac{j\pi}{2^n+1} < \frac{j\pi}{2^n} < \frac{(j+1)\pi}{2^n+1} < \pi,$$

valid for all natural numbers n and all $j = 1, 2, \cdots, 2^n - 1$, imply that the set

$$F_n = \Big\{\bigcup_{j=1}^{2^n-1} [-\frac{(j+1)\pi}{2^n+1}, -\frac{j\pi}{2^n}]\Big\} \bigcup [-\frac{\pi}{2^n+1}, \frac{\pi}{2^n+1}] \bigcup \Big\{\bigcup_{j=1}^{2^n-1} [\frac{j\pi}{2^n}, \frac{(j+1)\pi}{2^n+1}]\Big\} \qquad (3.1)$$

is a finite union of disjoint closed intervals within $(-\pi, \pi)$. For an $n \in \mathbb{N}$, define

$$m_n(\xi) = \sum_{k\in\mathbb{Z}} \chi_{F_n}(\xi + 2k\pi), \qquad \xi \in \mathbb{R}. \qquad (3.2)$$

Lemma 3.3 *For each $n \in \mathbb{N}$, the function m_n satisfies* (2.15).

Proof : A simple calculation shows that

$$\chi_{\{(F_n\cap[-\pi,0])+\pi\}}(\xi) + \chi_{F_n\cap[0,\pi]}(\xi) = \chi_{[0,\pi]}(\xi) \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}.$$

The π-periodicity of $|m_n(\cdot)|^2 + |m_n(\cdot+\pi)|^2$ gives us (1.5), which is equivalent to (2.15).

□

Lemma 3.4 *If $C_n = 2F_n$, then C_n has measure 2π and*

$$m_n(2^{-j}\xi) = 1 \qquad \textit{for } \ a.e.\ \xi \in C_n \quad \textit{and} \ \ j \in \mathbb{N}.$$

Proof : By Lemma 3.3 and remark 1 we have $|F_n| = \pi$, and, hence, the measure of the set C_n must be equal to 2π. If $\xi \in C_n$, $\frac{1}{2}\xi \in F_n$; thus, $m(\frac{\xi}{2}) = 1$. To prove the second statement of this lemma it is enough to show that if I is an interval in the union defining F_n (see 3.1), then $\frac{1}{2}I \subset F_n$. This is obviously true when $I = [-\frac{\pi}{2^n+1}, \frac{\pi}{2^n+1}]$. Now we assume

$$I = [\frac{j\pi}{2^n}, \frac{(j+1)\pi}{2^n+1}], \qquad j = 1, 2, \cdots, 2^n - 1,$$

and consider separately the cases where j is even or odd. If j is even, we can write $j = 2s$, and obtain

$$\frac{1}{2}I = \frac{1}{2}[\frac{2s\pi}{2^n}, \frac{(2s+1)\pi}{2^n+1}] \subset [\frac{s\pi}{2^n}, \frac{(s+1)\pi}{2^n+1}] \subset F_n.$$

When j is odd, we write $j = 2s + 1$, so that

$$\frac{1}{2}I = \frac{1}{2}[\frac{(2s+1)\pi}{2^n}, \frac{(2s+2)\pi}{2^n+1}] \subset [\frac{s\pi}{2^n}, \frac{(s+1)\pi}{2^n+1}] \subset F_n.$$

□

These last two lemmas together with Corollary 2.24 allow us to obtain a scaling function φ_n such that $\hat{\varphi}_n = \chi_{C_n}$, and an MSF wavelet ψ_n such that

$$\hat{\psi}_n(\xi) = e^{i\frac{\xi}{2}}\, \overline{m_n(\tfrac{\xi}{2}+\pi)}\, \hat{\varphi}_n(\tfrac{\xi}{2}) \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}. \tag{3.5}$$

This wavelet is band-limited; in fact, $\operatorname{supp} \hat{\psi}_n \subset [-4\pi, 4\pi]$. To see this observe that

$$\operatorname{supp} \hat{\varphi}_n = 2F_n \subset [-2\frac{2^n\pi}{2^n+1}, 2\frac{2^n\pi}{2^n+1}] \subset [-2\pi, 2\pi],$$

and use (3.5).

Lemma 3.6 *Let ψ_n be given by* (3.5). *Then we have*

(i) *if $\xi \in (-\frac{2\pi}{2^n+1}, \frac{2\pi}{2^n+1})$, then $\hat{\psi}_n(\xi) = 0$;*

(ii) *if $\xi \in (-\frac{2\pi}{2^n}, -\frac{2\pi}{2^n+1}) \cup (\frac{2\pi}{2^n+1}, \frac{2\pi}{2^n})$, then $|\hat{\psi}_n(\xi)| = 1$.*

Proof : In case (i) we have

$$\eta = \tfrac{\xi}{2} + \pi \in (\frac{2^n\pi}{2^n+1}, \frac{(2^n+2)\pi}{2^n+1}) = (\frac{2^n\pi}{2^n+1}, \pi] \cup (\pi, \frac{(2^n+2)\pi}{2^n+1}) \equiv X_1 \cup X_2.$$

On X_1, $m_n(\eta) = 0$ by (3.1) and (3.2). If $\eta \in X_2$, $\eta - 2\pi \in (-\pi, -\frac{2^n\pi}{2^n+1})$ and, hence,

$$m_n(\tfrac{\xi}{2}+\pi) = m_n(\eta) = m_n(\eta - 2\pi) = 0.$$

From (3.5) we obtain $\hat{\psi}_n(\xi) = 0$ when $\xi \in (-\frac{2\pi}{2^n+1}, \frac{2\pi}{2^n+1})$, and (i) is proved.

Observe that $|\hat{\psi}_n|$ is even, since $\hat{\varphi}_n$ is even and m_n is an even 2π-periodic function. Thus, to establish (ii) it suffices to consider $\xi \in (\frac{2\pi}{2^n+1}, \frac{2\pi}{2^n})$. For such a ξ,

$$\tfrac{\xi}{2} \in (\frac{\pi}{2^n+1}, \frac{\pi}{2^n}) \subset C_n, \qquad \text{and} \qquad \tfrac{\xi}{2} - \pi \in (-\frac{2^n\pi}{2^n+1}, -\frac{(2^n-1)\pi}{2^n}) \subset F_n.$$

Thus, by (3.5), $|\hat{\psi}(\xi)| = m_n(\frac{\xi}{2}+\pi)\hat{\varphi}_n(\frac{\xi}{2}) = m_n(\frac{\xi}{2}-\pi)\hat{\varphi}_n(\frac{\xi}{2}) = 1{\cdot}1 = 1.$

□

Remark 4 The interested reader can compare this result with Theorem 2.1 in [HWW] and observe that this is not one of the wavelets characterized by that theorem.

Remark 5 It was proved in [BSW] that for any band-limited wavelet ψ, such that $|\hat{\psi}|$ is continuous at zero, there is a neighborhood of the origin for which $\hat{\psi}$ is identically zero. The above lemma shows that this neighborhood can be as small as we wish, even for MSF wavelets.

When $n = 1$ we obtain example 2. The Shannon filter can be included in this family as the case $n = 0$. Figure 1 shows a computer graph of the wavelet ψ_2 for which $\hat{\psi}_2(\xi) = e^{i\frac{\xi}{2}}\chi_K(\xi)$, with

$$K = [-\tfrac{16}{5}\pi, -3\pi] \cup [-\tfrac{12}{5}\pi, -2\pi] \cup [-\tfrac{3}{5}\pi, -\tfrac{6}{5}\pi] \cup [-\tfrac{1}{2}\pi, -\tfrac{2}{5}\pi] \\ \cup [\tfrac{2}{5}\pi, \tfrac{1}{2}\pi] \cup [\tfrac{6}{5}\pi, \tfrac{3}{2}\pi] \cup [2\pi, \tfrac{12}{5}\pi] \cup [3\pi, \tfrac{16}{5}\pi].$$

Figure 1

4 A smoothing procedure

In this section we shall show how to obtain smooth approximations for a large number of MSF wavelets. The idea is to obtain a smoothing procedure for the corresponding low-pass filter that gives us a "new" low-pass filter for an MRA for which the scaling function can be chosen to be smooth.

Suppose that ψ is a band-limited MSF wavelet associated with an MRA for which its low-pass filter is given by

$$m(\xi) = \sum_{k\in\mathbb{Z}} \chi_F(\xi + 2k\pi),$$

where $F = \bigcup_{\ell=1}^{n} I_\ell$ is a finite union of disjoint intervals contained in $(-\pi, \pi)$. We choose the scaling function φ so that $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$.

Given $\varepsilon > 0$ and $r = 0, 1, 2, \cdots, \infty$, let s_ε be a C^r function defined on $\mathbb{R}$ such that $s_\varepsilon(x) = 0$ for all $x < -\varepsilon$, and

$$s_\varepsilon^2(x) + c_\varepsilon^2(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}, \tag{4.1}$$

where $c_\varepsilon(x) \equiv s_\varepsilon(-x)$. It is not difficult to construct such a function having any desired degree of smoothness. Details can be found in [AWW] and [Wic]. Figure 2 shows a sketch of the graphs of these functions.

Figure 2

If $I_\ell = (a_\ell, b_\ell)$, $\ell = 1, \cdots, n$, with $-\pi < a_1 < b_1 < a_2 < b_2 < \cdots < b_n < \pi$, we choose a positive ε small enough so that

$$-\pi < a_1 - \varepsilon < \cdots < a_\ell + \varepsilon < b_\ell - \varepsilon < b_\ell + \varepsilon < a_{\ell+1} - \varepsilon < \cdots < b_n + \varepsilon < \pi.$$

We assume $0 \in (a_{\ell_o}, b_{\ell_o})$ for some ℓ_o, and choose ε so that $\varepsilon \le \min\{-a_{\ell_o}, b_{\ell_o}\}$. Let m_ε be the 2π-periodic function given by

$$m_\varepsilon(\xi) = \sum_{k\in\mathbb{Z}} \Big\{ \sum_{\ell=1}^{n} s_\varepsilon(\xi - a_\ell + 2k\pi) c_\varepsilon(\xi - b_\ell + 2k\pi) \Big\}. \tag{4.2}$$

What we have done is to apply a smoothing precedure at each point of discontinuity of χ_F (see figure 3) and extend the new function 2π-periodically to $\mathbb{R}$.

Figure 3

We can apply Theorem 2.1 to m_ε. To see this observe that $m_\varepsilon(0) = 1$, and m_ε satisfies (1.5) since (4.1) holds. Let $K = \operatorname{supp} \hat{\varphi}$. Since ψ is band-limited, so is φ and, hence, K is compact and satisfies (2.2) and (2.3) (by Corollary 2.24 and remark 3).

By Theorem 2.1, if $m_\varepsilon \in C^{r+1}$ $(r \ge 0)$, we can construct a scaling function φ_ε by

$$\hat{\varphi}_\varepsilon(\xi) = \prod_{j=1}^{\infty} m_\varepsilon(2^{-j}\xi),$$

and a wavelet ψ_ε such that

$$\hat{\psi}_\varepsilon(\xi) = e^{i\frac{\xi}{2}}\, \overline{m_\varepsilon(\tfrac{\xi}{2} + \pi)}\, \hat{\varphi}_\varepsilon(\tfrac{\xi}{2}). \tag{4.3}$$

Moreover, both $\hat{\psi}_\varepsilon$ and $\hat{\varphi}_\varepsilon$ belong to C^r.

It is clear that $\|m - m_\varepsilon\|_{L^2(-\pi,\pi)} \longrightarrow 0$ as $\varepsilon \to 0$. We also have the following approximation result:

Theorem 4.4 *If $m, m_\varepsilon, \varphi, \varphi_\varepsilon, \psi$ and ψ_ε are as above, then we have*

(i) $\|\hat\varphi - \hat\varphi_\varepsilon\|_{L^2(\mathbb{R})} \longrightarrow 0$ *as* $\varepsilon \to 0$;

(ii) $\|\hat\psi - \hat\psi_\varepsilon\|_{L^2(\mathbb{R})} \longrightarrow 0$ *as* $\varepsilon \to 0$.

Proof : Fix $\varepsilon_o > 0$ and $j_o \in \mathbb{N}$ such that $m_\varepsilon(\xi) = 1$ for $\xi \in (-2^{-j_o}, 2^{-j_o}) \subset (a_{\ell_o}, b_{\ell_o})$ when $\varepsilon < \varepsilon_o$. This can be done since $0 \in (a_{\ell_o}, b_{\ell_o}) \subset F$. Since ψ is band-limited, so is φ and, thus, there exists a $J \in \mathbb{N}$ such that $\operatorname{supp}\hat\varphi = K \subset (-2^J, 2^J)$. We claim that if $k_o = j_o + J$, then

$$\hat\varphi_\varepsilon(\xi) = \prod_{j=1}^{k_o} m_\varepsilon(2^{-j}\xi) \qquad \text{for all } \ \xi \in K \quad \text{and all } \ \varepsilon < \varepsilon_o; \tag{4.5}$$

and

$$\hat\varphi(\xi) = \prod_{j=1}^{k_o} m(2^{-j}\xi) \qquad \text{for all } \ \xi \in K. \tag{4.6}$$

To see (4.5) observe that if $j > k_o = j_o + J$ and $\xi \in K$,

$$2^{-j}\xi \in 2^{-j}K \subset (-2^{J-j}, 2^{J-j}) \subset (-2^{-j_o}, 2^{-j_o});$$

and, hence, $m_\varepsilon(2^{-j}\xi) = 1$ for all $\varepsilon < \varepsilon_0$. A similar argument establishes (4.6) since $m \equiv 1$ on $(-2^{-j_o}, 2^{-j_o}) \subset (a_{\ell_o}, b_{\ell_o})$.

Formulae (4.5) and (4.6) imply

$$\Delta_K(\varepsilon) \equiv \|\hat\varphi_\varepsilon - \hat\varphi\|_{L^2(K)} \longrightarrow 0 \qquad \text{as } \ \varepsilon \to 0. \tag{4.7}$$

Thus, all we need to show is

$$\int_{\mathbb{R}-K} |\hat\varphi_\varepsilon(\xi)|^2 \, d\xi \longrightarrow 0 \qquad \text{as } \ \varepsilon \to 0,$$

because $\hat\varphi \equiv 0$ on $\mathbb{R} - K$. From

$$\sqrt{2\pi} = \|\hat\varphi\|_2 = \|\hat\varphi\|_{L^2(K)} \le \|\hat\varphi_\varepsilon - \hat\varphi\|_{L^2(K)} + \|\hat\varphi_\varepsilon\|_{L^2(K)} = \Delta_K(\varepsilon) + \|\hat\varphi_\varepsilon\|_{L^2(K)},$$

we deduce that, for ε small enough,

$$\int_K |\hat\varphi_\varepsilon(\xi)|^2 \, d\xi \ge [\sqrt{2\pi} - \Delta_K(\varepsilon)]^2.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}-K} |\hat\varphi_\varepsilon(\xi)|^2 \, d\xi &= \int_{\mathbb{R}} |\hat\varphi_\varepsilon(\xi)|^2 \, d\xi - \int_K |\hat\varphi_\varepsilon(\xi)|^2 \, d\xi \\ &\le 2\pi - [\sqrt{2\pi} - \Delta_K(\varepsilon)]^2 \longrightarrow 0, \qquad \text{as } \ \varepsilon \to 0. \end{aligned}$$

This shows part (i) of the theorem. Part (ii) follows immediately from (4.3) and part (i).

□

5 Two examples of a different nature

The smoothing procedure developed in section 3 will be applied to the wavelets ψ^a and ψ of examples 1 and 2, respectively. In the first case the new wavelets, ψ^a_ε, are band-limited and their Fourier transforms have supports that are very close to the support of $\widehat{\psi^a}$ when ε is small compared to a. In the second case there is a surprise: the new wavelets, ψ_ε, that approach ψ in $L^2(\mathbb{R})$, are not band-limited for any admissible choice of ε.

We first consider the wavelet ψ^a of example 1. In this case we can choose the scaling function φ^a satisfying

$$\widehat{\varphi^a}(\xi) = \chi_{[-a,2\pi-a]}(\xi),$$

and, thus,

$$m^a(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\frac{a}{2},\pi-\frac{a}{2}]}(\xi + 2k\pi).$$

We shall work out the details for the case $0 < a \leq \pi$. The case $\pi \leq a < 2\pi$ can be treated in a similar way. Let $0 < \varepsilon < \frac{a}{2}$ and apply our smoothing procedure to the low-pass filter m^a to obtain a new filter m^a_ε, a scaling function φ^a_ε with

$$\widehat{\varphi^a_\varepsilon}(\xi) = \prod_{j=1}^{\infty} m^a_\varepsilon(2^{-j}\xi),$$

and the wavelet ψ^a_ε such that

$$\widehat{\psi^a_\varepsilon}(\xi) = e^{i\frac{\xi}{2}}\, \overline{m^a_\varepsilon(\tfrac{\xi}{2}+\pi)}\, \widehat{\varphi^a_\varepsilon}(\tfrac{\xi}{2}). \tag{5.1}$$

Lemma 5.2 *Let $0 < a \leq \pi$ and $0 < \varepsilon \leq \frac{a}{6}$. Then $\widehat{\varphi^a_\varepsilon}$ has support contained in $I^a_\varepsilon = [-a-2\varepsilon, 2\pi - a + 2\varepsilon]$; moreover,*

$$\widehat{\varphi^a_\varepsilon}(\xi) = m^a_\varepsilon(\tfrac{\xi}{2})\chi_{I^a_\varepsilon}(\xi). \tag{5.3}$$

Proof : The graph in figure 4 is helpful for the proof of this theorem. Suppose $\xi \in 2^j[\pi - \frac{a}{2} + \varepsilon, 2\pi - a + 2\varepsilon]$, $j \geq 1$. Then the condition $\varepsilon \leq \frac{a}{6}$ implies

$$2^{-j}\xi \in [\pi - \tfrac{a}{2} + \varepsilon, 2\pi - a + 2\varepsilon] \subset [\pi - \tfrac{a}{2} + \varepsilon, 2\pi - \tfrac{a}{2} - \varepsilon].$$

But the last interval lies in the complement of $\operatorname{supp} m^a_\varepsilon$. Thus, $m^a_\varepsilon(2^{-j}\xi) = 0$, and it follows that

$$\widehat{\varphi^a_\varepsilon}(\xi) = \{\prod_{k=1}^{j-1} m^a_\varepsilon(2^{-k}\xi)\}\, m^a_\varepsilon(2^{-j}\xi)\, \{\prod_{k=j+1}^{\infty} m^a_\varepsilon(2^{-k}\xi)\} = 0$$

when $\xi \geq 2\pi - a + 2\varepsilon$ (the rigth hand endpoint of I^a_ε).

Now we let $\xi \in 2^j[-a-2\varepsilon, -\frac{a}{2}-\varepsilon]$, $j \geq 1$. Then the fact $\varepsilon \leq \frac{a}{6} \leq \frac{\pi}{6}$ leads us to

$$2^{-j}\xi \in [-a-2\varepsilon, -\tfrac{a}{2}-\varepsilon] \subset [-\tfrac{a}{2}-\pi+\varepsilon, -\tfrac{a}{2}-\varepsilon].$$

Again, the last interval lies in the complement of $\operatorname{supp} m^a_\varepsilon$. Hence, as before, $\widehat{\varphi^a_\varepsilon}(\xi) = 0$ for $\xi \leq -a-2\varepsilon$. This shows $\operatorname{supp}\widehat{\varphi^a_\varepsilon} \subset I^a_\varepsilon$. Equality (5.3) will follow from this if we can show that $m^a_\varepsilon(2^{-j}\xi) = 1$ when $j \geq 2$ and $\xi \in I^a_\varepsilon$. But this is the case since

$$2^{-j}[-a-2\varepsilon, 2\pi-a+2\varepsilon] \subset \tfrac{1}{4}[-a-2\varepsilon, 2\pi-a+2\varepsilon] \subset [-\tfrac{a}{2}+\varepsilon, \pi-\tfrac{a}{2}-\varepsilon]$$

for $j \geq 2$ (again, the last inclusion follows from $\varepsilon \leq \frac{a}{6} \leq \frac{\pi}{6}$).

□

Figure 4 : The support of m^a_ε.

In [HWW] we proved a theorem that characterizes all wavelets whose Fourier transforms are supported in the interval $S_a = [-\frac{8}{3}a, 4\pi - \frac{4}{3}a]$. The wavelets we have just constructed (by a different method) are members of this class. Since $\varepsilon \leq \frac{a}{6}$ and

$$\operatorname{supp}\widehat{\varphi^a_\varepsilon} \subset I^a_\varepsilon = [-a-2\varepsilon, 2\pi-a+2\varepsilon],$$

it follows from (5.1) that

$$\operatorname{supp}\widehat{\psi^a_\varepsilon} \subset 2I^a_\varepsilon = [-2a-4\varepsilon, 4\pi-2a+4\varepsilon] \subset S_a.$$

However, if we allow ε to exceed $\frac{a}{6}$ we obtain a completely different collection of wavelets that have much broader supports which increase unboundedly as ε approaches $\frac{a}{2}$. In fact, when $\varepsilon = \frac{a}{2}$ (the largest value that is possible in our procedure) the wavelet ψ^a_ε is not band-limited. Thus, the construction we are now presenting produces wavelets that, as those in [HWW], are "associated" with the MSF wavelet ψ^a, but they are really of a different nature. Those associated with Theorem 2.1 in [HWW] are "close" to ψ^a in the L^2 norm and have bands that are "metrically close" to $\operatorname{supp}\widehat{\psi^a}$. This last property does not hold for the wavelets ψ^a_ε we are now constructing.

It is easy to see that if $\varepsilon = \frac{a}{2}$, ψ^a_ε is not band-limited. Since $\widehat{\varphi^a_\varepsilon}$ is continuous and $\widehat{\varphi^a_\varepsilon}(0) = 1$, $\widehat{\varphi^a_\varepsilon}$ is nonzero in a neighborhood of 0. If ψ^a_ε were band-limited, $\widehat{\psi^a_\varepsilon}$ would have to be zero in a neighborhood of 0 (see [BSW]). Also

$$\widehat{\psi^a_\varepsilon}(\xi) = e^{i\frac{\xi}{2}}\,\overline{m^a_\varepsilon(\tfrac{\xi}{2}+\pi)}\,\widehat{\varphi^a_\varepsilon}(\tfrac{\xi}{2}).$$

These two facts would force $m^a_\varepsilon(\xi)$ to be identically zero in a neighborhood of π. But when $\varepsilon = \frac{a}{2}$, $m^a_\varepsilon(\xi) \neq 0$ for all $\xi \in (-a, \pi)$.

We now study the case $\varepsilon \leq \frac{a}{2}$. For technical reasons, that will become clear during the proof, we shall restrict ourselves to the case $a \leq \frac{\pi}{2}$.

Theorem 5.4 *Let $0 < 2\varepsilon \leq a \leq \frac{\pi}{2}$, and define*

$$j_\varepsilon \equiv j^a_\varepsilon = \begin{cases} \left[\log_2\left(\frac{a+2\varepsilon}{a-2\varepsilon}\right)\right] & \text{if } \varepsilon < \frac{a}{2}, \\ \infty & \text{if } \varepsilon = \frac{a}{2}. \end{cases}$$

Then, $\widehat{\varphi^a_\varepsilon}$ is identically zero on

$$E = (-\infty, -a-2\varepsilon] \cup \{\bigcup_{j=1}^{j_\varepsilon} [2^j\pi - 2^{j-1}a + 2^j\varepsilon, 2^{j+1}\pi - a - 2\varepsilon]\} \cup [2^{j_\varepsilon+1}\pi - 2^{j_\varepsilon}a + 2^{j_\varepsilon+1}\varepsilon, \infty).$$

Thus, φ^a_ε has a band that lies within $[-a-2\varepsilon, 2^{j_\varepsilon+1}\pi - 2^{j_\varepsilon}a + 2^{j_\varepsilon+1}\varepsilon]$ when $\varepsilon < \frac{a}{2}$. Moreover, if $s_\varepsilon(\xi) \neq 0$ for all $\xi \in (-\varepsilon, \varepsilon)$ (see (4.1)), then $\widehat{\varphi^a_\varepsilon}$ is nonzero at precisely the points of the complement of E.

Proof : Figure 4 will be helpful in the proof of this result. Suppose that $\xi \in 2^j[-a-2\varepsilon, -\frac{a}{2}-\varepsilon]$, $j \geq 1$. Since $0 < 2\varepsilon \leq a \leq \frac{\pi}{2}$, we have

$$2^{-j}\xi \in [-a-2\varepsilon, -\tfrac{a}{2}-\varepsilon] \subset [-\tfrac{a}{2}-\pi+\varepsilon, -\tfrac{a}{2}-\varepsilon].$$

But the last interval lies outside of $\operatorname{supp} m^a_\varepsilon$. Hence, $\widehat{\varphi^a_\varepsilon}(\xi) = 0$ for $\xi \leq -a-2\varepsilon$. Let

$$X_j = [2^j\pi - 2^{j-1}a + 2^j\varepsilon, 2^{j+1}\pi - a - 2\varepsilon], \qquad j \in \mathbb{Z}.$$

If $j = 1$ and $\xi \in X_1$, then $\frac{\xi}{2} \in [\pi - \frac{a}{2} + \varepsilon, 2\pi - \frac{a}{2} - \varepsilon]$, and this last interval lies in the complement of the support of m^a_ε. Hence, $\widehat{\varphi^a_\varepsilon}(\xi) = 0$ on X_1. We proceed by induction. Suppose $\widehat{\varphi^a_\varepsilon} \equiv 0$ on X_ℓ for all $\ell \leq j$. Then, for all $\xi \in X_{j+1}$,

$$\tfrac{\xi}{2} \in [2^j\pi - 2^{j-1}a + 2^j\varepsilon, 2^{j+1}\pi - \tfrac{a}{2} - \varepsilon] = X_j \cup [2^{j+1}\pi - \pi - \tfrac{a}{2} + \varepsilon, 2^{j+1}\pi - \tfrac{a}{2} - \varepsilon] \equiv X_j \cup G_j.$$

If $\frac{\xi}{2} \in X_j$, then $\widehat{\varphi^a_\varepsilon}(\frac{\xi}{2}) = 0$ by our inductive hypothesis. If $\frac{\xi}{2} \in G_j$, then $m^a_\varepsilon(\frac{\xi}{2}) = 0$ since $G_j = [-\pi - \frac{a}{2} + \varepsilon, -\frac{a}{2} - \varepsilon] \pmod{2\pi}$. Thus, $\widehat{\varphi^a_\varepsilon}(\xi) = m^a_\varepsilon(\frac{\xi}{2})\widehat{\varphi^a_\varepsilon}(\frac{\xi}{2}) = 0$ for all $\xi \in X_{j+1}$. We now show that

$$\bigcup_{j > j_\varepsilon} X_j = [2^{j_\varepsilon+1}\pi - 2^{j_\varepsilon}a + 2^{j_\varepsilon+1}\varepsilon, \infty),$$

and this clearly implies that $\widehat{\varphi^a_\varepsilon}$ is identically 0 on E. This equality follows if we can prove that the left endpoint of X_{j+1} is smaller than the right endpoint of X_j when $j > j_\varepsilon$; that is,

$$2^{j+1}\pi - 2^j a + 2^{j+1}\varepsilon \le 2^{j+1}\pi - a - 2\varepsilon \qquad \text{for all } \ j > j_\varepsilon.$$

This inequality is equivalent to $2^j \ge \frac{a+2\varepsilon}{a-2\varepsilon}$, which is true by our definition of j_ε.

It remains to show that $\widehat{\varphi^a_\varepsilon}$ is nonzero at each point of the complement of E. Clearly $\widehat{\varphi^a_\varepsilon}(\xi) \ne 0$ when $\xi \in (-a-2\varepsilon, 2\pi - a + 2\varepsilon)$, since $m^a_\varepsilon(2^{-j}\xi) \ne 0$ when $j \ge 1$. It suffices to show that $\widehat{\varphi^a_\varepsilon}(\xi) \ne 0$ for every $\xi \in E_j$, where

$$E_j = [2^{j+1}\pi - a - 2\varepsilon, 2^{j+1}\pi - 2^j a + 2^{j+1}\varepsilon], \qquad j = 1, 2, \cdots, j_\varepsilon.$$

Let us prove it by induction. If $\xi \in E_1 = [4\pi - a - 2\varepsilon, 4\pi - 2a + 4\varepsilon]$, then $\frac{\xi}{2} \in [2\pi - \frac{a}{2} - \varepsilon, 2\pi - a + 2\varepsilon]$. Thus, $\widehat{\varphi^a_\varepsilon}(\frac{\xi}{2}) \ne 0$ and $m^a_\varepsilon(\frac{\xi}{2}) \ne 0$, since

$$[2\pi - \tfrac{a}{2} - \varepsilon, 2\pi - a + 2\varepsilon] = [-\tfrac{a}{2} - \varepsilon, -a + 2\varepsilon] + 2\pi \subset \operatorname{supp} m^a_\varepsilon.$$

Now suppose $j > 1$ and $\widehat{\varphi^a_\varepsilon}(\xi) \ne 0$ for $\xi \in E_{j-1}$. Then, for $\xi \in E_j$,

$$\tfrac{\xi}{2} \in [2^j\pi - \tfrac{a}{2} - \varepsilon, 2^j\pi - 2^{j-1}a + 2^j\varepsilon] \subset E_{j-1}.$$

Thus, $\widehat{\varphi^a_\varepsilon}(\frac{\xi}{2}) \ne 0$ by our inductive assumption. We can also write

$$\tfrac{\xi}{2} \in [2^j\pi - \tfrac{a}{2} - \varepsilon, 2^j\pi - 2^{j-1}a + 2^j\varepsilon] = 2^j\pi + [-\tfrac{a}{2} - \varepsilon, -2^{j-1}a + 2^j\varepsilon] \subset \operatorname{supp} m^a_\varepsilon,$$

which shows $m^a_\varepsilon(\frac{\xi}{2}) \ne 0$. Hence, $\widehat{\varphi^a_\varepsilon}(\xi) = m^a_\varepsilon(\frac{\xi}{2})\widehat{\varphi^a_\varepsilon}(\frac{\xi}{2}) \ne 0$ for all $\xi \in E_j$.

□

Figure 5 is a three dimensional representation of the family of wavelets ψ^a_ε for $a = \pi$ and $0 \le \varepsilon \le \frac{\pi}{2}$. As ε increases from 0, the graph of ψ^a_ε approaches the viewer; the case $\varepsilon = 0$ is the Shannon wavelet. (This graph has been obtained in collaboration with J. Soria).

Figure 5

We now apply our smoothing procedure to the MSF wavelet ψ such that $\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_K$, where K is the particular simple disjoint union of four intervals described in example 2. In this case,

$$m(\xi) = \sum_{k\in\mathbb{Z}} \chi_F(\xi + 2k\pi),$$

where $F = [-\frac{2}{3}\pi, -\frac{1}{2}\pi] \cup [-\frac{1}{3}\pi, \frac{1}{3}\pi] \cup [\frac{1}{2}\pi, \frac{2}{3}\pi]$. With $0 < \varepsilon \leq \frac{1}{12}\pi$ we obtain a low-pass filter m_ε (see figure 6), a scaling function φ_ε such that

$$\hat{\varphi}_\varepsilon(\xi) = \prod_{j=1}^{\infty} m_\varepsilon(2^{-j}\xi),$$

and a wavelet ψ_ε satisfying

$$\hat{\psi}_\varepsilon(\xi) = e^{i\frac{\xi}{2}}\, \overline{m_\varepsilon(\tfrac{\xi}{2} + \pi)}\, \hat{\varphi}_\varepsilon(\tfrac{\xi}{2}),$$

which belongs to C^r $(r = 0, 1, 2, \cdots, \infty)$ if we choose $m_\varepsilon \in C^{r+1}$.

Figure 6

Theorem 5.5 *In the case we are considering, if* $m_\varepsilon \in C^{r+1}$ $(r = 0, 1, 2, \cdots, \infty)$ *and* $0 < \varepsilon \leq \frac{1}{12}\pi$*, then the scaling function* φ_ε *satisfies*

$$\hat{\varphi}_\varepsilon(\pm 2^\ell \tfrac{2}{3}\pi) \neq 0, \qquad \ell = 0, 1, 2, \cdots,$$

and $\hat{\varphi}_\varepsilon \in C^r$*. It follows that* $\hat{\psi}_\varepsilon \in C^r$ *and* ψ_ε *is not band-limited.*

Proof : For $\ell = 0, 1, 2, \cdots$

$$\hat{\varphi}_\varepsilon(2^\ell \tfrac{2}{3}\pi) = m_\varepsilon(2^{\ell-1}\tfrac{2}{3}\pi)\cdots m_\varepsilon(\tfrac{2}{3}\pi)\hat{\varphi}_\varepsilon(\tfrac{2}{3}\pi).$$

Observe that $\hat{\varphi}_\varepsilon(\frac{2}{3}\pi) = \frac{1}{\sqrt{2}}$ and $m_\varepsilon(\frac{2}{3}\pi) = \frac{1}{\sqrt{2}}$ (see figure 6). Moreover, the points $2^j\frac{2}{3}\pi \,(\text{mod } 2\pi)$, $j \geq 1$, belong to $\{-\frac{2}{3}\pi, \frac{2}{3}\pi\}$. The fact that m_ε is even and these observations show that $m_\varepsilon(2^{\ell-s}\frac{2}{3}\pi) = \frac{1}{\sqrt{2}}$ for all $s = 1, \cdots, \ell$. Consequently,

$$\hat{\varphi}_\varepsilon(2^\ell \tfrac{2}{3}\pi) = (\tfrac{1}{\sqrt{2}})^{\ell+1} \neq 0, \tag{5.6}$$

which finishes the proof of the theorem, since $\hat{\varphi}_\varepsilon$ is even.

□

Remark 6 Formula (5.6) shows that the decay of $|\hat{\varphi}_\varepsilon(\xi)|$, at infinity, is not faster than $C|\xi|^{-\frac{1}{2}}$. This behaviour is also shared by $\hat{\psi}_\varepsilon$. In fact, it is easy to see that

$$|\hat{\psi}_\varepsilon(2^\ell \tfrac{2}{3}\pi)| = (\tfrac{1}{\sqrt{2}})^{\ell+1}, \qquad \ell = 0, 1, 2, \cdots.$$

This shows that for wavelets ψ such that $|\hat{\psi}|$ is continuous on $\mathbb{R}$, the condition $|\hat{\psi}| = O((1+|\xi|)^{-\alpha-\frac{1}{2}})$, $\alpha > 0$, in P. Auscher's theorem (see [Aus] or Theorem 3.1 in [HWW]) is not necessary for ψ to be associated with an MRA.

Computer figures of the modulus of the Fourier transforms of the wavelet ψ_ε and the scaling function φ_ε for $\varepsilon = \frac{1}{12}\pi$ are given in figure 7.

Figure 7

Remark 7 The fact that φ_ε is not band-limited also follows from Proposition 5.12 in [BSW].

We now can amplify the discussion at the very end of the first section. The passage from the MSF low-pass filter m to the "smoother" filter m_ε is quite similar to the method developed in [BSW] and [HWW], that allows us to obtain a wavelet ψ_ε from an MSF wavelet ψ. In these earlier works, the support of $\hat{\psi}_\varepsilon$ is metrically close to the support of $\hat{\psi}$ (where ψ is either the Shannon wavelet or $\psi = \psi^a$); furthermore, $\|\psi_\varepsilon - \psi\|_2$ is small when ε is small. As we can see from the examples in this section, the last feature is still true; however, $\operatorname{supp}\hat{\psi}_\varepsilon$ is dramatically different from $\operatorname{supp}\hat{\psi}$. This difference arises in the MRA construction of a wavelet from its associated low-pass filter.

6 Invariant cycles and wavelets

In Theorem 5.5, the points $\frac{2}{3}\pi$ and $-\frac{2}{3}\pi$ play a special rôle. We consider other families of points that have similar properties with respect to the wavelets with continuous Fourier transforms.

Let $\rho : (-\pi, \pi] \longrightarrow (-\pi, \pi]$ be the transformation $x \mapsto 2x \,(\mathrm{mod}\, 2\pi)$. Suppose $\mathcal{O}(x) = \{\rho^n(x) : n = 0, 1, 2, \cdots\}$ is the corresponding orbit of x. We say that $x \in (-\pi, \pi]$ generates an invariant cycle for the transformation ρ if there exists a $k \in \mathbb{N}$, such that $\rho^k(x) = x$. The smallest such k is called the lenght of the cycle.

The point $x = \frac{2}{3}\pi$ generates an invariant cycle of lenght 2; and the point $x = \frac{2}{9}\pi$ generates an invariant cycle of lenght 6. The next lemma describes all the generators of invariant cycles.

Lemma 6.1 *The point $\xi \in (-\pi, \pi]$ generates an invariant cycle for ρ if and only if*

$$\xi = \frac{\ell}{2^k - 1} 2\pi \tag{6.2}$$

for some $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ such that $-2^{k-1} + 1 \leq \ell \leq 2^{k-1} - 1$.

Proof : Suppose $\xi \in (-\pi, \pi]$ generates an invariant cycle of lenght k. Then there exists an $\ell \in \mathbb{Z}$ such that $2^k\xi - 2\ell\pi = \xi$. Hence $\xi = \frac{\ell}{2^k-1}2\pi$, with $|\ell| \leq 2^{k-1} - 1$. Conversely, if $\xi = \frac{\ell}{2^k-1}2\pi$ for some $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ such that $|\ell| \leq 2^{k-1} - 1$, then we have

$$2^k\xi - 2\ell\pi = 2^k \frac{\ell}{2^k - 1} 2\pi - 2\ell\pi = \frac{\ell}{2^k - 1} 2\pi = \xi,$$

which shows $\rho^k(\xi) = \xi$.

□

The invariant cycles appear in the construction of MSF low-pass filters. One of these filters is given by the periodization of χ_F, where $F = [-\frac{2}{3}\pi, -\frac{1}{2}\pi] \cup [-\frac{1}{3}\pi, \frac{1}{3}\pi] \cup [\frac{1}{2}\pi, \frac{2}{3}\pi]$ (see example 2).

Another example is the family of wavelets described at the beginning of section 3. The supports of the associated filters involve the orbits of the points

$$\xi_n = \frac{1}{2^n+1} 2\pi = \frac{2^n-1}{2^{2n}-1} 2\pi, \qquad n \in \mathbb{N}, \tag{6.3}$$

which are of the form expressed in Lemma 6.1. Observe that, the orbit in question is

$$\mathcal{O}(\xi_n) = \{\frac{2\pi}{2^n+1}, \frac{4\pi}{2^n+1}, \dots, \frac{2^n\pi}{2^n+1}, -\frac{2\pi}{2^n+1}, \dots, -\frac{2^n\pi}{2^n+1}\},$$

which consists of the doubles of some of the endpoints of the set F_n given by (3.1).

In example 5 of [FW] a family of MSF wavelets that arise from MRA's was presented. This is the family $\{\psi_n : n \in \mathbb{N}\}$ for which $|\hat{\psi}_n| = \chi_{W_n}$, where

$$W_n = [-2^{n+1}\pi + 2\pi - \frac{2^{n+1}-2}{2^{n+1}-1}\pi, -2^{n+1}\pi + 2\pi) \cup [-2\pi + \frac{1}{2^{n-1}}\pi, -\frac{2^{n+1}-2}{2^{n+1}-1}\pi) \\ \cup [\frac{2}{2^{n+1}-1}\pi, \frac{1}{2^{n-1}}\pi) \cup [2\pi, 2\pi + \frac{2}{2^{n+1}-1}\pi).$$

It is easy to see that $\hat{\varphi}_n = \chi_{C_n}$, where

$$C_n = \Big\{\bigcup_{p=1}^{n} [-2^p\pi + \frac{2^p}{2^{n+1}-1}\pi, -2^p\pi + \frac{2^p}{2^n}\pi)\Big\} \\ \bigcup [-\frac{2^{n+1}-2}{2^{n+1}-1}\pi, \frac{2}{2^{n+1}-1}\pi) \bigcup \Big\{\bigcup_{p=1}^{n} [\frac{2^p}{2^n}\pi, \frac{2^{p+1}}{2^{n+1}-1}\pi)\Big\}.$$

The low-pass filter m_n is given by the periodization of the set $F_n = \frac{1}{2}C_n$. The set F_n is not contained in $[-\pi, \pi]$, but it is not difficult to see that

$$m_n(\xi) = \sum_{k \in \mathbb{Z}} \chi_{S_n}(\xi + 2k\pi),$$

where

$$S_n = [-\pi + \frac{1}{2^{n+1}-1}\pi, -\pi + \frac{1}{2^n}\pi) \\ \cup [-\frac{2^n-1}{2^{n+1}-1}\pi, \frac{1}{2^{n+1}-1}\pi) \cup [\frac{1}{2^n}\pi, \frac{2^n}{2^{n+1}-1}\pi), \tag{6.4}$$

and the set S_n is contained in $[-\pi, \pi]$. This filter is related to the orbit generated by the point

$$\frac{1}{2^{n+1}-1} 2\pi. \tag{6.5}$$

We are now ready to prove the main result of this section.

Theorem 6.6 *Suppose that φ is a band-limited scaling function for an* MRA *with low-pass filter m, and $|\hat{\varphi}|$ is continuous. If $\xi \in (-\pi, \pi]$ generates an invariant cycle for ρ and $\mathcal{O}(\xi)$ is its orbit, then*

$$\prod_{\eta \in \mathcal{O}(\xi)} m(\eta) = 0.$$

Proof : We prove the theorem in the case $\xi > 0$. Since $|\hat{\varphi}|$ is continuous, the orthonormality of the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is equivalent to

$$\sum_{\ell \in \mathbb{Z}} |\hat{\varphi}(\xi + 2\ell\pi)|^2 = 1 \qquad \text{for all } \ \xi \in \mathbb{R}.$$

(see (2.14)). The assumption that $\hat{\varphi}$ has compact support implies the existence of the largest positive $k \in \mathbb{Z}$, such that $\hat{\varphi}(\xi + 2k\pi) \neq 0$. Let $\mathcal{O}(\xi) = \{\xi_1, \xi_2, \cdots, \xi_p\}$ be the orbit of ξ, where $\xi_n = \rho^{n-1}(\xi)$, $1 \leq n \leq p$, and $\rho^p(\xi) = \xi$. Using the equality $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ (see (2.9)), and the 2π-periodicity of m, we obtain

$$\begin{aligned}
&\hat{\varphi}(2k\pi + \xi) \prod_{\eta \in \mathcal{O}(\xi)} m(\eta) = \hat{\varphi}(2k\pi + \xi) \prod_{j=1}^{p} m(\xi_j) \\
&= \hat{\varphi}(2k\pi + \xi) m(2k\pi + \xi) m(4k\pi + 2\xi) \cdots m(2^p k\pi + 2^{p-1}\xi) \\
&= \hat{\varphi}(4k\pi + 2\xi) m(4k\pi + 2\xi) \cdots m(2^p k\pi + 2^{p-1}\xi) = \cdots = \hat{\varphi}(2^{p+1}k\pi + 2^p\xi).
\end{aligned}$$

By Lemma (6.1), $\xi = \frac{\ell}{2^p - 1} 2\pi$ for $1 \leq \ell \leq 2^{p-1} - 1$. Thus, $2\ell\pi + \xi = 2^p\xi$, and, hence,

$$\hat{\varphi}(2k\pi + \xi) \prod_{\eta \in \mathcal{O}(\xi)} m(\eta) = \hat{\varphi}((2^p k + \ell)2\pi + \xi).$$

Since $2^p k + \ell > k$, the right hand side of the above equality is zero. The result follows from here since $\hat{\varphi}(\xi + 2k\pi) \neq 0$.

□

Remark 8 The ideas of the proof of this theorem are already contained in the proof of Proposition 5.12 in [BSW] where the case $\xi = \frac{2}{3}\pi$ is treated.

Remark 9 In the proof of Theorem 5.5 we exhibited a filter m_ε and an invariant cycle generated by $\xi = \frac{2}{3}\pi$ for which

$$\prod_{\eta \in \mathcal{O}(\xi)} m_\varepsilon(\eta) \neq 0.$$

In fact, we showed that neither φ_ε nor ψ_ε, in this case, can be band-limited because of this result. In Theorem 6.6, of course, one of our hyphothesis is that φ is band-limited.

Theorem 6.6 can be written using a wavelet instead of a scaling function

Corollary 6.7 *Suppose that ψ is a band-limited wavelet associated with an* MRA*, and $|\hat{\psi}|$ is continuous. If $\xi \in (-\pi, \pi]$ generates an invariant cycle for ρ and $\mathcal{O}(\xi)$ denotes its orbit, then*

$$\prod_{\eta \in \mathcal{O}(\xi)} \hat{\psi}(2(\eta - \pi)) = 0.$$

Proof : Use (2.10) and (2.12).

□

Theorem (6.6) allows us to show that if we apply our smoothing procedure, described at the beginning of section 4, to χ_{F_n}, where F_n is given by (3.1) and to χ_{S_n}, where S_n is given by (6.4), we obtain non band-limited wavelets with smooth Fourier transforms. To see this we only need to observe that $m_\varepsilon(\eta) \neq 0$ for all η belonging to an invariant cycle: for F_n take $\eta \in \mathcal{O}(\xi_n)$ where ξ_n is given by (6.3); for S_n take $\eta \in \mathcal{O}(\xi_n)$ where ξ_n is given by (6.5).

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